IDENTITIES IN THE UNIVERSAL ENVELOPES OF LIE ALGELRAS

Dedicated to the memory of Hanna Neumann

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Communicated by M. F. Newman

(Received 14 July 1972)

1. Introduction

It is well known (see Latyšev [1] for finite dimensional case) that the universal envelope of Lie Alaebra $\mathfrak{g}$ over a commutative field $\mathfrak{k}$ of characteristic 0 is a *PI*-alaebra (i.e. possesses a nontrivial identity) if and only if this Lie algebra is abelian. On the other hand the recent results due to Passman [2] describe the conditions under which the group algebra of a group over an arbitrary commutative field is a *PI*-algebra. A. L. Šmel'kin suggested that I should find necessary and sufficient conditions for a Lie algebra $\mathfrak{g}$ over a field of nonzero characteristic under which its universal envelope $U\mathfrak{g}$ should be a *PI*-algebra. These conditions are given in the following theorem.

THEOREM 1.1. *The universal envelope $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ over a field $\mathfrak{k}$ of characteristic $p > 0$ is a PI-algebra if and only if $\mathfrak{g}$ possesses an abelian ideal $\mathfrak{a}$ of finite codimension, the adjoint representation of $\mathfrak{g}$ being algebraic of a bounded degree.*

Since $U\mathfrak{g}$ is a semisimple algebra Theorem 1.1 implies the following immediate consequence.

COROLLARY 1.2. *The restrictions of the Theorem 1.1 describe the class of those Lie algebras all of whose irreducible representations are of bounded degree.*

We recall now some basic definitions and results which will be necessary in the proof of 1.1. Fix commutative field $\mathfrak{k}$ (of characteristic $p > 0$).

We use $[x, y]$ to denote the product of elements x, y of $\mathfrak{g}$. An associative algebra U is called a universal envelope of the Lie algebra $\mathfrak{g}$ if there is given a map $\varepsilon\colon \mathfrak{g} \to U\mathfrak{g}$ which is k-linear and satisfies the equation

$$\varepsilon([x, y]) = \varepsilon(x)\varepsilon(y) - \varepsilon(y)\varepsilon(x)$$

for all $x, y \in \mathfrak{g}$, and such that if α maps $\mathfrak{g}$ into A where A is an arbitrary associative algebra over $\mathfrak{k}$ (α enjoys the same properties as ε) then there exists the unique homomorphism $\phi\colon U\mathfrak{g} \to A$ such that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & U\mathfrak{g} \arrow[dl, dashed, "\phi"] \\ A & \end{tikzcd}$$

is commutative. If $(x_i)_{i \in I}$ is a certain linear basis of $\mathfrak{g}$, I being in some way linear ordered, then according to Poincaré-Birkhoff-Witt theorem (cf. [3], or [4], p. 159) one may take the following system of monomials for the linear basis of $U\mathfrak{g}$:

$$(1) \qquad \varepsilon(x_{i_1})\varepsilon(x_{i_2}) \cdots \varepsilon(x_{i_n}), \quad i_1 \leqq i_2 \cdots \leqq i_n, \quad n \geqq 0.$$

It is not difficult to deduce from this theorem that the graduated algebra gr $U\mathfrak{g}$ which is canonically associated with $U\mathfrak{g}$ (see [4], p. 163) is isomorphic to the symmetrical algebra of the linear space $\mathfrak{g}$ over $\mathfrak{k}$ or simply to the polynomial algebra $\mathfrak{k}[\mathfrak{g}] = \mathfrak{k}[(x_i)_{i \in I}]$.

It is well known that $U\mathfrak{g}$ has no zero divisors and that $U\mathfrak{g}$ can always be embedded in a skew field (for these facts see [4] and [5]). In the proof of the key Proposition 3.3 an important role plays the following assertion.

THEOREM 1.3. (*Amitsur* [6]). *Let A be and arbitrary associative PI-algebra without zero divisors over a commutative field $\mathfrak{k}$. Then A possesses a (right) quotient field D which is finite dimensional over its centre Z.*

We remind the reader that a skew field D is called right quotient of A if $A \subseteq D$ and each element from D equals to a fraction $a_1 a_2^{-1}$ for some $a_1, a_2 \in A$.

The proof of the Lemma 3.2 is based on an interesting result due to Neumann.

THEOREM 1.4. (*Neumann* [7]). *Let $\mathfrak{g}$ be a Lie algebra over a commutative field $\mathfrak{k}$ such that the codimension of the centralizer $\mathfrak{c}_{\mathfrak{g}}(x)$ of each element $x \in \mathfrak{g}$ in $\mathfrak{g}$ is bounded by the fixed number b. Then*

$$\dim \mathfrak{g}^2 \leqq b^2.$$

We remind the reader that $\mathfrak{g}^2$ (the commutator subalgebra of $\mathfrak{g}$) is the linear subspace of $\mathfrak{g}$ spanned by the products $[g, h]$, $g, h \in \mathfrak{g}$, and

$$\mathfrak{c}_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} \mid [x, y] = 0\}$$

Since the above-mentioned map ε is an injection we may assume from now on that it is an identical map, that is, that $\mathfrak{g} \leqq U\mathfrak{g}$. Let the symbol $ad_{\mathfrak{g}}x$ where $x \in U\mathfrak{g}$ denote the $\mathfrak{k}$-linear map of $\mathfrak{g}$ into $U\mathfrak{g}$ given by

$$g\, ad_{\mathfrak{g}} x = [g, x] = gx - xg, \ g \in \mathfrak{g}.$$

LEMMA 1.5. *Let* $\mathfrak{k}$ *be of characteristic* $p > 0$. *If* $x \in \mathfrak{g}$ *then*

(i) $(ad_{\mathfrak{g}}x)^{p^k} = ad_{\mathfrak{g}}x^{p^k}$, $k > 1$;

(ii) *if* $ad_{\mathfrak{g}}x$ *is algebraic over* $\mathfrak{k}$ *of degree at most* n *then* x *is algebraic over the centre of* $U\mathfrak{g}$ *of degree at most* p^n.

