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INFERENCE IN MILDLY EXPLOSIVE AUTOREGRESSIONS UNDER UNCONDITIONAL HETEROSKEDASTICITY

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Mildly explosive autoregressions have been extensively employed in recent theoretical and applied econometric work to model the phenomenon of asset market bubbles. An important issue in this context concerns the construction of confidence intervals for the autoregressive parameter that represents the degree of explosiveness. Existing studies rely on intervals that are justified only under conditional homoskedasticity/heteroskedasticity. This paper studies the problem of constructing asymptotically valid confidence intervals in a mildly explosive autoregression where the innovations are allowed to be unconditionally heteroskedastic. The assumed variance process is general and can accommodate both deterministic and stochastic volatility specifications commonly adopted in the literature. Within this framework, we show that the standard heteroskedasticity- and autocorrelation-consistent estimate of the long-run variance converges in distribution to a nonstandard random variable that depends on nuisance parameters. Notwithstanding this result, the corresponding t-statistic is shown to still possess a standard normal limit distribution. To improve the quality of inference in small samples, we propose a dependent wild bootstrap-t procedure and establish its asymptotic validity under relatively weak conditions. Monte Carlo simulations demonstrate that our recommended approach performs favorably in finite samples relative to existing methods across a wide range of volatility specifications. Applications to international stock price indices and U.S. house prices illustrate the relevance of the advocated method in practice.

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1. INTRODUCTION

Over the past decade, the mildly explosive autoregressive (MEA) framework has emerged as a popular econometric device for modeling the phenomenon of asset market bubbles. The framework has been extensively utilized to develop a multitude of procedures for detecting and dating the origination and termination of bubbles as well as constructing asymptotically valid confidence intervals for the size of the bubble. These methods equip policymakers with a powerful set of econometric tools that can be employed to mitigate the potentially adverse consequences of a bubble and thereby maintain economic and financial stability. The techniques have been successfully applied to a variety of asset prices including stock prices, house prices, cryptocurrencies, and commodity prices. For detailed reviews of this literature, see, inter alia, Phillips and Shi (2020) and Skrobotov (2023).

Introduced by Phillips and Magdalinos (2005, 2007a), the MEA framework posits that the autoregressive parameter which represents the degree of explosiveness (i.e., the size of the bubble) evolves as a function of the sample size (T) according to $\rho_T = 1 + c/k_T,\ c > 0,\ k_T \to \infty,\ k_T/T \to 0,$ as $T \to \infty$. The motivation for adopting this specification emanated from the fact that modeling the parameter as fixed and independent of the sample size (i.e., $\rho_T = \rho > 1$) precludes the application of an invariance principle so that the limit distribution of its least squares estimate depends on the underlying error distribution that is typically unknown in practice (Anderson, 1959). Wang and Yu (2015) show that in the fixed parameter autoregression of order one with an intercept, the standard t-statistic on the autoregressive coefficient has a nonstandard limit distribution that depends on the initial value of the stochastic process as well as the true values of the model parameters. On the other hand, modeling the autoregressive coefficient as local-to-unity ($\rho_T = 1 + c/T$) facilitates the application of functional central limit theory and leads to a limit distribution that is not reliant on the error distribution but depends on the local-to-unity parameter c which cannot be consistently estimated (Phillips, 1987). Phillips and Magdalinos (2007a) show that the MEA framework permits the application of an invariance principle that induces a Cauchy limit distribution for the least squares estimate without assuming Gaussian errors.

While the Cauchy limit was initially derived assuming independently and identically distributed (i.i.d.) innovations, subsequent work has demonstrated the same limit distribution continues to hold under weak or strong dependence in the errors (Magdalinos, 2012), anti-persistent errors (Lui, Xiao, and Yu, 2021), and conditional heteroskedasticity (Arvanitis and Magdalinos, 2018). Fei (2018) and Liu and Peng (2019) show that the inclusion of a fixed nonzero drift in the first-order MEA model with i.i.d. errors leads to a standard normal limit distribution for the t-statistic. Assuming weakly dependent and conditionally homoskedastic innovations, Guo, Sun, and Wang (2019) show that the t-statistic based on a heteroskedasticity- and autocorrelation-robust estimate of the standard error follows a Student's t distribution in large samples. They further establish the

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invariance of the limit distribution to a possible drift in the process, regardless of whether the drift dominates the explosive stochastic component. Chan, Li, and Peng (2012) develop an empirical likelihood-based confidence interval in the first-order autoregressive model with i.i.d. errors that is asymptotically valid for stationary, unit-root, near-integrated, and fixed parameter explosive processes, thereby providing a unified approach to inference.

