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SOME EXTENSIONS OF ASKEY-WILSON'S Q-BETA INTEGRAL AND THE CORRESPONDING ORTHOGONAL SYSTEMS

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ABSTRACT. A seven-parameter extension of Askey and Wilson's four parameter q-beta integral is written in a symmetric form as the sum of multiples of two very-well-poised balanced basic hypergeometric ${}_{10}\phi_9$ series. Two special cases are considered in which the evaluation of the integral gives single terms by the q-Dixon formula in one case and by a special case of the Verma-Jain formula in the other. An orthogonal polynomial system is obtained in the first case and a system of biorthogonal rational function is obtained in the second. It is also shown that the biorthogonal system represents a generalization of Rogers' q-ultraspherical polynomials.

1. **Introduction.** A basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$$(1.1)\quad {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n,$$

where

$$(1.2)\quad (a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n,$$

$$(a; q)_n = \begin{cases} 1 & , \text{if } n = 0 \\ (1 - a)(1 - aq) \ldots (1 - aq^{n-1}) & , \text{if } n = 1, 2, \ldots . \end{cases}$$

We shall assume throughout the paper that the base q is less than 1 in absolute value. To ensure convergence of the series (1.1) we shall also assume that $|z| < 1$ unless the series terminates which happens when one of the numerator parameters $a_1, \ldots, a_{r+1}$ is of the form q^{-k}, k a nonnegative integer.

The series (1.1) is called balanced if $z = q$ and $b_1 b_2 \ldots b_r = q a_1 a_2 \ldots a_{r+1}$. It is a nearly-poised series of the first kind if $qa_1 \neq a_2 b_1 = a_3 b_2 = \ldots = a_{r+1} b_r$, a nearly-poised series of the second kind if $qa_1 = a_2 b_1 = \ldots = a_r b_{r-1} \neq a_{r+1} b_r$. The series (1.1) is well-poised if $qa_1 = a_2 b_1 = \ldots = a_{r+1} b_r$; if, in addition,

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$a_2 = -a_3 = q\sqrt{a_1}$ then the series is called very-well-poised. We shall use the notation ${}_{r+3}W_{r+2}$ for a very-well-poised ${}_{r+3}\phi_{r+2}$ series, that is,

$$(1.3)\quad {}_{r+3}W_{r+2}(a;\, a_1, a_2, \dots, a_r;\, q, z)$$
$$= {}_{r+3}\phi_{r+2}\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & a_1, & a_2, \dots, a_r \\ & \sqrt{a}, & -\sqrt{a}, & qa/a_1, & qa/a_2, \dots, qa/a_r \end{matrix};\, q, z\right].$$

In [6] the author derived the following integral representation of a balanced very-well-poised ${}_{10}\phi_9$ series:

$$(1.4)\quad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty h(\cos\theta, b) h(\cos\theta, c)}{\prod_{j=1}^{6} h(\cos\theta, a_j)} d\theta$$
$$= A\left\{\frac{(c^2; q)_\infty}{(c/b; q)_\infty} \prod_{j=1}^{6} \frac{(ba_j; q)_\infty}{(b/a_j; q)_\infty}\right.$$
$$\times\, {}_{10}W_9(b^2q^{-1};\, bcq^{-1}, b/a_1, b/a_2, b/a_3, b/a_4, b/a_5, b/a_6;\, q, q)$$
$$+ \frac{(b^2; q)_\infty}{(b/c; q)_\infty} \prod_{j=1}^{6} \frac{(ca_j; q)_\infty}{(c/a_j; q)_\infty}$$
$$\left.\times\, {}_{10}W_9(c^2q^{-1};\, bcq^{-1}, c/a_1, c/a_2, c/a_3, c/a_4, c/a_5, c/a_6;\, q, q)\right\},$$

where $bc = a_1a_2a_3a_4a_5a_6$ and

$$(1.5)\quad A = \frac{2\pi \prod_{j=1}^{6} (b/a_j, c/a_j; q)_\infty}{(q, b^2, c^2; q)_\infty \prod_{1 \leqq j < k \leqq 6} (a_ja_k; q)_\infty}.$$