The proof of this Lemma can be found in Jacobson [8].

The following well known assertion will be used implicitly at several points and its proof may be found for example in [2]:

LEMMA 1.6. If a ring R satisfies a non-zero polynomial identity of degree d then R satisfies a non-zero homogeneous multilinear identity of degree d.

It is necessary to note here that the ideas of this work are often similar to those of Passman's [2]. However, this is hardly true for the methods.

2. Sufficient conditions

THEOREM 2.1. *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathfrak{k}$ *of characteristic* $p > 0$. *Suppose that for any* $x \in \mathfrak{g}$ $ad_{\mathfrak{g}}x$ *is algebraic of degree at most* N_1, *and* $\mathfrak{g}$ *contains an abelian subalgebra* $\mathfrak{h}$ *of codimension* N_2. *Then* $U\mathfrak{g}$ *is PI-algebra of the degree not exceeding* $2p^{N_1}N_2$.

PROOF. Let $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{h}$ be the direct sum of its linear subspaces $\mathfrak{f}$ and $\mathfrak{h}$, and $f_1, f_2, \cdots, f_{N_2}$, be a linear basis of $\mathfrak{f}$ and $(h_\alpha)_{\alpha \in M}$ be linear basis of $\mathfrak{h}$ with M in some way linear ordered. Then the following system of elements may be regarded as a linear basis of the universal envelope $U\mathfrak{g}$ of $\mathfrak{g}$:

(2) $$h_{\alpha_1}^{l_1} \cdot h_{\alpha_2}^{l_2} \cdots h_{\alpha_s}^{l_s} f_1^{k_1} f_2^{k_2} \cdots f_t^{k_t}; \ k_i, l_k \geqq 0 \text{ and } \alpha_1 < \alpha_2 < \cdots < \alpha \text{ in } M.$$

By the above-mentioned theorem due to Cohn $U\mathfrak{g}$ can be embedded in a skew field D. Let S be subring of $U\mathfrak{g}$ generated by the universal envelope $U\mathfrak{h}$ of $\mathfrak{h}$ and the centre $\mathfrak{z}(U\mathfrak{g})$ of $U\mathfrak{g}$ and let K be the commutative subfield of D generated by S. Consider D as a left vector space over K. Let V be the K-subspace spanned by $U\mathfrak{g}$. We prove that V has finite dimension over K and that this dimension does not exceed $p^{N_1}N_2$. This is true since the linear map $ad_{\mathfrak{g}}f_i$ for each $i = 1, 2, \cdots, N_2$ is algebraic of degree at most N_1 and therefore using Lemma 1.5 one easily observes that $U\mathfrak{g}$ is spanned as a left S-module by a finite system of monomials

$$f^{k_1} f_2^{k_2} \cdots f_{N_2}^{k_{N_2}}, \ k_i \leqq p^{N_1}, \ i = 1, 2, \cdots, N_2.$$

Thus we certainly have $\dim_K V \leqq p^{N_1}N_2$.

Now let $B = End_K V$ be the ring of all linear endomorphisms of V over K. B is a matrix ring of degree not more that $p^{N_1}N_2$. It is well known that B is a PI-algebra of degree $2p^{N_1}N_2$. But the fact that the left multiplications in D commu e with the right ones and the absence of zero divisors show that $U\mathfrak{g}$ is a subring of B and that is why it inherits the standard identity $S_{2p^{N_1}N_2} = 0$ whose coefficients lie in the prime subfield of K. This proves 2.1.

NOTE. It is easily seen from the proof of 2.1 that the boundedness of the degree of $ad_{\mathfrak{g}}\mathfrak{g}$ is not essential.

3. Necessary conditions

PROPOSITION 3.1. *Let the commutator subalgebra $\mathfrak{g}^2$ of Lie algebra $\mathfrak{g}$ be finite-dimensional. If $U\mathfrak{g}$ is a PI-algebra then $\mathfrak{g}$ possesses an abelian ideal of finite codimension.*