While the aforementioned inference methods are justified under conditional heteroskedasticity, they rule out the presence of unconditional heteroskedasticity. A plethora of empirical studies, however, document that several macroeconomic and financial time series exhibit time-varying unconditional volatility profiles. For example, Sensier and van Dijk (2004) find that approximately 80% of 214 macroeconomic time series over the period of 1959 to 1999 were subject to a break in unconditional volatility with the break date estimated at 1984 and associated with a reduction in volatility for a large number of series (the so-called "Great Moderation"). Harvey et al. (2016) reject the null hypothesis of stationary volatility in the prices of two types of crude oil, three precious metals (gold, silver, and platinum), and two non-ferrous metals (aluminum and copper) using a battery of four tests developed in Cavaliere and Taylor (2007b). Based on the rejection patterns of the tests, they conclude that a single discrete break volatility model or a trending volatility model might be appropriate for these series. Using the same set of tests, Astill et al. (2018) also find statistically significant evidence against stationary volatility for three out of five major stock price indices: the FTSE All Share index (UK), the Nasdaq Composite index (USA), and the Nikkei 225 index (Japan). Kurozumi, Skrobotov, and Tsarev (2023) use estimated variance profiles to document the presence of time-varying volatility in the 12 largest cryptocurrencies by capitalization.

Motivated by these considerations, this paper studies the problem of constructing asymptotically valid confidence intervals in a mildly explosive autoregression with weakly dependent innovations that are allowed to be unconditionally heteroskedastic. Our framework adopts a general specification for the volatility process that can accommodate both deterministic and stochastic volatility (SV) specifications commonly employed in the literature (see Section 2). Within this framework, we show that the standard heteroskedasticity- and autocorrelation-consistent (HAC) estimate of the long-run variance converges in distribution to a nonstandard random variable that depends on nuisance parameters. Notwithstanding this result, the corresponding t-statistic is shown to still possess a standard normal limit distribution. To improve the quality of inference in small samples, we propose a dependent wild bootstrap-t procedure that can simultaneously account for time-varying unconditional volatility and weak dependence in the errors. The large sample validity of the proposed approach is established under relatively weak conditions. The theoretical analysis does, however, rule out the possibility that the sign of the current shock affects future volatility, commonly referred to as leverage. This is due to the fact that conditional on the data, the bootstrap innovations are independent over time. Monte Carlo simulations demonstrate that our proposed approach performs favorably in finite samples relative to existing methods across a wide range of volatility specifications. In particular, the dependent wild bootstrap confidence interval is shown to be adept at maintaining coverage close to the nominal confidence level while controlling average length both for data generating processes (DGPs) with and without leverage effects. The relevance of the proposed method in practice is illustrated in two empirical applications concerning international stock price indices and U.S. house prices, respectively.

The bootstrap approach has been employed in prior work concerning inference in fixed parameter explosive autoregressive processes. Basawa et al. (1989) establish the asymptotic validity of the standard i.i.d. bootstrap in the first-order autoregressive process with i.i.d. errors. More recently, Cavaliere, Nielsen, and Rahbek (2020) develop bootstrap-based inference procedures in noncausal autoregressions with heavy-tailed innovations. They show that the asymptotic distribution of the least squares estimate in this framework is non-pivotal in that it depends on the tail behavior of the innovations. To address this issue, three alternative choices for the bootstrap are considered, namely, the wild bootstrap, the permutation bootstrap, and a permutation wild bootstrap. Sufficient conditions for the validity of each of these choices in large samples are provided. In contrast, the goal of the present paper is to study the properties of asymptotic and wild bootstrap procedures for conducting inference within the MEA framework with weakly dependent errors and time-varying volatility.

Our paper is also closely related to a strand of the literature that studies stable and unit-root autoregressions under unconditional heteroskedasticity. Working in a stable autoregressive framework with deterministic volatility, Phillips and Xu (2006) develop inference procedures based on a nonparametric kernel-based estimate of the variance function, while Xu and Phillips (2008) employ the estimated variance function to propose adaptive least squares estimation of the autoregressive coefficients and demonstrate via simulations the efficiency gains achievable over ordinary least squares estimation. Gonçalves and Kilian (2004) propose a wild bootstrap approach to inference in stable autoregressions under conditional heteroskedasticity of unknown form, while Xu (2008) extends their work by showing that the wild bootstrap remains valid under time-varying unconditional volatility. Xu (2008) also studies the effects of allowing for deterministic and SV on the consistency, rate of convergence, and limit distributions of the least squares estimates. The non-robustness of standard unit-root tests to unconditional heteroskedasticity was demonstrated, inter alia, by Cavaliere (2005) and Cavaliere and Taylor (2007a), who show that the limit distributions of these tests are non-pivotal and depend on the time-varying variance profile. Cavaliere and Taylor (2007a) and Beare (2018) propose unit-root tests that employ a nonparametric estimate of the variance profile where the former uses critical values simulated from the limit distribution while the latter is based on standard null asymptotic critical values. Cavaliere and Taylor (2008, 2009) develop wild bootstrap tests of the unit-root hypothesis and establish their asymptotic validity, while Boswijk and Zu (2018) propose adaptive wild bootstrap unit-root tests based on nonparametric