The infinite products and the h functions above are defined by

$$(1.6)\quad (a; q)_\infty = \lim_{n\to\infty} (a; q)_n = \prod_{k=0}^{\infty} (1 - aq^k)$$

and

$$(1.7)\quad h(x, a) = \prod_{k=0}^{\infty} (1 - 2axq^k + a^2q^{2k}) = (ae^{i\theta}, ae^{-i\theta}; q)_\infty,\ x = \cos\theta.$$

The above representation is valid provided $\max|a_j| < 1$, $j = 1, \dots, 6$ and b/c is not the form q^k, $k = 0, \pm 1, \pm 2, \dots$. Formula (1.4) is not stated quite in this form in [6]; it is deduced from ([6], Eq. 2.9) by applying an iteration of Bailey's

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[3] four-term transformation formula for balanced very-well-poised ${}_{10}\phi_9$ series. Apart from the obvious symmetry on both sides of (1.4) it is much easier to remember than formula (2.9) of [6]. The restriction about b/c is essential for the right side of (1.4) but not for the left. In case the parameters do not satisfy this restriction then we can always transform the right side by Bailey's transformation formula [3] and get an expression where the products $(b/c;\, q)_\infty$ and $(c/b;\, q)_\infty$ disappear from the denominators.

The purpose of this paper is to consider some special cases of (1.4) that, to my knowledge, have not been studied elsewhere. The most important special case of (1.4) is, of course, Askey and Wilson's [2] extension of the beta integral

$$(1.8)\quad \int_0^\pi \frac{(e^{2i\theta},\, e^{-2i\theta};\, q)_\infty}{\prod_{j=1}^{4} h(\cos\theta,\, a_j)}\, d\theta = \frac{2\pi(a_1a_2a_3a_4;\, q)_\infty}{(q,\, a_1a_2,\, a_1a_3,\, a_1a_4,\, a_2a_3,\, a_2a_4,\, a_3a_4;\, q)_\infty}.$$

This is the $b = a_6$, $a_5 = 0$ case of (1.4). Askey and Wilson [2] showed that the polynomials

$$(1.9)\quad P_n(\cos\theta;\, a_1,\, a_2,\, a_3,\, a_4|q) = (a_1a_2,\, a_1a_3,\, a_1a_4;\, q)_n a_1^{-n}$$
$$\times\, {}_4\phi_3\left[\begin{matrix} q^{-n}, & a_1a_2a_3a_4q^{n-1}, & a_1e^{i\theta}, & a_1e^{-i\theta} \\ & a_1a_2, & a_1a_3, & a_1a_4 \end{matrix}\; ;\, q,\, q\right]$$

are orthogonal with respect to the weight function in the integral of (1.8). If we set $a_6 = 0$ in (1.4) we obtain Nassrallah and Rahman's [5] integral representation of a very-well-poised ${}_8\phi_7$ series:

$$(1.10)\quad \int_0^\pi \frac{(e^{2i\theta},\, e^{-2i\theta};\, q)_\infty h(\cos\theta,\, b)}{\prod_{j=1}^{5} h(\cos\theta,\, a_j)}\, d\theta$$
$$= 2\pi \frac{(a_1a_2a_3a_4a_5/b;\, q)_\infty \prod_{j=1}^{5} (ba_j;\, q)_\infty}{(q,\, b^2;\, q)_\infty \prod_{1\leqq j<k\leqq 5} (a_ja_k;\, q)_\infty}$$
$$\times\, {}_8W_7(b^2q^{-1};\, b/a_1,\, b/a_2,\, b/a_3,\, b/a_4,\, b/a_5;\, q,\, a_1a_2a_3a_4a_5/b).$$

If we set $b = a_1a_2a_3a_4a_5$ then, via a transformation of the ${}_8\phi_7$ series in (1.10), we are led to a generalization of (1.8), namely,

$$(1.11)\quad \int_0^\pi \frac{(e^{2i\theta},\, e^{-2i\theta};\, q)_\infty h(\cos\theta,\, a_1a_2a_3a_4a_5)}{\prod_{j=1}^{5} h(\cos\theta,\, a_j)}\, d\theta$$

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$$= \frac{2\pi \prod_{j=1}^{5} \left(\frac{a_1a_2a_3a_4a_5}{a_j}; q\right)_\infty}{(q; q)_\infty \prod_{1 \leqq j < k \leqq 5} (a_ja_k; q)_\infty}.$$

This was stated by the author in [6] who also found a system of biorthogonal rational functions representable by balanced very-well-poised ${}_{10}\phi_9$ series.