PROOF. We denote by the symbol $|\mathfrak{b} : \mathfrak{a}|$ the codimension of a linear subspace $\mathfrak{a}$ in a linear space $\mathfrak{b}$. Let $x \in \mathfrak{g}^2$ and $\dim \mathfrak{g}^2 < \infty$. Then $|\mathfrak{g}: \mathfrak{c}_{\mathfrak{g}}(x)| < \infty$. For, since subspace $[x, \mathfrak{g}] \subseteq \mathfrak{g}^2$ it is finite dimensional. If $g_1, g_2, \cdots, g_n$ are such that $[x, g_1], [x, g_2], \cdots, [x, g_n]$ is a linear basis of $[x, \mathfrak{g}]$ then for each $g \in \mathfrak{g}$ one can find $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathfrak{k}$ such that $g + \alpha_1 g_1 + \cdots + \alpha_n g_n \in \mathfrak{cg}(x)$, i.e. $|\mathfrak{g}: \mathfrak{c}_{\mathfrak{g}}(x)| \leqq n$. Now suppose that $e_1, e_2, \cdots, e_k$ form a linear basis of $\mathfrak{g}^2$ and put

$$\mathfrak{c} = \bigcap_{i=1}^{k} \mathfrak{c}_{\mathfrak{g}}(e_i). \tag{3}$$

Then $\mathfrak{c}$ is a class two nilpotent algebra and an ideal of finite codimension in $\mathfrak{g}$. For at first for any $h \in \mathfrak{g}^2$, $[h, \mathfrak{c}] = 0$. Further if $h \in \mathfrak{g}^2$, $g \in \mathfrak{g}$, $c \in \mathfrak{c}$ then

$$[[g, c], h] = [[g, h], c] + [[h, c], g] = 0$$

since $[h, g] \in \mathfrak{g}^2$. Finally if $c_1, c_2, c_3 \in \mathfrak{c}$, then $[c_1, c_2] \in \mathfrak{g}^2$ and so $[[c_1, c_2], c_3] = 0$. The finiteness of $|\mathfrak{g}: \mathfrak{c}|$ is evident from (3).

We consider now this algebra $\mathfrak{c}$. Since $\mathfrak{c}$ is a subalgebra of $\mathfrak{g}$ it follows that $U\mathfrak{c}$ is a subalgebra of $U\mathfrak{g}$ and so a PI-algebra. We are going to show that this implies $\mathfrak{z}(\mathfrak{c})$ (the centre of $\mathfrak{c}$) be of finite codimension in $\mathfrak{c}$. Then $\mathfrak{z}(\mathfrak{c})$ is an abelian ideal in $\mathfrak{g}$ of finite codimension.

Since $\dim \mathfrak{c}^2 < \infty$ and $\mathfrak{c}^2 \subseteq \mathfrak{z}(\mathfrak{c})$ one easily verifies that $\mathfrak{c}$ is isomorphic to a subdirect product of a finite number of its homomorphic images with one-dimensional commutator subalgebra. So it is enough to prove that in an algebra $\mathfrak{d}$ with $\dim \mathfrak{d}^2 = 1$, $\mathfrak{d}^2 \subseteq \mathfrak{z}(\mathfrak{d})$ and $U\mathfrak{d}$ a PI-algebra the number $|\mathfrak{d} : \mathfrak{z}(\mathfrak{d})|$ is finite. Let on the contrary $|\mathfrak{d}: \mathfrak{z}(\mathfrak{d})| = \infty$. Choose in $\mathfrak{d}$ a set of elements

$$x_k, y_k, z, \quad k = 1, 2, 3, \cdots$$

with the following properties

(4) $[x_k, x_l] = [y_k, y_l] = [x_k, z] = [y_l, z] = 0,\ [x_k, y_l] = \delta_{kl} z, k,\ l = 0, 1, \cdots.$

This may be done in the following way. Let z be a basis of $\mathfrak{d}^2$ and x_1, y_1 be two elements of $\mathfrak{d} = \mathfrak{d}_0$ such that $[x_1, y_1] = z$

If $x_1, x_2, \cdots, x_n,\ y_1, y_2 \cdots y_n,\ \mathfrak{d}_{n+1}$ are already chosen put

$$\mathfrak{d}_n = c_{\mathfrak{d}_{n-1}}(x_n) \cap c_{\mathfrak{d}_{n-1}}(y_n)$$

and choose in $\mathfrak{d}_n$ (which is always nonabelian because of conditions on $|\mathfrak{d}: \mathfrak{z}(\mathfrak{d})|$ the elements x_{n+1}, y_{n+1} such that $[x_{n+1}, y_{n+1}] = z$. Easily verified that this system satisfies the conditions (4). Now let $\mathfrak{h}$ be the subalgebra of $\mathfrak{d}$ with the linear basis $x_1, x_2, \cdots, x_d, y_2, y_3, \cdots, y_{d+1}, z$. Since $U\mathfrak{h} \subseteq U\mathfrak{d}$ the former algebra satisfies the identities of the latter one. Namely there exists a non-commutative polynomial of the degree d of the kind

(5) $$X_1 \cdot X_2 \cdots X_d + \sum_{1 \neq \sigma \in S_d} a_\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(d)}$$

which is identically zero in $U\mathfrak{h}$. (Here S_d is the symmetrical group of the degree d on the symbols $1, 2, \cdots, d$).