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volatility estimation and show that they achieve the same asymptotic power envelope as in the known volatility case. Harvey et al. (2016) show that the recursive right-tailed unit-root tests proposed by Phillips, Wu, and Yu (2011) for detecting explosive behavior are not robust to nonstationary volatility and present a wild bootstrap approach to inference that is effective at controlling size while retaining power against locally explosive (as opposed to mildly explosive) alternatives. It is useful to note that the use of the bootstrap in this strand of the literature is primarily motivated by the non-pivotal nature of the limit distributions of standard test statistics that depend on the unknown volatility process. In contrast, the dependent wild bootstrap adopted in the current paper is motivated by its ability to provide a more reliable approximation to the finite-sample distribution of the standard HAC-based t-statistic than that afforded by its standard normal limit distribution.

In a recent contribution, Phillips (2023) develops a unified approach to estimation and inference in nonstationary time series with autoregressive roots near unity. His approach allows both local and mild departures from unity and entails consistent estimation of a localizing rate parameter that characterizes such departures. Confidence intervals for the rate parameter facilitate classification of the process as local-to-unity, mildly integrated, or mildly explosive. This approach can be viewed as complementary to the approach that constructs confidence intervals for the autoregressive parameter conditional on pretest evidence against a unit root (e.g., Phillips et al., 2011, 2015). The analysis in Phillips (2023), while allowing for weak dependence in the errors, rules out unconditional heteroskedasticity in the errors, which is the primary focus of our paper. Extending the approach in Phillips (2023) to allow for heteroskedasticity in the noise component is an interesting avenue for future research but outside the scope of the present paper.

The rest of the paper is organized as follows. Section 2 lays out the modeling framework and associated assumptions. Section 3 discusses methods for constructing asymptotic confidence intervals for the autoregressive parameter. Section 4 proposes a dependent wild bootstrap-t procedure for inference and develops its large sample properties. Section 5 presents a set of Monte Carlo experiments comparing the finite-sample adequacy of the different methods in terms of coverage and expected length for a variety of volatility specifications. Section 6 details an empirical application to illustrate the practical relevance of the proposed approach. Section 7 concludes. The Supplementary Material includes four appendixes: Appendix A contains a set of technical lemmas required for the proofs of the main results, Appendix B contains the proofs of the main results in Sections 3 and 4, Appendix C contains additional Monte Carlo results, and Appendix D contains an additional empirical application to U.S. house prices. As a matter of notation, $\stackrel{p}{\to}$ denotes convergence in probability, $\stackrel{w}{\to}$ denotes weak convergence, and $\stackrel{w}{\to}_p$ denotes weak convergence in probability under the bootstrap measure.

2. THE MODEL AND ASSUMPTIONS

Consider a scalar random variable y_t generated by the following mildly autoregressive process with possibly nonzero drift:

$$y_t = \mu_T + \rho_T y_{t-1} + u_t, \quad t = 1, ..., T, \tag{1}$$

$$\rho_T = 1 + \frac{c}{k_T}, \quad c > 0, \tag{2}$$

$$u_t = C(L)e_t = \sum_{j=0}^{\infty} c_j e_{t-j}, \quad e_t = \sigma_t \varepsilon_t. \tag{3}$$

The DGP in (1)–(3) allows for weakly dependent errors modeled via the polynomial $C(.)$ and for conditional as well as unconditional heteroskedasticity modeled through the volatility function σ_t. A special case of this DGP with $\sigma_t = \sigma \ \forall \ t$ was considered by Phillips and Magdalinos (2007b) and Guo et al. (2019). Specifically, our analysis is based on the following assumptions:

Assumption 1. (a) $k_T = T^{\alpha}$, $0 < \alpha < 1$; (b) $\mu_T \sqrt{k_T} \to \nu \in [0, \infty]$ as $T \to \infty$.

Assumption 2. (a) The lag polynomial satisfies $C(z) \neq 0$ for all $|z| \leq 1$, $C(1) \in (0, \infty)$, and $\sum_{j=0}^{\infty} j|c_j| < \infty$; (b) $\varepsilon_t \sim iid(0,1)$ with $\mathbb{E}(\varepsilon_t^{4+\kappa_1}) \leq K_1 < \infty$ for some $\kappa_1 > 0$; (c) for some strictly positive deterministic sequence $\{a_T\}$, $\{\sigma_t\}$ satisfies $a_{k_T}^{-1} \sigma_{\lfloor k_T r \rfloor} \xrightarrow{w} g(r)$, $r \in [0, +\infty)$, $\int_0^1 g(r)^2 dr > 0$ a.s., $\int_0^{\infty} g(r)^2 dr < \infty$ a.s., $g(1) > 0$ a.s., and $\sup_t \mathbb{E}(a_{k_T}^{-1} \sigma_t)^{4+\kappa_2} \leq K_2 < \infty$ for some $\kappa_2 > 0$; (d) σ_t is independent of ε_s for any t and s; (e) the initial value y_0 is independent of $\{u_t\}_{t=1}^T$ and $a_{k_T}^{-1} y_0 = o(k_T^{1/2})$.