The special case we wish to consider first is obtained by setting $a_1 = i\lambda$, $a_2 = -i\lambda$, $a_3 = \mu$, $a_4 = -\mu$, $a_5 = \nu$, $b = -q/\nu$ in (1.10) which reduces to

$$(1.12)\quad \int_0^\pi \frac{|(e^{2i\theta}; q)_\infty|^2 h(\cos\theta, -q/\nu)}{|(-\lambda^2 e^{2i\theta}, \mu^2 e^{2i\theta}; q^2)_\infty|^2 h(\cos\theta, \nu)} d\theta$$

$$= \frac{2\pi(-q; q)_\infty(-q^2/\nu^2, \lambda^2\mu^2\nu^2; q^2)_\infty}{(q; q)_\infty(-\lambda^2\nu^2, \mu^2\nu^2, -\lambda^2\mu^2, \lambda^2, -\mu^2; q^2)_\infty},$$

by the q-Dixon theorem ([8], 3.3.1.5), where we assume that λ, μ, ν are real and less than 1 in absolute value. In Section 2 we will obtain a system of polynomials which are orthogonal with respect to the weight function in the integral of (1.12). In Section 3 we will prove that

$$(1.13)\quad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty h(\cos\theta, a^2bq)h(\cos\theta, a^2c)}{h(\cos\theta, a)h(\cos\theta, -a)h(\cos\theta, a\sqrt{q})h(\cos\theta, -a\sqrt{q})h(\cos\theta, b)h(\cos\theta, c)} d\theta$$

$$= \frac{2\pi(a^2, qa^2, a^4bc; q)_\infty}{(1 - a^2b^2)(q, a^4, bc; q)_\infty},$$

and will derive a system of biorthogonal rational functions with respect to the weight function in (1.13), in Section 4.

We would like to point out that by setting $bc = q$ in (1.4) we obtain the evaluation of a six-parameter integral:

$$(1.14)\quad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty h(\cos\theta, b)h(\cos\theta, q/b)d\theta}{h(\cos\theta, a_1)h(\cos\theta, a_2)h(\cos\theta, a_3)h(\cos\theta, a_4)h(\cos\theta, a_5)h\left(\cos\theta, \frac{q}{a_1a_2a_3a_4a_5}\right)}$$

$$= B\left\{\frac{\left(\frac{q^2}{ba_1a_2a_3a_4a_5}; q\right)_\infty}{\left(\frac{a_1a_2a_3a_4a_5}{b}; q\right)_\infty} \prod_{j=1}^{5} \frac{(qa_j/b; q)_\infty}{(q/ba_j; q)_\infty} - \frac{b^2}{q}\right.$$

$$\left.\times \frac{\left(\frac{bq}{a_1a_2a_3a_4a_5}; q\right)_\infty}{(ba_1a_2a_3a_4a_5/q; q)_\infty} \prod_{j=1}^{5} \frac{(ba_j; q)_\infty}{(b/a_j; q)_\infty}\right\},$$

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where

$$(1.15)\quad B = \frac{2\pi(a_1a_2a_3a_4a_5/b,\ ba_1a_2a_3a_4a_5/q;\ q)_\infty \prod_{j=1}^{5}(b/a_j,\ q/ba_j;\ q)_\infty}{(q,\ b^2/q,\ q^2/b^2;\ q)_\infty \prod_{1\leqq j<k\leqq 5}(a_ja_k;\ q)_\infty \prod_{m=1}^{5}\left(\frac{qa_m}{a_1a_2a_3a_4a_5};\ q\right)_\infty}.$$

However, this integral is probably not as interesting as those in (1.11), (1.12) and (1.13). First, it does not seem to be possible to combine the two terms on the right side of (1.14) into a single one. Secondly, the range of the parameters is too restrictive, since for (1.14) to be valid we must have $\max|a_j| < 1$ *and* $|q| < |a_1a_2a_3a_4a_5|$. Nonetheless, (1.14) is more general than (1.11) as can be seen by setting $b = a_1a_2a_3a_4a_5$ in (1.14).

