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ON THE DEGREE SEQUENCE OF AN EVOLVING RANDOM GRAPH PROCESS AND ITS CRITICAL PHENOMENON

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Abstract

In this paper we focus on the problem of the degree sequence for a random graph process with edge deletion. We prove that, while a specific parameter varies, the limit degree distribution of the model exhibits critical phenomenon.

Keywords: Degree sequence; power law; critical phenomenon; real-world networks

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1. Introduction and statement of the results

In the past decade, a lot of effort has been devoted to studying large-scale real-world networks and modeling their properties. For a general introduction to this topic, we refer the reader to [1], [3], [7], [20], and [24]. Although the study of real-world networks as graphs can be traced back a long time, such as the classical model proposed by Erdös and Rényi [15] and Gilbert [16], recent influential activity perhaps started with the work of Watts and Strogatz [25] about the 'small-world phenomenon' published in 1998. Another influential work may be due to the scale-free model proposed by Bollobás and Albert [5] in 1999. Since then, various forms of scale-free phenomenon have been widely revealed. In particular, power-law degree distributions have been extensively investigated. Many new models have been introduced to circumvent the shortcomings of the classical models introduced by Erdös and Rényi [15] and Grilbert [16]. One class of these new models aimed to explain the underlying causes for the emergence of power-law degree distributions; see, for example, [8], [9], [11], [12], [13], [14], and [18].

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Besides the power-law degree distributions (see [2] and [10]), other forms of the degree distributions can also be observed in real-world networks (see [4] and [22]). For example, Gaussian distributions can be observed in the acquaintance network of Mormons [6], and exponential distributions can be observed in the power grid of southern California [25]. On the other hand, the degree distribution of the network of world airports [4] interpolates between Gaussian and exponential distributions, whereas the degree distribution of the citation network in high energy physics [19] interpolates between exponential and power-law distributions. For more forms of degree distributions, we refer the reader to [21].

Different models often lead to different forms of degree distributions. An interesting problem arises naturally: does there exist some dynamically evolving random graph process which brings forth various degree distributions by continuous changing of its parameters only? To the best of the authors' knowledge, it seems that the problem and its answer have not been formulated in a mathematically rigorous manner. In this paper we focus on a model with edge deletions and provide precise analysis to show, while a parameter varies, that the model exhibits various degree distributions.

We begin by introducing our model and then state our main results. Consider the following process which generates a sequence of graphs $G_t = (V_t, E_t)$, $t \geq 1$. Write $v_t = |V_t|$ and $e_t = |E_t|$.

Time step 1. Let G_1 consist of an isolated vertex x_1.

Time step $t \geq 2$. (i) With probability $\alpha_1 > 0$ we add a vertex x_t to G_{t-1}. We then add m random edges incident with x_t. In the case in which $e_{t-1} > 0$, the m random neighbors $w_1, w_2, \dots, w_m$ are chosen independently. For $1 \leq i \leq m$ and $w \in V_{t-1}$,

$$\mathrm{P}(w_i = w) = \frac{d_w(t-1)}{2e_{t-1}}, \tag{1.1}$$

where $d_w(t-1)$ denotes the degree of vertex w at the beginning of substep t. Thus, neighbors are chosen by *preferential attachment.* In the case in which $e_{t-1} = 0$, we add a new vertex x_t and join it to a randomly chosen vertex in V_{t-1}.

(ii) With probability $\alpha - \alpha_1 \geq 0$ we add m random edges to existing vertices. If $e_{t-1} > 0$ then both endpoints are chosen independently with the same probabilities as in (1.1). Otherwise, we do nothing.

(iii) With probability $1 - \alpha \geq 0$ we delete $\min\{m, e_{t-1}\}$ randomly chosen edges from E_{t-1}.

Now we assume that

$$\tfrac{1}{2} < \alpha \leq 1, \qquad 0 < \alpha_1 \leq \alpha. \tag{1.2}$$

For given α and α_1 satisfying (1.2), define

$$\alpha_c := 4\alpha - 2, \qquad \eta := \frac{m\alpha_c}{2}, \tag{1.3}$$

and choose $\epsilon = \epsilon(\alpha, \alpha_1) \in (0, \eta)$ such that

$$\rho_\epsilon := \max\left\{\frac{m(\alpha_c - \alpha_1)}{2(\eta - \epsilon)}, \frac{1}{2}\right\} < 1. \tag{1.4}$$

Let

$$\beta = \frac{\alpha_c}{\alpha_c - \alpha_1}, \qquad \gamma = 1 - \frac{\alpha_1 - \alpha_c}{2(1-\alpha)}, \qquad \theta = \frac{2\alpha_c - \alpha_1}{2\alpha_c}, \qquad \mu = \frac{\alpha_c}{2(1-\alpha)}. \tag{1.5}$$

To obtain our main results, besides (1.2), we assume that

$$\alpha_1 < 2\alpha_c. \tag{1.6}$$

Now, let $D_k(t)$ be the number of vertices with degree $k \geq 0$ in G_t and let $\bar{D}_k(t)$ be the expectation of $D_k(t)$.