Assumption 3. The sequence $\{a_T\}$ satisfies $a_T \propto T^{\gamma}$, where γ is a constant with $\gamma \in [0, \infty)$.

Assumption 1(a) characterizes the mildly explosive framework developed by Phillips and Magdalinos (2007a, 2007b) whereby the explosive root approaches unity at a sufficiently slow rate relative to the sample size.[1] Assumption 1(b) specifies the drift component μ_T following Guo et al. (2019) and allows the drift to be small ($\nu \in [0, \infty)$) or large ($\nu = \infty$). Given that the magnitude of the drift is typically unknown in practice, potential model misspecification can be avoided by including a constant in the estimated regression.

Assumption 2(a) imposes conditions on the lag polynomial that ensures that the errors u_t are weakly dependent and admits a Beveridge–Nelson decomposition (see Phillips and Solo, 1992). Assumption 2(b) specifies the innovations ε_t to be i.i.d. with bounded fourth moments. While we adopt the i.i.d. assumption to simplify the theoretical analysis, we expect that the results in the paper will continue to hold under the weaker condition that $\{\varepsilon_t, \mathcal{F}_t\}$ is a martingale difference

[1] While the formulation in Phillips and Magdalinos (2007a) only requires $k_T \to \infty$, $k_T/T \to 0$, Assumption 1(a) has been adopted in several studies (see, e.g., Phillips and Magdalinos, 2007b; Magdalinos, 2012; Arvanitis and Magdalinos, 2018).

sequence with respect to $\mathcal{F}_t = \sigma\text{-}field\{\varepsilon_s, s \le t\}$, satisfying (i) $\mathbb{E}(\varepsilon_t^2) = 1$ for all t, (ii) $T^{-1}\sum_{t=1}^{T}\varepsilon_t^2 \xrightarrow{p} 1$, and (iii) $\sup_t \mathbb{E}(\varepsilon_t^{4+\kappa_1}) \le K_1 < \infty$ for some $\kappa_1 > 0$. Assumption 2(c) states that the appropriately scaled volatility process weakly converges to a function $g(.)$ that satisfies three conditions: (i) a lower bound condition ($\int_0^1 g(r)^2 dr > 0$ a.s.), (ii) square integrability ($\int_0^\infty g(r)^2 dr < \infty$ a.s.), and (iii) a positivity condition at one ($g(1) > 0$ a.s.). Condition (i) facilitates the application of a central limit theorem to the partial sums of the errors $\{u_t\}$, conditional on $\{\sigma_t\}$ (see Theorem 2(d) below). Conditions (ii) and (iii) ensure the non-degeneracy of the random variables appearing in the limit distribution of the autoregressive parameter estimate (see Section 3.2). Conditions (i) and (ii) were also imposed by Cavaliere and Taylor (2009) in their analysis of unit-root tests under heteroskedasticity (see their Assumption 2). Assumption 2(c) allows for a wide class of volatility processes including a variety of deterministic and nondeterministic specifications for $\{\sigma_t\}$ commonly employed in the literature. In the deterministic case, the assumption allows single and multiple volatility shifts, linearly and polynomially trending volatility (with an appropriate choice of a_T), and smooth transition breaks. In the nondeterministic case, the class of volatility processes allowed includes nonstationary autoregressive SV models (Hansen, 1995), SV models with jumps (Georgiev, 2008), nonstationary nonlinear heteroskedastic models with stochastically trending volatility, and near-integrated GARCH models (see Cavaliere and Taylor, 2009, for a detailed discussion of the class of volatility processes permissible under this assumption). Assumption 2(d) precludes the possibility that the sign of the current shock affects future volatilities, often referred to as leverage. This assumption is needed to ensure the asymptotic validity of the proposed dependent wild bootstrap approach since the wild bootstrap innovations cannot replicate any leverage effects that may be present in the original data. Nevertheless, we examine the sensitivity of the various methods to the failure of this assumption via simulations in Section 5. Assumption 2(e) guarantees the invariance of the limit theory to the initial condition. Defining $\mathcal{G}_{t-1} = \sigma\text{-}field\{\sigma_{s+1}, \varepsilon_s, s \le t-1\}$, we have $\sigma_t^2 = Var(y_t|\mathcal{G}_{t-1})$, i.e., the conditional variance of the time series y_t is represented by the process σ_t^2.