2. **Orthogonal polynomials corresponding to (1.12).** Let us denote

$$(2.1)\quad V(\cos\theta;\ \lambda,\ \mu,\ \nu) = \frac{|(e^{2i\theta};\ q)_\infty|^2 h(\cos\theta,\ -q/\nu)}{|(-\lambda^2e^{2i\theta},\ \mu^2e^{2i\theta};\ q)_\infty|^2 h(\cos\theta,\ \nu)},$$

and

$$(2.2)\quad g(\lambda,\ \mu,\ \nu) = \frac{2\pi(-q;\ q)_\infty(-q^2/\nu^2,\ \lambda^2\mu^2\nu^2;\ q^2)_\infty}{(q;\ q)_\infty(-\lambda^2\nu^2,\ \mu^2\nu^2,\ -\lambda^2\mu^2,\ \lambda^2,\ -\mu^2;\ q^2)_\infty}.$$

Then, for nonnegative integers k and l we have

$$(2.3)\quad \int_0^\pi V(\cos\theta;\ \lambda,\ \mu,\ \nu)(-\lambda^2e^{2i\theta},\ -\lambda^2e^{-2i\theta};\ q^2)_k(\mu^2e^{2i\theta},\ \mu^2e^{-2i\theta};\ q^2)_l d\theta$$
$$= \int_0^\pi V(\cos\theta;\ \lambda q^k,\ \mu q^l,\ \nu)d\theta$$
$$= g(\lambda,\ \mu,\ \nu)\frac{(-\lambda^2\mu^2;\ q^2)_{k+l}(\lambda^2,\ -\lambda^2\nu^2;\ q^2)_k(-\mu^2,\ \mu^2\nu^2;\ q^2)_l}{(\lambda^2\mu^2\nu^2;\ q^2)_{k+l}},$$

by (1.12). Hence

$$(2.4)\quad \int_0^\pi V(\cos\theta;\ \lambda,\ \mu,\ \nu)(\mu^2e^{2i\theta},\ \mu^2e^{-2i\theta};\ q^2)_l$$
$$\times\ {}_4\Phi_3\left[\begin{matrix} q^{-2m}, & \lambda^2\mu^2\nu^2q^{2m-2}, & -\lambda^2e^{2i\theta}, & -\lambda^2e^{-2i\theta} \\ & -\lambda^2\mu^2, & \lambda^2, & -\lambda^2\nu^2 \end{matrix};\ q^2,\ q^2\right]d\theta$$
$$= g(\lambda,\ \mu,\ \nu)\frac{(-\mu^2,\ \mu^2\nu^2,\ -\lambda^2\mu^2;\ q^2)_l}{(\lambda^2\mu^2\nu^2;\ q^2)_l}$$
$$\times\ {}_3\phi_2\left[\begin{matrix} q^{-2m}, & \lambda^2\mu^2\nu^2q^{2m-2}, & -\lambda^2\mu^2q^{2l}, \\ & \lambda^2\mu^2\nu^2q^{2l}, & -\lambda^2\mu^2, \end{matrix};\ q^2,\ q^2\right]$$

$$= g(\lambda, \mu, \nu)\frac{(-\mu^2, \mu^2\nu^2, -\lambda^2\mu^2; q^2)_l(-\nu^2, q^{2+2l-2m}; q^2)_m}{(\lambda^2\mu^2\nu^2; q^2)_{l+m}(-q^{2-2m}/\lambda^2\mu^2; q^2)_m},$$

by the q-Saalschütz formula ([8], IV.4). Since, by Sears' transformation formula [7],

$$(2.5)\quad P_m(\cos 2\theta; \lambda, \mu, \nu|q)$$
$$\equiv {}_4\phi_3\left[\begin{matrix} q^{-2m}, & \lambda^2\mu^2\nu^2q^{2m-2}, & -\lambda^2e^{2i\theta}, & -\lambda^2e^{-2i\theta} \\ & -\lambda^2\mu^2, & \lambda^2, & -\lambda^2\nu^2 \end{matrix}; q^2, q^2\right]$$
$$= \frac{(-\mu^2, \mu^2\nu^2; q^2)_m}{(\lambda^2, -\lambda^2\nu^2; q^2)_m}\left(-\frac{\lambda^2}{\mu^2}\right)^m$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-2m}, & \lambda^2\mu^2\nu^2q^{2m-2}, & -\mu^2e^{2i\theta}, & \mu^2e^{-2i\theta} \\ & -\lambda^2\mu^2, & -\mu^2, & \mu^2\nu^2 \end{matrix}; q^2, q^2\right]$$