The main results of this paper are as follows.

Theorem 1.1. *Assume that (1.2) and (1.6) hold. Then α_c defined in (1.3) is a critical point for the degree sequence of the model satisfying the following conditions.*

1. *If $\alpha_1 < \alpha_c$ then there exists a constant $C_1 = C_1(m, \alpha, \alpha_1)$ such that, for any $\nu \in (0, 1 - \rho_\epsilon)$,*
$$\left|\frac{\bar{D}_k(t)}{t} - C_1 k^{-1-\beta}\right| = O(t^{\rho_\epsilon+\nu-1}) + O(k^{-2-\beta}). \tag{1.7}$$
2. *If $\alpha_1 > \alpha_c$ then there exists a constant $C_2 = C_2(m, \alpha, \alpha_1)$ such that*
$$\left|\frac{\bar{D}_k(t)}{t} - C_2 \gamma^k k^{-1+\beta}\right| = O(t^{-\theta}) + O(\gamma^k k^{-2+\beta}). \tag{1.8}$$
3. *If $\alpha_1 = \alpha_c$ then there exists a constant $C_c = C_c(m, \alpha, \alpha_1)$ such that, for any $\nu \in (0, \frac{1}{2})$,*
$$\left|\frac{\bar{D}_k(t)}{t} - C_c u_c(k)\right| = O(t^{-1/2+\nu}) \tag{1.9}$$
uniformly in k, where $u_c(k) = \int_0^1 t^{k-1} \mathrm{e}^{-\mu/(1-t)}\, \mathrm{d}t$.

With help of computer calculation, we know that $u_c(k)$ satisfies

$$\lim_{k\to\infty} \frac{\ln u_c(k)}{-k} = \lim_{k\to\infty} \frac{-\ln k}{\ln u_c(k)} = 0.$$

Remark 1.1. Compared with the model $G(\alpha_1, \alpha - \alpha_1, 0, 1 - \alpha, m)$ in [12], Theorem 1.1 extends the range of power laws from the case $\alpha_1 < 2\alpha - 1$ to $\alpha_1 < 4\alpha - 2 = \alpha_c$, where α_c is the critical point. Theorem 1.1 also extends the results of [14] with $\alpha_0 = 1 - \alpha$; actually, Cooper [14] showed that, in case of α_0 being small enough, the model possesses power-law degree distributions.

Based on Theorem 1.1, we can obtain the following two corollaries, which provide a complete distinction with respect to the parameters between the degree sequences for the present model.

Corollary 1.1. *If the parameters satisfy*

1. *$\alpha > \frac{2}{3}$, or*
2. *$\alpha \leq \frac{2}{3}$ and $\alpha_1 < \alpha_c$,*

then the present random graph process has power-law degree sequence (1.7).

Corollary 1.2. *Assume that $\alpha \leq \frac{2}{3}$.*

1. *If $\alpha_c < \alpha_1 < 2\alpha_c$ then the present random graph process has exponential degree sequence (1.8).*
2. *If $\alpha_1 = \alpha_c$ then the present random graph process has critical degree sequence (1.9).*

Remark 1.2. When $\alpha > \frac{2}{3}$, for any α_1, the inequality $\alpha_1 \leq \alpha < \alpha_c = 4\alpha - 2$ holds always; therefore, part 1 of Corollary 1.1 follows from part 1 of Theorem 1.1. Parts 2 of Corollary 1.1 and Corollary 1.2 are straightforward from Theorem 1.1.

Remark 1.3. No result has been obtained for the case in which $\alpha \leq \frac{2}{3}$ and $2\alpha_c \leq \alpha_1 \leq \alpha$. Clearly, this case can only appear when $\alpha \leq \frac{4}{7}$. It is natural to conjecture that the model possesses an exponential degree sequence in this case.

2. Proof of Theorem 1.1

We first note that the methodology of our proof follows the standard procedure, which can be found in [13] and [14].