The specification for a_T adopted in Assumption 3 nests all of the volatility models in the examples considered by Cavaliere and Taylor (2009). Specifically, when $\gamma = 0$ and $a_T = 1$, it incorporates the models in their Examples 1–4, and when $\gamma > 0$, it incorporates the models in their Examples 5 and 6. Without loss of generality, we henceforth directly set $a_T = T^\gamma$ instead of $a_T = \psi T^\gamma$, $\psi > 0$, since ψ is not identified but could be absorbed into the unknown volatility function $g(\cdot)$. As will be seen in the subsequent analysis, a_T will have a nonnegligible effect on the asymptotic theory, a feature also observed by Xu (2008) in the context of stationary autoregressive models with nonstationary volatility.

Remark 1. While the errors $\{u_t\}$ are weakly dependent (or short memory) in our framework, our assumptions allow $\{\sigma_t\}$ to be a stationary long memory process. To see this, note that regardless of whether $\{\sigma_t\}$ is short memory or stationary

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long memory, the independence of $\{\sigma_t\}$ and $\{\varepsilon_t\}$ (Assumption 2(d)) ensures that $\{e_t\}$ is serially uncorrelated, i.e., $Cov(e_t, e_s) \neq 0$ for all $t \neq s$. Consequently, as long as the process $\{a_{k_T}^{-1}\sigma_t\}$ has uniformly bounded second moments (as ensured by Assumption 2(c)), the memory structure of $\{u_t\}$ is entirely determined by the conditions imposed on the lag polynomial $C(.)$ in Assumption 2(a).

Remark 2. Assumption 3 allows a_T to diverge with T when $\gamma > 0$. This possibility opens up an interesting connection between the properties of the process $\{u_t\}$ in our framework and those of a stationary long memory process (i.e., $u_t \sim I(d)$ such that $\Delta^d u_t \sim I(0)$, with $0 < d < 0.5$). Specifically, in our framework, the partial sums satisfy $\sum_{t=1}^{[Tr]} u_t = O_p(T^{\gamma+1/2})$, whereas if $u_t \sim I(d)$, $\sum_{t=1}^{[Tr]} u_t = O_p(T^{d+1/2})$ (see, e.g., Davidson and De Jong, 2000). Thus, if $d = \gamma$, the two processes possess the same signal strength. The difference arises in their limit distributions: in our framework (see Theorem 2 below), $[vu_T]^{-1/2} a_T^{-1} T^{-1/2} \sum_{t=1}^{[Tr]} u_t \xrightarrow{w} B(r)$, where $vu_T = Var(a_T^{-1} T^{-1/2} \sum_{t=1}^{[Tr]} u_t)$ and $B(.)$ denotes a standard Brownian motion on $[0,1]$; if $u_t \sim I(d), [vu_T]^{-1/2} a_T^{-1} T^{-1/2} \sum_{t=1}^{[Tr]} u_t \xrightarrow{w} B_d(r)$, where $B_d(.)$ denotes a fractional Brownian motion on $[0,1]$. A potentially interesting extension of our framework would be to explicitly allow for long memory in $\{u_t\}$ via appropriate conditions on $C(.)$ as in Magdalinos (2012). The exploration of this extension is left for future research. We thank an anonymous referee for his/her suggestion to include this discussion.

3. ASYMPTOTIC CONFIDENCE INTERVALS

The objective of the paper is to analyze the properties of alternative methods for constructing confidence intervals for the autoregressive parameter ρ_T in (1) in the potential presence of unconditional heteroskedasticity of the form specified in Assumption 2. This section first discusses existing methods based on an asymptotic approximation to the sampling distribution of the least squares estimate of ρ_T or the corresponding t-statistic under the assumption that the innovations are unconditionally homoskedastic, i.e., $\mathbb{E}(e_t^2) = \sigma^2$ for all t. Subsequently, we consider the standard t-statistic based on the usual heteroskedasticity- and autocorrelation-consistent (HAC) estimate of the long-run variance of u_t (e.g., Andrews, 1991). We show that despite the nonstandard nature of the limit distribution of the HAC estimate, the t-statistic is still asymptotically standard normal. In what follows, we denote $z_t = (1, y_{t-1})'$ and $\iota_2 = (0,1)'$.

3.1. Existing Inference Methods

Phillips and Magdalinos (2007a) considered a version of (1)–(3)with no drift under the assumption that the errors u_t are i.i.d. and square-integrable. They establish that the following limit theory holds as $T \to \infty$:

$$\frac{k_T \rho_T^T}{2c}(\tilde{\rho}_T - \rho_T) \xrightarrow{w} \mathcal{C} \text{ and } \frac{\rho_T^T}{\rho_T^2 - 1}(\tilde{\rho}_T - \rho_T) \xrightarrow{w} \mathcal{C}, \tag{4}$$

where $\tilde{\rho}_T = (\sum_{t=1}^{T} y_{t-1}^2)^{-1}(\sum_{t=1}^{T} y_{t-1}y_t)$ denotes the least squares estimate and $\mathcal{C}$ denotes a standard Cauchy random variable. It follows that a $100(1-\delta)\%$ confidence interval for ρ_T can be constructed as

$$\left(\tilde{\rho}_T \pm \frac{\tilde{\rho}_T^2 - 1}{\tilde{\rho}_T^T}\mathcal{C}_\delta\right), \tag{5}$$

where $\mathcal{C}_\delta$ is the two-tailed δ percentile critical value of the standard Cauchy distribution. For instance, a 95% confidence interval will use the critical value $C_{0.05} = 12.7$ compared to the corresponding Gaussian critical value of 1.96. We will refer to (5) as the PM interval.