and since $(q^{2+2l-2m}; q^2)_m$ vanishes unless $l \geqq m$, it follows that

$$(2.6)\quad \int_0^\pi V(\cos\theta; \lambda; \mu, \nu)P_n(\cos 2\theta; \lambda, \mu, \nu|q)P_m(\cos 2\theta; \lambda; \mu, \nu|q)d\theta = 0$$

if $n < m$. By symmetry, (2.6) is also true if $n > m$. To evaluate the integral in (2.6) for $m = n$ we take the first ${}_4\phi_3$ series on the right side of (2.5) for one P_m and the second ${}_4\phi_3$ series for the other. So we find that

$$(2.7)\quad \int_0^\pi V(\cos\theta; \lambda, \mu, \nu)P_n(\cos 2\theta; \lambda, \mu, \nu|q)P_m(\cos 2\theta; \lambda, \mu, \nu|q)d\theta$$
$$= \frac{\delta_{m,n}}{h_n},$$

where

$$(2.8)\quad h_n = \frac{(q; q)_\infty(\lambda^2, -\mu^2, -\lambda^2\mu^2, -\lambda^2\nu^2, \mu^2\nu^2; q^2)_\infty}{2\pi(-q; q)_\infty(-q^2/\nu^2, \lambda^2\mu^2\nu^2; q^2)_\infty}$$
$$\times \frac{(\lambda^2\mu^2\nu^2q^{-2}; q^2)_n(1 - \lambda^2\mu^2\nu^2q^{4n-2})(\lambda^2, -\lambda^2\mu^2, -\lambda^2\nu^2; q^2)_n}{(q^2; q^2)_n(1 - \lambda^2\mu^2\nu^2q^{-2})(\mu^2\nu^2, -\nu^2, -\mu^2; q^2)_n}\lambda^{-4n}.$$

Formula (2.6) is a q-analogue of the orthogonality relation

$$(2.9)\quad \int_0^1 (1 - r^2)^\alpha r^{2\beta+1}P_m^{(\alpha,\beta)}(2r^2 - 1)P_n^{(\alpha,\beta)}(2r^2 - 1)dr = 0 \quad m \neq n$$

for the Jacobi polynomials

$$(2.10)\quad P_m^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_m}{m!}{}_2F_1\left(-m, m + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\right).$$

It may be noted that (2.9) was used by Koornwinder [4] to obtain his addition formula for the Jacobi polynomials.

3. **Proof of (1.13).** Verma and Jain [9] obtained the following transformation formula

$$(3.1)\quad {}_5\phi_4\left[\begin{matrix} a, & b, & c, & d, & e, \\ & aq/b, & aq/c, & aq/d, & f \end{matrix}; q, q\right]$$

$$+\frac{(a, b, c, d, e, q/f, aq^2/bf, aq^2/cf, aq^2/df; q)_\infty}{(aq/b, aq/c, aq/d, f/q, aq/f, bq/f, cq/f, dq/f, eq/f; q)_\infty}$$

$$\times\ {}_5\phi_4\left[\begin{matrix} eq/f, & aq/f, & bq/f, & cq/f, & dq/f \\ & q^2/f, aq^2/bf, & aq^2/cf, & aq^2/df \end{matrix}; q, q\right]$$

$$= \frac{(\lambda q/a, \lambda q/e, q\lambda^2/a, q/f; q)_\infty}{(\lambda q, aq/f, eq/f, aq/\lambda f; q)_\infty}$$

$$\times\ {}_{12}W_{11}(\lambda; \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, \lambda b/a, \lambda c/a, \lambda d/a, e, aq/f; q, q)$$

$$+\frac{(a, e, \lambda b/a, \lambda c/a, \lambda d/a, q/f, a^2q^2/\lambda bf, a^2q^2/\lambda cf, a^2q^2/\lambda df, aq^3/f^2; q)_\infty}{(aq/b, aq/c, aq/d, aq/f, bq/f, cq/f, dq/f, eq/f, \lambda f/aq, a^2q^3/\lambda f^2; q)_\infty}$$