For times s and t with $1 \leq s \leq t$, let $d_{x_s}(t)$ be the degree of vertex x_s in G_t. If x_s is not added in time step s, i.e. at time step s one of the other two substeps is executed, set $d_{x_s}(t) = 0$.

Cooper *et al.* [14] derived

$$|v_t - \alpha_1 t| \leq ct^{1/2} \log t \quad \text{quite surely}$$

for any constant $c > 0$. We say that an event happens *quite surely* (q.s.) if the probability of the complimentary set of the event is $O(t^{-K})$ for any $K > 0$. The estimate for e_t can be derived by the similar argument that

$$|e_t - \eta t| \leq ct^{1/2} \log t \quad \text{q.s.} \tag{2.1}$$

for any constant $c > 0$.

By a standard argument in large deviation theory (see [23]), we can further show that, for any $\epsilon > 0$, there exist $c_1, c_2 > 0$ such that

$$\mathrm{P}(|e_t - \eta t| \geq \epsilon t) \leq c_1 \exp\{-c_2 t\} \tag{2.2}$$

for all $t \geq 1$.

The following is our bounding for $d_{x_s}(t)$; note that our result is based on the exact estimation (2.2) for e_t.

Lemma 2.1. *For any $\alpha \in (\frac{1}{2}, 1]$ and $\alpha_1 \in (0, \alpha]$,*

$$d_{x_s}(t) \leq \left(\frac{t}{s}\right)^{\rho_\epsilon} (\log t)^3 \quad q.s., \tag{2.3}$$

where ρ_ϵ is given in (1.4).

Proof. Fix $s \leq t$. Suppose that x_s is added in time step s. Let $X_\tau = d_{x_s}(\tau)$ for $\tau = s, s+1, \ldots, t$, and let Y be the $\{1, 2, 3\}$-valued random variable with

$$\mathrm{P}(Y = 1) = \alpha_1, \qquad \mathrm{P}(Y = 2) = \alpha - \alpha_1, \qquad \mathrm{P}(Y = 3) = 1 - \alpha.$$

Conditional on $X_\tau = x$ and $e_\tau \geq m$, we have

$$X_{\tau+1} = x + \mathbf{1}_{\{Y=1\}} B\left(m, \frac{x}{2e_\tau}\right) + \mathbf{1}_{\{Y=2\}} B\left(2m, \frac{x}{2e_\tau}\right) - \mathbf{1}_{\{Y=3\}} S\left(m, \frac{x}{e_\tau}\right), \tag{2.4}$$

where $B(m, p)$ is the binomial random variable with parameter (m, p) and $S(m, x/e_\tau)$ is the super geometric random variable with parameter (e_τ, x, m).

If e_τ in (2.4) is substituted by $\eta\tau$, (2.3) can be derived by a standard argument which is given in the proof of Lemma 3.1 of [14]. Actually, with the exact estimate (2.2) of e_t, (2.3) follows from a random modification of such a standard argument. Details can be found in [26].

Now we follow the basic procedures in [14] to establish the recurrence for $\bar{D}_k(t)$. Set $D_{-1}(t) = 0$ for all $t \ge 1$. For $k \ge 0$, we have

$$\begin{aligned}\bar{D}_k(t+1) &= \bar{D}_k(t)\\ &\quad + (2\alpha - \alpha_1)m\,\mathrm{E}\biggl(-\frac{kD_k(t)}{2e_t} + \frac{(k-1)D_{k-1}(t)}{2e_t} + O\biggl(\frac{\Delta_t}{e_t}\biggr)\biggm| e_t > 0\biggr)\mathrm{P}(e_t > 0)\\ &\quad + (1-\alpha)m\,\mathrm{E}\biggl(\frac{(k+1)D_{k+1}(t)}{e_t} - \frac{kD_k(t)}{e_t} + O\biggl(\frac{\Delta_t}{e_t}\biggr)\biggm| e_t \ge m\biggr)\mathrm{P}(e_t \ge m)\\ &\quad + \alpha_1\,\mathbf{1}_{\{k=m\}}\,\mathrm{P}(e_t > 0) + O(\mathrm{P}(e_t = 0)) + O(\mathrm{P}(e_t < m)),\end{aligned}$$

where Δ_t denotes the maximum degree in G_t and the term $O(\Delta_t/e_t)$ accounts for the probability that we create larger than one degree changes for some vertices at time step $t+1$. By (2.2) and Lemma 2.1, we have

$$\frac{\Delta_t}{e_t} \le O(t^{\rho_\epsilon - 1}(\log t)^3) \quad \text{q.s.} \tag{2.5}$$