Phillips and Magdalinos (2007b) showed that (4) remains valid even when the errors u_t are weakly dependent and satisfy Assumption 1(a) while imposing conditional homoskedasticity by assuming $e_t \sim iid(0,\sigma^2)$. Magdalinos (2012) extended the validity of (4) to include error processes that can be strongly dependent (i.e., exhibiting long memory), thereby demonstrating the robustness of the interval (5) to a general dependence structure in the innovation sequence. More recently, Arvanitis and Magdalinos (2018) established that the Cauchy limit theory is also invariant to a wide class of stationary conditionally heteroskedastic error processes with weak or strong dependence.[2]

Based on the limit result (4) Guo et al. (2019) show that under the assumption that the errors u_t are i.i.d., the OLS t-statistic that does not correct for heteroskedasticity or autocorrelation has a standard normal limiting distribution. In the no drift case ($\mu_T = 0$), this t-statistic is given by

$$t_{PM} = \frac{\tilde{\rho}_T - \rho_T}{\sqrt{\tilde{s}_T^2 \left(\sum_{t=1}^{T} y_{t-1}^2\right)^{-1}}} \xrightarrow{w} N(0,1),$$

where $\tilde{s}_T^2 = (T-1)^{-1}\sum_{t=1}^{T}(y_t - \tilde{\rho}_T y_{t-1})^2$. When the estimated regression includes a constant, Guo et al. (2019) establish, under Assumptions 1 and 2(e), that

$$t_{MED} = \frac{\hat{\rho}_T - \rho_T}{\sqrt{\hat{s}_T^2 \iota_2' \left(\sum_{t=1}^{T} z_t z_t'\right)^{-1} \iota_2}} \xrightarrow{w} N(0,1),$$

where $(\hat{\mu}_T, \hat{\rho}_T)' = \left(\sum_{t=1}^{T} z_t z_t'\right)^{-1} \sum_{t=1}^{T} z_t y_t$, and $\hat{s}_T^2 = (T-2)^{-1}\sum_{t=1}^{T}(y_t - \hat{\mu}_T - \hat{\rho}_T y_{t-1})^2$.

In the case of weakly dependent errors u_t given by (3) where $e_t \sim iid(0,\sigma^2)$ with finite fourth moments and $C(.)$ satisfies Assumption 2(a), Guo et al. (2019) develop an inference procedure based on an orthonormal series long-run variance

[2]Lee (2018) considers a framework in which u_t is assumed to be strong mixing with exponentially decaying coefficients and finite fourth moments. In contrast, Arvanitis and Magdalinos (2018) does not require strong mixing and instead relies on an L_1-mixingale condition on e_t which does not place restrictions on the moments of u_t higher than order 2.

estimator that accounts for the dependence structure. Specifically, they propose the t -statistic

$$\tilde{t}_{MED} = \frac{\hat{\rho}_T - \rho_T}{\sqrt{\hat{\lambda}_K^2 \iota_2' \left(\sum_{t=1}^T z_t z_t'\right)^{-1} \iota_2}}, \tag{6}$$

where $\hat{\lambda}_K^2$ is the estimate of the long-run variance of u_t constructed from the estimated residuals $\hat{u}_t = y_t - \hat{\mu}_T - \hat{\rho}_T y_{t-1}$ as

$$\hat{\lambda}_K^2 = \frac{1}{K} \sum_{j=1}^{K} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi_j\left(\frac{t}{T}\right) \hat{u}_t \right]^2. \tag{7}$$

In (7), K is an even constant and $\phi_j(x) = \sqrt{2}\sin(2\pi jx)$ and $\phi_{2j-1}(x) = \sqrt{2}\cos(2\pi jx)$ are the Fourier basis functions. Guo et al. (2019) show that under the fixed-K asymptotics where $T \to \infty$ for a given K, $\tilde{t}_{MED} \xrightarrow{w} t_K$, where t_K is the Student's t distribution with K degrees of freedom.[3] The choice of K is data-dependent and based on the asymptotic mean squared error criterion implemented using the AR(1) plug-in procedure. This value of K is then rounded to the closest even number between 4 and T (see Phillips, 2005).

All of the aforementioned confidence intervals are predicated upon the assumption of unconditionally homoskedastic innovations. In Section 5, we will examine their finite-sample performance for DGPs that fail this assumption via simulations. These simulation results would allow us to assess the degree to which these methods are sensitive to the underlying homoskedasticity assumption.