$$\times\ {}_{12}W_{11}(a^2q^2/\lambda f^2; qa^{3/2}/\lambda f, -qa^{3/2}/\lambda f,$$

$$(qa)^{3/2}/\lambda f, -(qa)^{3/2}/\lambda f, \lambda q/a, aq/f, bq/f, cq/f, dq/f; q, q),$$

where $\lambda = qa^2/bcd$ and $f = ea^2/\lambda^2$, as a nonterminating extension of Bailey's formula ([8], 3.4.16) transforming a terminating balanced nearly-poised ${}_5\phi_4$ series of the second kind into a terminating balanced very-well-poised ${}_{12}\phi_{11}$ series. Verma and Jain's original statement of the formula has a misprint, so another display here may be helpful to some readers. To derive (1.13) all we need is to set $d = 1$ in (3.1) and observe that the left side equals 1 resulting in the summation formula

$$(3.2)\quad {}_{10}W_9(\beta; \sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{\alpha q}, -\sqrt{\alpha q}, \gamma, \beta\alpha, q\beta^2/\alpha\gamma; q, q)$$

$$+\frac{(\alpha, \gamma, \beta q, \beta/\alpha, \beta q^2/\gamma, \beta q/\alpha\gamma, q^3\beta^4/\alpha^3\gamma^2; q)_\infty}{(\alpha q, \beta q/\alpha, \beta q/\gamma, q\beta^2/\alpha, q\beta^2/\gamma\alpha^2, \alpha\gamma/q\beta, q^3\beta^3/\alpha^2\gamma^2; q)_\infty}$$

$$\times\ {}_{10}W_9\Big(q^2\beta^3/\alpha^2\gamma^2; \frac{\beta q}{\gamma\sqrt{\alpha}}, -\frac{\beta q}{\gamma\sqrt{\alpha}}, \frac{\beta q^{3/2}}{\gamma\sqrt{\alpha}}, -\frac{\beta q^{3/2}}{\gamma\sqrt{\alpha}},$$

$$\beta q/\alpha, q\beta^2/\gamma\alpha^2, q\beta^2/\alpha\gamma; q, q\Big)$$

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$$= \frac{(\beta q, q\beta^2/\alpha^2, q\beta^2/\alpha\gamma, q\beta/\alpha\gamma; q)_\infty}{(q\beta/\alpha, q\beta/\gamma, q\beta^2/\alpha, q\beta^2/\alpha^2\gamma; q)_\infty}.$$

On the other hand, by (1.4), the left side of (1.13) equals

$$(3.8)\quad 2\pi\frac{(qa^2b^2, a^2, qa^2, qa^2, qa^2b/c, a^6c^2, a^2bc; q)_\infty}{(q, a^4c^2, a^4, qa^4, a^2b^2, bc, bq/c; q)_\infty}$$

$$\times \{{}_{10}W_9(a^4c^2/q; ac/\sqrt{q}, -ac/\sqrt{q}, ac, -ac, a^2, a^2c/b, a^4bc; q, q)$$

$$+ \frac{(a^4c^2, q^2a^6b^2, qa^2bc, a^2c^2/q, a^2c/b, bq/c; q)_\infty}{(q^2a^4b^2, a^6c^2, a^2c^2, a^2bc, qa^2, qa^2b/c, c/bq; q)_\infty}$$

$$\times {}_{10}W_9(a^4b^2q; ab\sqrt{q}, -ab\sqrt{q}, abq, -abq, qa^2, qa^2/bc, a^4bc; q, q)\}$$

which, together with (3.2), immediately yields (1.13).