Map $X_1 \to x_1 y_2,\ X_2 \to x_2 y_3, \cdots, X_d \to x_d y_{d+1}$. Order the basis of $\mathfrak{h}$ in such a way that

(6) $$x_1 < x_2 < \cdots < x_d < y_2 < y_3 < \cdots < y_{d+1} < z.$$

According to (1) then the following system of monomials forms a linear basis of $U\mathfrak{h}$

(7) $$x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} \cdot y_2^{l_2} y_3^{l_3} \cdots y_{d+1}^{l_{d+1}} z^t,\ k_i, l_j,\ t \geqq 0.$$

Direct calculations show that the element

$$x_{\sigma(1)} y_{\sigma(1)+1} x_{\sigma(2)} y_{\sigma(2)+1} \cdots x_{\sigma(d)} y_{\sigma(d)+1}$$

for each $\sigma \in S_d$ can be represented as the following sum:

$$\sum_{(\varepsilon_{ij})} x_{\sigma(1)} x_{\sigma(2)}^{\varepsilon_{12}} \cdots x_{\sigma(d)}^{\varepsilon_{1d}} (y_{\sigma(1)+1} (ad\, x_{\sigma(2)})^{\varepsilon_{22}} \cdots (ad\, x_{\sigma(d)})^{\varepsilon_{2d}}) \cdot (y_{\sigma(d-1)+1} (ad\, x_{\sigma(d)})^{\varepsilon_{dd}}) y_{\sigma(d)+1}.$$

(8)

Here each member of the sum is completely determined by the upper triangular matrix (ε_{ij}) in each column of which all the elements but exactly one equal zero and this nonzero element equals unity. In the sum (8) all nonzero members equal to certain basic elements of the kind (7). This follows from the multiplication table (4). Easily seen that in (8) only one element may have degree $d-1$ on z. *One obtains it by taking* $\varepsilon_{dd} = \varepsilon_{d-1.d-1} = \cdots = \varepsilon_{22} = 1$. *When* $\sigma = 1$ *this element equals* $x_1 y_{d+1} z^{d-1}$. Now if $\sigma \neq 1$ the element equals

$$x_{\sigma(1)}(y_{\sigma(1)+1}adx_{\sigma(2)})(y_{\sigma(2)+1}adx_{\sigma(3)})\cdots(y_{\sigma(d-1)+1}adx_{\sigma(d)})y_{\sigma(d)+1} = 0$$

for in this case one can find $1 \leqq i \leqq d-1$ with $\sigma(i)+1 \neq \sigma(i+1)$, and so one of the factors

$$y_{\sigma(i)+1}adx_{\sigma(i+1)} = [y_{\sigma(i)+1}, x_{\sigma(i+1)}] = \delta_{\sigma(i)+1,\sigma(i+1)}z = 0.$$

This shows that (5) is not an identity in $U\mathfrak{g}$. Thus the assumption $|\mathfrak{d}:\varepsilon(\mathfrak{d})| = \infty$ is not true, and consequently $|\mathfrak{c}:\mathfrak{z}(\mathfrak{c})| < \infty$. So $\mathfrak{z}(\mathfrak{c})$ is an abelian ideal of finite codimension in $\mathfrak{g}$. This proves proposition 3.1.

We consider now in Lie algebra $\mathfrak{g}$ the following subset

$$\Delta_n = \{x \in \mathfrak{g} \mid |\mathfrak{g}:{}_f(x)| \leqq n\}.$$

It is clear that Δ_n is closed with respect to multiplication by the elements of the basic field $\mathfrak{k}$, but in general it is not even a subspace. However we say that the elements $x_1, x_2, \cdots, x_m$ are linear dependent mod Δ_n if there exist not all zero $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathfrak{k}$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m \in \Delta_n$$

LEMMA 3.2. *Suppose that for Lie algebra* $\mathfrak{g}$ *there exist positive integers* m, n *such that any* m *elements are linear dependent modulo* Δ_n. *Then* $\mathfrak{g}$ *possesses a subalgebra* $\mathfrak{t}$ *of finite codimension with* $\dim \mathfrak{t}^2 < \infty$.

PROOF. Let m be the least number with the described property. Then there are in $\mathfrak{g}$ elements $e_1, e_2, \cdots, e_{m-1}$ linear independent mod Δ_n. Let E be linear subspace spanned by these elements, and $\mathfrak{t}$ be subalgebra of $\mathfrak{g}$ generated by Δ_n. Put $S = \mathfrak{t} \cap E$. Then $\mathfrak{t} = \Delta_n + S$ since for any $t \in \mathfrak{t}$ we have

$$t + \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{m-1} e_{m-1} \in \Delta_n.$$

Now choose in S a linear basis $f_1, f_2, \cdots, f_q$, $q \leqq m-1$. Each of the elements $f_1, f_2, \cdots, f_q$ may be written as a linear combination of bounded length of some products of bounded length of elements of Δ_n. Surely any element of $\mathfrak{t}$ also enjoys this property.

From the obtained fact one can easily derive that the centralizers of all elements from $\mathfrak{t}$ in $\mathfrak{g}$ have codimensions in $\mathfrak{g}$ bounded by certain number N. Of course N exceeeds the codimension of the centralizer of each element of $\mathfrak{t}$ in $\mathfrak{t}$. Now it rests to apply Neumann's theorem from the introduction to get $\mathfrak{t}^2 \leqq N^2$.

PROPOSIITON 3.3. *Let* $\mathfrak{g}$ *be Lie algebra over commutative field* $\mathfrak{k}$, $U\mathfrak{g}$ *its universal envelope. If* $U\mathfrak{g}$ *is a PI-algebra, then there exist positive integers* m, n *such that any* m *elements of* $\mathfrak{g}$ *are linear dependent modulo* Δ_n.