With the help of (2.1), (2.2), (2.5), and Lemma 2.1, we obtain the recurrence for $\bar{D}_k(t)$ as follows: $\bar{D}_{-1}(t) = 0$ for all $t > 0$ and, for $k \ge 0$,

$$\begin{aligned}\bar{D}_k(t+1) &= \bar{D}_k(t) + (A_2(k+1) + B_2)\frac{\bar{D}_{k+1}(t)}{t} + (A_1 k + B_1 + 1)\frac{\bar{D}_k(t)}{t}\\ &\quad + (A_0(k-1) + B_0)\frac{\bar{D}_{k-1}(t)}{t} + \alpha_1\,\mathbf{1}_{\{k=m\}} + O(t^{\rho_\epsilon - 1}(\log t)^3),\end{aligned} \tag{2.6}$$

where

$$A_2 = \frac{1-\alpha}{2\alpha - 1}, \qquad A_1 = -\frac{2-\alpha_1}{2(2\alpha-1)}, \qquad A_0 = \frac{2\alpha - \alpha_1}{2(2\alpha-1)},$$
$$B_2 = B_0 = 0, \quad \text{and} \quad B_1 = -1.$$

Note that the term $O(t^{\rho_\epsilon - 1}(\log t)^3)$ in (2.6) is independent of k, which follows from the fact that $e_t = O(t)$ and $kD_k(t) \le 2e_t = O(t)$ uniformly in k.

Recurrence (2.6) corresponds to the following recurrence in k: $d_{-1} = 0$ and, for $k \ge -1$,

$$(A_2(k+2) + B_2)d_{k+2} + (A_1(k+1) + B_1)d_{k+1} + (A_0 k + B_0)d_k = -\alpha_1\,\mathbf{1}_{\{k=m-1\}}\,. \tag{2.7}$$

The following lemma shows that (2.7) is a good approximation to (2.6).

Lemma 2.2. *Let d_k be a solution to (2.7) such that $|d_k| \le C/k$ for $k > 0$ and a constant C. Then*

1. *if $\alpha_1 \le \alpha_c$, for any $\nu \in (0, 1-\rho_\epsilon)$, there exists a constant $M_1 > 0$ such that*

$$|\bar{D}_k(t) - t d_k| \le M_1 t^{\rho_\epsilon + \nu} \tag{2.8}$$

for all $t \ge 1$ and $k \ge -1$,

2. *if* $\alpha_c < \alpha_1 < 2\alpha_c$, *there exists a constant* $M_2 > 0$ *such that*

$$|\bar{D}_k(t) - td_k| \le M_2 t^{1-\theta} \tag{2.9}$$

for all $t \ge 1$ *and* $k \ge -1$, *where* θ *is given in (1.5).*

Proof. This proof follows the methodology of Cooper *et al.* [14]. Specifically, we provide some details in order to analyze the case in which

$$0 < \varepsilon_0 := A_2 + B_1 + 1 - A_0 = \frac{\alpha_1 - \alpha_c}{\alpha_c} < 1,$$

which provides very important evidence for revealing the critical phenomenon. Note that Cooper *et al.* [14] only dealt with the case in which $\varepsilon_0 \le 0$.

Let $\Theta_k(t) = \bar{D}_k(t) - td_k$, and let $k_0 = k_0(t) = \lfloor t^{\rho_\epsilon}(\log t)^3 \rfloor$. Lemma 2.1 implies that

$$0 \le \bar{D}_k(t) \le t^{-10} \quad \text{for } k \ge k_0(t). \tag{2.10}$$

1. Equation (2.10) and $d_k \le C/k$ imply that (2.8) holds for $k \ge k_0$ uniformly, i.e. there exists a constant $N_1 > 0$, independent of k and t, such that

$$|\bar{D}_k(t) - td_k| = |\Theta_k(t)| \le N_1 t^{\rho_\epsilon}$$

for all $k \ge k_0(t)$ and $t \ge 1$.