4. **Biorthogonal rational functions corresponding to (1.13).** From formula (1.13) it is clear that for nonnegative integers j and k,

$$(4.1)\quad \int_0^\pi U(\cos\theta; a, b, c)\frac{(be^{i\theta}, be^{-i\theta}; q)_j(ce^{i\theta}, ce^{-i\theta}; q)_k}{(qa^2be^{i\theta}, qa^2be^{-i\theta}; q)_j(a^2ce^{i\theta}, a^2ce^{-i\theta}; q)_k}d\theta$$

$$= \int_0^\pi U(\cos\theta; a, bq^j, cq^k)d\theta = f(a, b, c)\frac{(bc; q)_{j+k}(1-a^2b^2)}{(a^4bc; q)_{j+k}(1-a^2b^2q^{2j})},$$

where

(4.2) $U(\cos\theta; a, b, c)$

$$= \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty h(\cos\theta, qa^2b)h(\cos\theta, a^2c)}{h(\cos\theta, a)h(\cos\theta, -a)h(\cos\theta, a\sqrt{q})h(\cos\theta, -a\sqrt{q})h(\cos\theta, b)h(\cos\theta, c)}$$

and

$$(4.3)\quad f(a, b, c) = \int_0^\pi U(\cos\theta; a, b, c)d\theta$$

$$= \frac{2\pi(a^2, qa^2, a^4bc; q)_\infty}{(1-a^2b^2)(q, a^4, bc; q)_\infty}.$$

Denoting

(4.4) $f_n(\cos\theta)$

$$= \frac{1-a^2}{1-a^2q^n}\frac{b^{-n}}{(q; q)_n}$$

$$\times {}_6\phi_5\left[\begin{matrix} abq, & -abq, & be^{i\theta}, & be^{-i\theta}, & a^4bcq^{n-1}, & q^{-n} \\ ab, & -ab, & qa^2be^{-i\theta}, & qa^2be^{i\theta}, & bc & \end{matrix}; q, q\right]$$

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and

$$(4.5)\quad g_n(\cos\theta) = \frac{c^{-n}}{(q;q)_n}\,{}_4\phi_3\left[\begin{matrix} ce^{i\theta}, & ce^{-i\theta}, & a^4bcq^{n-1}, & q^{-n} \\ a^2ce^{-i\theta}, & a^2ce^{i\theta}, & bc & \end{matrix};q,q\right],$$

it can be shown by using the q-Saalschütz formula ([8], IV. 4) and the q-Vandermonde formula ([8], IV. 1) that

$$(4.6)\quad \int_0^\pi U(\cos\theta;a,b,c)f_n(\cos\theta)g_m(\cos\theta)d\theta$$
$$= f(a,b,c)\frac{1-a^2}{1-a^2q^n}\frac{(a^4;q)_n(1-a^4bcq^{-1})}{(a^4bcq^{-1};q)_n(q;q)_n(1-a^4bcq^{2n-1})}\delta_{m,n}.$$

The reason for the rather strange-looking normalization in (4.4) and (4.5) is as follows. We know that for $b = c = 0$, $U(\cos\theta; a, b, c)$ is the weight function for the continuous q-ultraspherical polynomials which have different basic hypergeometric representations – as a ${}_2\phi_1$, ${}_3\phi_2$, and as a balanced ${}_4\phi_3$ series. The representation that is relevant for our purposes is a ${}_3\phi_2$ series found by Askey and Ismail [1]

$$(4.7)\quad C_n(\cos\theta;a^2|q) = \frac{(a^4;q)_n}{(q;q)_n}\left(\frac{e^{-i\theta}}{a^2}\right)^n {}_3\phi_2\left[\begin{matrix} q^{-n}, & a^2, & a^2e^{2i\theta}, \\ & a^4, & 0 \end{matrix};q,q\right].$$

We will show that both $f_n(\cos\theta)$ and $g_n(\cos\theta)$ reduce to $C_n(\cos\theta; a^2|q)$ as $b \to 0$ and $c \to 0$. Since the ${}_4\phi_3$ series in (4.5) is balanced, we find that, by Sears' formula [7]

$$(4.8)\quad g_n(\cos\theta) = \frac{(a^4, a^2be^{i\theta};q)_n}{(q, bc, a^2ce^{-i\theta};q)_n}\left(\frac{e^{-i\theta}}{a^2}\right)^n$$
$$\times\, {}_4\phi_3\left[\begin{matrix} q^{-n}, & a^4bcq^{n-1}, & a^2, & a^2e^{2i\theta} \\ & a^4, & a^2ce^{i\theta}, & a^2be^{i\theta}, \end{matrix};q,q\right]$$