PROOF. Since $U\mathfrak{g}$ has no zero divisors we may apply Amitsur's theorem from the introduction of the article. Let the dimension of the quotient field D of $U\mathfrak{g}$ over its centre Z equal $m-1$. Then for any elements $x_1, x_2, \cdots, x_m$ from $\mathfrak{g}$ there are elements $A_1B_1^{-1}, A_2B_2^{-1}, \cdots, A_mB_m^{-1}$ from the centre $Z, A_i, B_i \in U\mathfrak{g}$, $1 \leqq i \leqq m$ such that

$$x_1A_1B_1^{-1} + x_2A_2B_2^{-1} + \cdots + x_mA_mB^{-1} = 0. \tag{9}$$

Not all $A_iB_i^{-1}$, $i = 1, \cdots, m$ in this relation equal 0. If x is an element of $\mathfrak{g}$, then commuting (9) with x we get

$$[x_1, x]A_1B_1^{-1} + [x_2, x]A_2B_2^{-1} + \cdots + [x_m, x]A_mB_m^{-1} = 0. \tag{10}$$

Further

$$A_iB_i^{-1}B_1B_2 \cdots B_m = B_1B_2 \cdots B_{i-1}A_iB_i^{-1}B_iB_{i+1} \cdots B_m$$
$$= B_1B_2 \cdots B_{i-1}A_iB_{i+1} \cdots B_m, \quad i = 1, 2, \cdots, m.$$

Thus multiplying both parts of (10) by $B_1B_2 \cdots B_m$ from the right we obtain

$$[x_1, x]C_1 + [x_2, x]C_2 + \cdots + [x_m, x]C_m = 0 \tag{11}$$

in which $C_i \in U\mathfrak{g}$ and not all C_i, $i = 1, 2, \cdots, m$ equal zero. The nonzero homogeneous part of the highest degree of (11) gives us a nontrivial relation

$$[x_1, x]f_1 + [x_2, x]f_2 + \cdots + [x_m, x]f_m = 0 \tag{12}$$

in the associated graduated algebra which is, as it is mentioned above, the symmetrical algebra of $\mathfrak{g}$, or the polynomial ring $\mathfrak{k}[\mathfrak{g}]$. Thus in (12) all the elements $f_1, f_2, \cdots, f_m \in \mathfrak{k}[\mathfrak{g}]$.

Let $U = (U_1, U_2, \cdots, U_m)$ – be a string of indeterminates, A be a matrix with elements

$$a_i = [x, x_i] \quad i = 1, 2, \cdots, m; \; x \in \mathfrak{g}.$$

The system of equations over $\mathfrak{k}[\mathfrak{g}]$

$$UA = 0$$

has, as it is visible from (12), the nontrivial solution $(f_1, f_2, \cdots, f_m)$. Thus all the minors of matrix A of order > 1 must equal zero for some $1 \leqq m - 1$. Let l be the least number with this property. Then there exists a nontrivial minor

$$|P| = \begin{vmatrix} a_{i_1y_1} & a_{i_1y_2} & \cdots & a_{i_1y_l} \\ a_{i_2y_1} & a_{i_2y_2} & \cdots & a_{i_2y_2} \\ a_{i_ly_1} & a_{i_ly_2} & \cdots & a_{i_ly_l} \end{vmatrix} \neq 0.$$

If x is any element of $\mathfrak{g}$ then the following minor equals zero

$$\left|\begin{array}{ccc|c} & & & a_{i_1x} \\ & P & & a_{i\,x} \\ & & & a_{i_lx} \\ \hline a_{i_l+1y_1} & a_{i_l+1y_2} & \cdots\; a_{i+1yl} & a_{i_l+1x} \end{array}\right|$$

In this minor i_{l+1} is one of the numbers $1,2,\cdots,m$ but not $i_1,i_2,\cdots,i_l$ is taken. The decomposition of this minor by the last column gives since $|P| \neq 0$ a non-trivial relation (here we remember what is a_{ix})

$$[x_1,x]g_1+[x_2,x]g_2+\cdots+[x_m,x]g_m = 0. \tag{13}$$

In this relation $g_1,g_2,\cdots,g_m$ are again elements from $\mathfrak{k}[\mathfrak{g}]$ but now there exists a subspace $V \subseteq \mathfrak{g}$ with $\dim V \leqq l(l+1)$ and such that $g_1,g_2,\cdots,g_m \in \mathfrak{k}[V]$. This subspace is spanned by the elements $a_{i_k y_j} = [x_{i_k}, y_j] \in \mathfrak{g}$, $1 < k \leqq l+1$, $1 \leqq j \leqq l$. Without loss of generality we may assume that $g_{m^2} \neq 0$.

So, let elements $x_1, x_2, \cdots, x_m$ be linear independent modulo Δ_{m^2}. Then it does not exist i, $1 \leqq i \leqq m$, with $[x_i,x] \in V$ for all $x \in \mathfrak{g}$, for then obviously $|\mathfrak{g}: \mathfrak{c}_m(x_i)| \leqq m^2$, since $\dim V \leqq m^2$ and hence $x_i \in \Delta_{m^2}$. Thus we can find $y_0 \in \mathfrak{g}$ such that $[x_1,y_0] = e \notin V$. Choose in $\mathfrak{g}$ a linear basis including a basis of V and e and put $[x_i,y_0] = \alpha_i e + h_i$, $i = 2,3,\cdots,m$; h_i written on the elements of this basis do not contain e. Putting $x = y_0$ in (13) we obtain

$$eg_1+\alpha_2 eg_2+\cdots+\alpha_m eg_m+h_2g_2+\cdots+h_mg_m = 0.$$

Using standard linear basis of a polynomial ring we get

$$g_1+\alpha_2g_2+\cdots+\alpha_mg_m = 0. \tag{14}$$

Now (13) with the help of (14) may be transformed into

$$[x_2-\alpha_2x_1, x g_2+[x_3-\alpha_3x_1,x]g_3+\cdots+[x_m-\alpha_mx_1,x]g_m = 0. \tag{15}$$