3.2. HAC-Based Inference

We now consider an asymptotic approach to inference that, in contrast to the extant methods described in Section 3.1, remains valid even in the presence of unconditional heteroskedasticity of the form allowable under Assumption 2. Our approach is simply based on the standard t-statistic that employs an HAC estimate of the standard errors to account for heteroskedasticity and autocorrelation (e.g., Newey and West, 1987; Andrews, 1991). In order to define this statistic, we introduce the following notation. Let $\bar{y} = T^{-1}\sum_{t=1}^{T} y_t$, $\bar{y}_{-1} = T^{-1}\sum_{t=0}^{T-1} y_t$, $\bar{u} = T^{-1}\sum_{t=1}^{T} u_t$, $\dot{y}_{t-1} = y_{t-1} - \bar{y}_{-1}$, $\dot{u}_t = u_t - \bar{u}$, for $t = 1, \ldots, T$, and

$$\hat{\Omega} = \sum_{j=-(T-1)}^{T-1} w(j/b_T)\hat{\Gamma}(j), \quad \hat{\Gamma}(j) = T^{-1}\sum_{t=1}^{T-|j|} \dot{y}_{t-1}\hat{u}_t \dot{y}_{t-1+|j|}\hat{u}_{t+|j|}, \tag{8}$$

with $\hat{u}_t$ the residuals defined as in (7), $w(\cdot)$ is a kernel function, and b_T is the bandwidth. The conditions on $w(\cdot)$ and b_T will be specified later. Then,

[3] Under joint asymptotics where $K \to \infty$ as $T \to \infty$ with $K/T \to 0$, $\tilde{t}_{MED} \xrightarrow{w} N(0,1)$.

letting $Q_T = T^{-1}\sum_{t=1}^{T}\dot{y}_{t-1}^2$, and $\hat{\Lambda} = T^{-1}Q_T^{-2}\hat{\Omega}$, the HAC-based t-statistic can be expressed as

$$t_{hac} := \frac{\hat{\rho}_T - \rho_T}{\hat{\Lambda}^{\frac{1}{2}}}. \tag{9}$$

We will now establish that under Assumptions 1 and 2 accompanied by suitable conditions on $w(\cdot)$ and b_T, the statistic t_{hac} has a standard normal limit distribution. To this end, we first decompose y_t in (1) into two parts, i.e., $y_t = d_t + \mu_T(\rho_T^t - 1)k_T/c$, where d_t follows

$$d_t = \rho_T d_{t-1} + u_t, \quad d_0 = y_0. \tag{10}$$

Now d_t is a mildly explosive process without drift, while $\mu_T(\rho_T^t - 1)k_T/c$ is a deterministic nonlinear trend component when $\mu_T \neq 0$. The following result derives the limits of two random quantities which will be useful in the subsequent analysis, where $MN(0,V_x)$ and $MN(0,V_y)$ denote mixed Gaussian random variables with mixing variates V_x and V_y, respectively.

THEOREM 1. *Denote* $X_T := a_T^{-1}k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}u_t$ *and* $Y_T := a_{k_T}^{-1}k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-t}u_t$. *Under Assumptions 1 and 2,* $[X_T, Y_T] \xrightarrow{w} [X, Y]$, *where X and Y are independent random variables such that* $X \sim MN(0,V_x)$, $Y \sim MN(0,V_y)$ *with* $V_x = \frac{C(1)^2 g(1)^2}{2c}$, $V_y = C(1)^2\int_0^\infty e^{-2cr}g(r)^2 dr$.

Remark 3. In the conditionally homoskedastic case where $g(r) = \sigma, \forall r \in [0, +\infty)$, Theorem 1 degenerates to the results in Phillips and Magdalinos (2007a), as $V_x = V_y = \frac{C(1)^2\sigma^2}{2c}$.

Next, we obtain the limit distribution of the least squares estimate $\hat{\rho}_T$ of ρ_T in (1). Note that

$$\hat{\rho}_T - \rho_T = \frac{\sum_{t=1}^{T}(y_{t-1} - \bar{y}_{-1})u_t}{\sum_{t=1}^{T}(y_{t-1} - \bar{y}_{-1})^2} = \frac{\sum_{t=1}^{T}y_{t-1}u_t - T^{-1}\sum_{t=1}^{T}y_{t-1}\sum_{t=1}^{T}u_t}{\sum_{t=1}^{T}y_{t-1}^2 - T^{-1}(\sum_{t=1}^{T}y_{t-1})^2}. \tag{11}$$

The following theorem presents the limits of the sample statistics appearing in (11).