which obviously goes to $C_n(\cos\theta; a^2|q)$ in the limit $b, c \to 0$. To find the limit of $f_n(\cos\theta)$ we first observe that

$$(1-a^2)(1-a^2b^2q^{2j})$$
$$= (1-a^2bq^je^{i\theta})(1-a^2bq^je^{-i\theta}) - a^2(1-bq^je^{i\theta},)(1-bq^je^{-i\theta})$$

and therefore

$$(4.9)\quad f_n(\cos\theta) = \frac{1-a^2}{1-a^2q^n}\frac{b^{-n}}{(q;q)_n}\left\{\frac{(1-a^2be^{i\theta})(1-a^2be^{-i\theta})}{(1-a^2)(1-a^2b^2)}\right.$$
$$\times\, {}_4\phi_3\left[\begin{matrix} q^{-n}, & a^4bcq^{n-1}, & be^{i\theta}, & be^{-i\theta} \\ & bc, & a^2be^{i\theta}, & a^2be^{-i\theta} \end{matrix};q,q\right]$$

$$- \frac{a^2(1 - be^{i\theta})(1 - be^{-i\theta})}{(1 - a^2)(1 - a^2b^2)}$$

$$\times {}_4\phi_3\left[\begin{matrix} q^{-n}, & a^4bCq^{n-1}, & bqe^{i\theta}, & bqe^{-i\theta} \\ & bc, & qa^2be^{i\theta}, & qa^2be^{-i\theta} \end{matrix} ; q, q\right]\Bigg\}.$$

Since both the ${}_4\phi_3$ series on the right side of (4.9) are balanced we may apply Sears' formula on both of them to obtain

$$(4.10)\quad f_n(\cos\theta) = \frac{(a^4; q)_n(e^{-i\theta}/a^2)^n}{(q; q)_n(bc; q)_n(1 - a^2q^n)}\Bigg\{\frac{(1 - a^2be^{i\theta})(1 - a^2be^{-i\theta})}{(1 - a^2b^2)}$$

$$\times {}_4\phi_3\left[\begin{matrix} q^{-n}, & a^4bcq^{n-1}, & a^2, & a^2e^{2i\theta} \\ & a^4, & a^2be^{i\theta}, & a^2ce^{i\theta} \end{matrix} ; q, q\right]$$

$$- \frac{a^2q^n(1 - be^{i\theta})(1 - be^{-i\theta})}{(1 - a^2b^2)}$$

$$\times {}_4\phi_3\left[\begin{matrix} q^{-n}, & a^4bcq^{n-1}, & a^2, & a^2e^{2i\theta} \\ & a^4, & qa^2be^{i\theta}, & a^2ce^{i\theta}/q \end{matrix} ; q, q\right]\Bigg\}.$$

It is now clear that the limit of $f_n(\cos\theta)$ is also $C_n(\cos\theta; a^2|q)$ as $b, c \to 0$.

REFERENCES

1. R. Askey and M. E. H. Ismail, *A generalization of ultraspherical polynomials*, Studies in Pure Mathematics (P. Erdös, ed.), Boston, Birkhäuser, (1982), pp. 56-78.

2. R. Aksey and J. Wilson, *Some basic hypergeometric polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Sec. **54** (1985), #319.

3. W. N. Bailey, *Well-poised basic hypergeometric series*, Quart. J. Math. (Oxford), **18** (1947), pp. 157-166.

4. Tom H. Koornwinder, *Jacobi polynomials, III. An analytic proof of the addition formula*, SIAM J. Math. Anal. **6** (1975), pp. 533-543.

5. B. Nassrallah and M. Rahman, *Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials*, SIAM J. Math. Anal. **16** (1985), pp. 186-197.

6. M. Rahman, *An integral representation of a ${}_{10}\phi_9$ and continuous biorthogonal rational functions*, Can. J. Math. **38** (1986), pp. 605-618.

7. D. B. Sears, *On the transformation theory of basic hypergeometric functions*, Proc. Lond. Math. Soc. **53** (1951), pp. 158-180.

8. L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, London and New York, 1966.

9. A. Verma and V. K. Jain, *Transformations of nonterminating hypergeometric series, their contour integrals and applications to Rogers-Ramanujan identities*, J. Math. Anal. Appl. **87** (1982), pp. 9-44.

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