In the relation (15) none of the elements $x_i-\alpha_i x_1$, $i = 2,3,\cdots,m$ satisfies $[x_i-\alpha_i x_1, x] \in V$ for all $x \in \mathfrak{g}$ for otherwise x_i and x_1 are linear dependent modulo Δ_{m^2}. Hence with (15) we may operate just like with (13). This chain may be prolonged only till obtaining the equality

$$[x_m+\gamma_{m-1}x_{m-1}+\cdots+\gamma_1x_1,x]g_m = 0 \tag{16}$$

for all $x \in \mathfrak{g}$. Since in (16) $g_m \neq 0$ we have

$$[x_m+\gamma_{m-1}x_{m-1}+\cdots+\gamma_1x_1,x] = 0 \text{ for all } x \in \mathfrak{g}.$$

But in this case the elements $x_1,x_2,\cdots,x_m$ are linear dependent modulo the centre of algebra $\mathfrak{g}$ which is of course contained in Δ_{m^2}. This gives the desired contradiction.

Thus any m elements are linear dependent mod Δ_{m^2} and we may put $n = m^2$ in the conclusion of 3.3.

COROLLARY 3.4. *Let $\mathfrak{g}$ be a Lie algebra over a commutative field $\mathfrak{k}$ such that its universal envelope is a PI-algebra. Then $\mathfrak{g}$ possesses an abelian subalgebra of finite codimension.*

This follows from 3.3 3.2 and 3.1.

Now we prove

LEMMA 3.5. *Let $\mathfrak{g}$ be Lie algebra over a commutative field $\mathfrak{k}$ which has an abelian subalgebra $\mathfrak{a}$ of finite codimension. Then $\mathfrak{g}$ possesses an abelian ideal $\mathfrak{h}$ with $|\mathfrak{g}:\mathfrak{h}| < \infty$.*

PROOF. We define subspaces $\mathfrak{b}_k$, $k = 0, 1, 2 \cdots$ in $\mathfrak{g}$ in such a way: $\mathfrak{b}_0 = \mathfrak{a}$, and if $[\mathfrak{a}, \mathfrak{g}^k]$ is the subspace of $\mathfrak{g}$ spanned by the elements $[a, g_1, g_2, \cdots, g_k]$, $a \in \mathfrak{a}$, $g_i \in \mathfrak{g}$, $i = 1, 2, \cdots, k$ then

$$\mathfrak{b}_k = \mathfrak{b}_{k-1} + [\mathfrak{a}, \mathfrak{g}^k].$$

We have $\mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots \subseteq \mathfrak{b}_k \subseteq \mathfrak{b}_{k+1} \subseteq \cdots$

Since $\mathfrak{b}_0 = \mathfrak{a}$, $|\mathfrak{g}:\mathfrak{a}| < \infty$ one can find $S \geqq 0$ with $\mathfrak{b}_s = \mathfrak{b}_{s+1}$. We assert that $\mathfrak{b}_s$ is an ideal in $\mathfrak{g}$. For

$$[\mathfrak{b}_s, \mathfrak{g}] = [\mathfrak{b}_{s-1}, \mathfrak{g}] + [\mathfrak{a}, \mathfrak{g}^{s+1}] \subseteq \mathfrak{b}_s + [\mathfrak{a}, \mathfrak{g}^{s+1}] \subseteq \mathfrak{b}_{s+1} = \mathfrak{b}_s.$$

Evidently $0 \leqq s \leqq m$, where $m = |\mathfrak{g}:\mathfrak{a}|$.

Define further the chain of abelian subalgebras $\mathfrak{a}_l$, $l = 0, 1, 2, \cdots$. Put $\mathfrak{a}_0 = \mathfrak{a}$. Let $g_1, g_2, \cdots, g_m$ be elements of $\mathfrak{g}$ linear independent modulo $\mathfrak{a}$ (remember that $m = |\mathfrak{g}:\mathfrak{a}|$). If $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{t-1}$ are already defined, put

$$\mathfrak{a}_t(g_j) = \{x \in \mathfrak{a}_{t-1} \mid [x, g_j] \in \mathfrak{a}_{t-1}\},\ j = 1, 2, \cdots, m;\ \mathfrak{a}_t = \bigcap_{j=1}^{m} \mathfrak{a}_t(g_j).$$

It is clear that

$$[\mathfrak{a}_t, \mathfrak{g}] \leqq \mathfrak{a}_{t-1}.$$

Show that

(a) $|\mathfrak{g}:\mathfrak{a}_t| < \infty$, $t = 0, 1, 2, \cdots$,

(b) $[\mathfrak{a}_t, \mathfrak{b}_t] = 0$, $t = 0, 1, 2, \cdots$.

Both assertions are proved using induction by t. The base of the induction $t = 0$ is obvious.