Let L be the hidden constant in $O(t^{\rho_\epsilon - 1}(\log t)^3)$ of (2.6). For any $\nu \in (0, 1 - \rho_\epsilon)$, let $R \ge L$, satisfying

$$Lt^{\rho_\epsilon - 1}(\log t)^3 \le Rt^{\rho_\epsilon + \nu - 1}$$

for all $t \ge 1$. Let $N_2 = R/(\rho_\epsilon + \nu) + 1$, take $\sigma > 0$ such that

$$1 - \frac{R}{N_2} - (1 + \sigma)(1 - \rho_\epsilon - \nu) \ge 0, \tag{2.11}$$

and take $\delta \in (0, 1)$ such that

$$\delta^{1+\sigma} < \mathrm{e}^{-1} < \delta.$$

Let $t_1 > 0$ be an integer such that

$$k_0(t) \le -\frac{t}{A_1} = \frac{2(2\alpha - 1)}{2 - \alpha_1} t \tag{2.12}$$

and

$$\delta^{1+\sigma} \le \left(1 - \frac{1}{t+1}\right)^{t+1}, \qquad \left(1 - \frac{1 - R/l}{t+1}\right)^{(t+1)/(1-R/l)} \le \delta, \tag{2.13}$$

for all $t \ge t_1$ and $l \ge N_2$.

Now, for the above t_1, let $N_3 \ge N_1$, satisfying

$$|\Theta_k(t)| \le N_3 t^{\rho_\epsilon + \nu} \quad \text{for all } 1 \le t \le t_1 \text{ and } k \ge -1. \tag{2.14}$$

Take

$$M_1 = \max\{N_2, N_3\}. \tag{2.15}$$

We will prove that (2.8) holds for the above M_1 by induction. Our inductive hypothesis is

$$\mathcal{H}_t^1\colon |\Theta_k(t)| \le M_1 t^{\rho_\epsilon+\nu} \quad \text{for all } k \ge -1.$$

Note that (2.14) and (2.15) imply that $\mathcal{H}_t^1$ holds for $1 \le t \le t_1$.

It follows from (2.6) and (2.7) that

$$\begin{aligned}\Theta_k(t+1) = \Theta_k(t) &+ A_2(k+1)\frac{\Theta_{k+1}(t)}{t} + (A_1 k + B_1 + 1)\frac{\Theta_k(t)}{t} \\ &+ A_0(k-1)\frac{\Theta_{k-1}(t)}{t} + O(t^{\rho_\epsilon - 1}(\log t)^3). \end{aligned} \tag{2.16}$$

For $t \ge t_1$, by (2.12), we have $t + A_1 k + B_1 + 1 \ge 0$ and then (2.16) implies that

$$|\Theta_k(t+1)| \le (t+\varepsilon_0) M_1 t^{\rho_\epsilon+\nu-1} + R t^{\rho_\epsilon+\nu-1}.$$

Since $\alpha_1 \le \alpha_c$, we have $\varepsilon_0 \le 0$. Then, combining (2.11), (2.13), and (2.15), we have

$$\frac{(t+\varepsilon_0) M_1 t^{\rho_\epsilon+\nu-1} + R t^{\rho_\epsilon+\nu-1}}{M_1 (t+1)^{\rho_\epsilon+\nu}} \le \delta^{(1 - R/M_1 - (1+\sigma)(1-\rho_\epsilon-\nu))/(t+1)} \le 1.$$

The induction hypothesis $\mathcal{H}_{t+1}^1$ has been verified and the proof of part 1 is complete.

2. In this case we have $\alpha_c < \alpha_1 < 2\alpha_c$, and then, for some $\nu \in (0, \frac{1}{2})$, $\varepsilon_0 \le \rho_\epsilon + \nu < 1 - \theta$ (note that in this case $\rho_\epsilon = \frac{1}{2}$). Proceeding as in the proof of part 1, for certain $\sigma > 0$ and $\delta \in (\mathrm{e}^{-1}, 1)$, we have

$$\frac{(t+\varepsilon_0) M_2 t^{-\theta} + R t^{-\theta}}{M_2 (t+1)^{1-\theta}} \le \delta^{(1-\varepsilon_0 - R/M_2 - (1+\sigma)\theta)/(t+1)} \le 1$$

for sufficient large t and M_2. This is enough for an inductive proof of (2.9).

Remark 2.1. Lemma 2.2 remains open for the case in which $\varepsilon_0 \ge 1$, and this leads to condition (1.6).

Proof of Theorem 1.1. Theorem 1.1 is proved in two steps. Firstly, the Laplace method (see [17]) is used to solve (2.7) in the following three cases: (i) $\alpha_1 < \alpha_c$, (ii) $\alpha_1 > \alpha_c$, and (iii) $\alpha_1 = \alpha_c$. Secondly, the resulting solutions of (2.7) are checked to see if they satisfy the requirements of Lemma 2.2. For details, we refer the reader to [26].

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