Let $|\mathfrak{g}:\mathfrak{a}_{t-1}| = n$, and let j be one of $1, 2, \cdots, m$.

If elements $x_1, x_2, \cdots, x_{n+1} \in \mathfrak{a}_{t-1}$ then there exist not all zero $\alpha_1, \alpha_2, \cdots, \alpha_{n+1} \in \mathfrak{k}$ such that $\alpha_1[x_1, g_j] + \alpha_2[x_2, g_j] + \cdots + \alpha_{n+1}[x_{n+1}, g_j] \in \mathfrak{a}_{t-1}$, or $[\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n+1} x_{n+1}, g_j] \in \mathfrak{a}_{t-1}$. Now it is clear that $|\mathfrak{a}_{t-1}:\mathfrak{a}_t(g_j)| \leqq n$ for $j = 1, 2, \cdots, m$, and hence $|\mathfrak{a}_{t-1}:\mathfrak{a}_t| \leqq nm$. In particular we get from here such bound: $|\mathfrak{g}:\mathfrak{a}_t| \leqq m^{t+1}$. So (a) is proved.

(b): Let already $[\mathfrak{a}_{t-1}, \mathfrak{b}_{t-1}] = 0$. Then

$$\begin{aligned}[\mathfrak{a}_t, \mathfrak{b}_t] &= [\mathfrak{a}_t, \mathfrak{b}_{t-1} + [\mathfrak{a}, \mathfrak{g}^t]] = [\mathfrak{a}_t, \mathfrak{b}_{t-1}] + [\mathfrak{a}, \mathfrak{g}^t, \mathfrak{a}_t] = \\ &= [\mathfrak{a}_t, \mathfrak{b}_{t-1}] + [\mathfrak{a}, \mathfrak{g}^{t-1}, \mathfrak{a}_t, \mathfrak{g}] + [[\mathfrak{a}, \mathfrak{g}^{t-1}], [\mathfrak{g}, \mathfrak{a}_t]] \subseteq \\ &\subseteq [\mathfrak{a}_{t-1}, \mathfrak{b}_{t-1}] + [\mathfrak{b}_{t-1}, \mathfrak{a}_{t-1}, \mathfrak{g}] + [\mathfrak{b}_{t-1}, \mathfrak{a}_{t-1}] = 0.\end{aligned}$$

Consider now the ideal $\mathfrak{b}_s$ of the algebra $\mathfrak{g}$ and by $\mathfrak{h}$ denote the centre of the ideal $\mathfrak{b}_s$. Since by (b) $[\mathfrak{a}_s, \mathfrak{b}_s] = 0$ we have $\mathfrak{h} \supseteq \mathfrak{a}_s$ and since $|\mathfrak{g}:\mathfrak{a}_s| \leqq m^{s+1}$ we have $|\mathfrak{g}:\mathfrak{h}| \leqq m^{s+1} < \infty$. $\mathfrak{h}$ further is an ideal in $\mathfrak{g}$ since $[g,h] \in \mathfrak{h}$ for any $g \in \mathfrak{g}$, $h \in \mathfrak{h}$, for if $b \in \mathfrak{b}_s$, then $[[g,h],b] = [[g,b],h] + [g,[h,b]] = 0$, since $[g,b] \in \mathfrak{b}_s$. Hence $\mathfrak{h}$ is an abelian ideal in $\mathfrak{g}$ with $\dim \mathfrak{g}/\mathfrak{h} \leqq m^{m+1}$. Lemma is proved.

To complete the proof of the main theorem it is enough to prove the following proposition.

PROPOSITION 3.6. *Let* $\mathfrak{g}$ *be Lie algebra over a commutative field* $\mathfrak{k}$ *and* $U\mathfrak{g}$ *be a PI-algebra. Then the adjoint representation of* $\mathfrak{g}$ *is algebraic of bounded degree.*

PROOF. Suppose that $U\mathfrak{g}$ is a PI-algebra of the degree d, in which the following relation is identically true:

$$P(x_1, x_2, \cdots, x_d) = \sum_{\sigma \in S_d} a_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(d)} = 0, \tag{17}$$

where S_d is a symmetrical group of degree d on $1, 2, \cdots, d$, $a_\sigma \in k$ and $a_1 \neq 0$.

We suppose further that there are the elements v and g in $\mathfrak{g}$ such that the elements $v_0 = v$, $v_1 = v(\operatorname{ad} g)^1, \cdots, v_{N+1} = v(\operatorname{ad} g)^{N+1}$ are linear independent $N = p^{2d} + p^d + p^{d-1} + \cdots + p^2 + p + 1$. Obviously it follows that $\{v_i\ i=0,1 \cdots, N; g\}$ is also linear independent. We include then this system to the linear basis of $\mathfrak{g}$ (denote the complement by $(e_j)_{j \in J}$) and order the obtained basis in such a way that $g, v_0, v_1, \cdots, v_N$ should form the initial segment.

The basis of $U\mathfrak{g}$ then is the system of monomials

$$g^n v_0^{n_0} v_1^{n_1} \cdots v_N^{n_N} e_1^{l_1} e_2^{l_2} \cdots e_t^{l_t} \tag{18}$$

where almost all $n, n_0, n_1, \cdots, n_N, l_1, l_2, \cdots, l_t, \cdots$ equal zero and the rest are positive integers.

Show now that the element

$$P(g^p v_{p^d+1}, g^{p^2} v_{p^d+2}, \cdots, g^{p^d} v_{p^{2d}}) \tag{19}$$

is a nonzero element of $U\mathfrak{g}$ in contradiction with (17).

The element (19) is the sum of the monomials of the sort

$$B(\sigma, d) = g^{p^{\sigma(1)}} v_{p^d+\sigma(2)} g^{p^{\sigma(2)}} v_{p^d+\sigma(2)} \cdots g^{p^{\sigma(d)}} v_{p^d+\sigma(d)} \tag{20}$$

Let us prove that the element (20) equals to the sum

$$B(\sigma, d) = \tag{21}$$

$$= \sum_{(\varepsilon_{ij})} g^{p^{\sigma(1)} + \varepsilon_{12} p^{\sigma(2)} + \cdots + \varepsilon_{1d} p^{\sigma(d)}} v_{p^d + \sigma(1) + \varepsilon_{22} p^{\sigma(2)} + \cdots + \varepsilon_{2d} p^{\sigma(d)}} v_{p^d + \sigma(d-1) + \varepsilon_{dd} p^{\sigma(d)}} v_{p^d + \sigma(d)}.$$

Here each element of the sum is determined by the upper triangular matrix (ε_{ij}) such that in each column there is exactly one nonzero element, and it equals 1.

The formula (21) is proved by induction by d. The base is trivial. The general case is dealt with by transforming of the initial segment of (20) without $g^{p^{\sigma(d)}}v_{p^{d}+\sigma(d)}$ with the help of (21) and then using formuli

$$[v, a^p] = b(ad\, a)^p \quad \text{(Lemmma 1.5)}$$

and

$$[b_1 b_2 \cdots b_s, a] = \sum_{i=1}^{s} b_1 \cdots b_{i-1}[b_i, a]b_{i+1} \cdots b_s.$$

This completes the proof of (21).

Now since $g, v_0, v_1, \cdots, v_N$ are linear independent in $\mathfrak{g}$ the elements

(22) $$v_{p^d+k+\varepsilon_1 p+\varepsilon_2 p^2+\cdots+\varepsilon_d p^d}, \text{ where } 1 \leqq k \leqq d, \varepsilon_s = 0, 1,$$

are linear independent at different values of $(k, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_d)$ and they are the part of the fixed basis of $U\mathfrak{g}$.

We denote by $\bar{M}$ the basic element of the highest degree in the expression of the monomial M through the base (18). It is clear that if T any member of the sum then $\bar{T}$ differs from T only by the order of its factors of the sort (22). Easily seen that in the basic expression of $B(\sigma, d)$ as (21) through (18) we may separate only one basic element which enjoys the following properties:

(i) its degree on the elements of the type (22) equals d;

(ii) among its factors of the type (22) there is only one element with $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_d = 0$.

This elements equals

$$C(\sigma, d) = g^{p^{\sigma(1)}} v_{p^d+\sigma(1)+p^{\sigma(2)}} v_{p^d+(2)+p^{\sigma(3)}} \cdots v_{p^d+\sigma(d)}$$

So it rests to compare the elements $C(\sigma, d)$ at different $\sigma \in S_d$. Let $C(\sigma, d) = C(1, d)$. Then at first $p^{\sigma(1)} = p$, i.e. $\sigma(1) = 1$. But then $p^{d+\sigma(1)} + p^{\sigma(2)} = p^{d+1} + p^{\sigma(2)} = p^{d+1} + p^2$ because of the mentioned above property of the elements (22) i.e. $\sigma(2) = 2$. Now it is clear how to get $\sigma = 1$, which means that in the basic expression of the element (19) through (18) there is the basic element $C(1, d)$ with coefficient $a_1 \neq 0$.

Thus we have proved that for any two elements v, $g \in \mathfrak{g}$ there exists such a polynomial $\phi_v(t) \in \mathfrak{k}[t]$ such that $v\phi_v(ad\, g) = 0$ and $\deg \phi_v(t) \leqq pN+1$. Now we decompose $\mathfrak{g}$ as a periodic $\mathfrak{k}[t]$ module (through $v\ \theta\ t = v\, ad\, g$) in the direct sum of $\mathfrak{k}[t]$-primary components.

(23) $$\mathfrak{g} = \sum_{\mu} \mathfrak{g}_\mu \quad (\mu - \text{any irreducible in } \mathfrak{k}[t])$$

It is clear that at most $N+1$ different primary components may be nonzero in (23), for otherwise the element $v = v_1 + v_2 + \cdots + v_{N+2}$, $v_i \neq 0$, $i = 1\ 2, \cdots, N+2+1$ from different components cannot be annihiliated by any polynomial of degree $N+1$. Since every $\mathfrak{g}_{\mu_i}$, is annihilated by μ_i^{N+1} it becomes true that all of the $\mathfrak{g}$ is annihilated by the polynomial $\psi(t)$ whose degree does not exceed $(N+1)^2$. Thus for the element $g \in \mathfrak{g}$ its image $ad\,g$ in the adjoint representation is algebraic of the degree at most $(N+1)^2$ which proves the proposition and the theorem 1.1.

Acknowledgements

I thank Prof. A. L. Šmel'kin for putting the problem and useful discussion, I thank also I. L. Guseva, A. Yu. Ol'šanskii and D. I. Eidelkind for the discussion.

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