THEOREM 2. *Under Assumptions 1–3, defining* $1/\infty = 0$, *we have the following joint convergence results:*

$$\text{(a)} \quad (a_{k_T}^2\mu_T^2 k_T^3 \rho_T^{2T})^{-1}\sum_{t=1}^{T}y_{t-1}^2 \xrightarrow{w} \begin{cases} \frac{Y^2}{2c\nu^2}, & \gamma > 0, \\ \frac{1}{2c}(\frac{Y}{\nu} + \frac{1}{c}), & \gamma = 0, \end{cases}$$

$$\text{(b)} \quad (a_{k_T}\mu_T k_T^2 \rho_T^{T})^{-1}\sum_{t=1}^{T}y_{t-1} \xrightarrow{w} \begin{cases} \frac{Y}{c\nu}, & \gamma > 0, \\ \frac{1}{c}(\frac{Y}{\nu} + \frac{1}{c}), & \gamma = 0, \end{cases} \tag{12}$$

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(c) $(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{w} \begin{cases} \frac{XY}{\nu}, & \gamma > 0, \\ X(\frac{Y}{\nu} + \frac{1}{c}), & \gamma = 0, \end{cases}$

(d) $a_T^{-1} T^{-1/2} \sum_{t=1}^{T} u_t \xrightarrow{w} U \sim MN(0, \sigma_u^2), \quad \sigma_u^2 = C(1)^2 \int_0^1 g(r)^2 dr,$

U is independent of X and Y.

Now, using the results of Theorem 2, it follows that

$$
\begin{aligned}
&(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \left(\sum_{t=1}^{T} y_{t-1} u_t - T^{-1} \sum_{t=1}^{T} y_{t-1} \sum_{t=1}^{T} u_t \right) \\
&= (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} y_{t-1} u_t - T^{-1/2} k_T^{1/2} (a_{k_T} \mu_T k_T^2 \rho_T^T)^{-1} \sum_{t=1}^{T} y_{t-1} \times a_T^{-1} T^{-1/2} \sum_{t=1}^{T} u_t \\
&= (a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-1} \sum_{t=1}^{T} y_{t-1} u_t - O_p(T^{-1/2} k_T^{1/2}) \xrightarrow{w} X \left[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \right], \qquad \textbf{(13)}
\end{aligned}
$$

$$
\begin{aligned}
&(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \left(\sum_{t=1}^{T} y_{t-1}^2 - T^{-1} (\sum_{t=1}^{T} y_{t-1})^2 \right) \\
&= (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T} y_{t-1}^2 - T^{-1} k_T [(a_{k_T} \mu_T k_T^2 \rho_T^T)^{-1} \sum_{t=1}^{T} y_{t-1}]^2 \\
&= (a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} \sum_{t=1}^{T} y_{t-1}^2 - O_p(T^{-1} k_T) \xrightarrow{w} \frac{1}{2c} \left[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \right]^2. \qquad \textbf{(14)}
\end{aligned}
$$

We thus have the asymptotic distribution of $\hat{\rho}_T$ as stated in the following corollary.

COROLLARY 1. *Under Assumptions 1–3,*

$$
a_T^{-1} a_{k_T} \mu_T k_T^{3/2} \rho_T^T (\hat{\rho}_T - \rho_T) \xrightarrow{w} \frac{2cX}{\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0)}. \qquad \textbf{(15)}
$$

Remark 4. In the special case $\nu = \infty$, the limit distribution in (15) reduces to $2c^2 X$, which implies $\hat{\rho}_T$ is asymptotically mixed normal. A similar result assuming $\sigma_t = \sigma$ in (3) was obtained in Guo et al. (2019) where X reduces to an $N(0, C(1)^2 \sigma^2 / 2c)$ random variable. In the more general case where the volatility structure follows Assumption 2, the mixing variate V_x takes a more complex form that depends on the unknown function $g(\cdot)$.

In order to obtain the limit distribution of $\hat{\mu}_T$, we make the following additional assumption that restricts the rate at which volatility grows with the sample size.

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Assumption 3′. $\gamma < 1/2$.

The upper bound on the rate of growth of a_T specified in Assumption 3′ ensures the consistency of the intercept estimate $\hat{\mu}_T$. Interestingly, this upper bound condition coincides with that in Xu (2008) to ensure the consistency of the intercept estimate in the stationary autoregressive framework. To obtain the distribution of $\hat{\mu}_T$, observe that

$$a_T^{-1}T^{1/2}(\hat{\mu}_T - \mu_T) = a_T^{-1}T^{1/2}\left(T^{-1}\sum_{t=1}^{T}u_t - T^{-1}\sum_{t=1}^{T}y_{t-1}(\hat{\rho}_T - \rho_T)\right)$$
$$= a_T^{-1}T^{-1/2}\sum_{t=1}^{T}u_t - T^{-1/2}k_T^{1/2} \times (a_{k_T}\mu_T k_T^2\rho_T^T)^{-1}\sum_{t=1}^{T}y_{t-1} \times a_T^{-1}a_{k_T}\mu_T k_T^{3/2}\rho_T^T(\hat{\rho}_T - \rho_T)$$
$$= a_T^{-1}T^{-1/2}\sum_{t=1}^{T}u_t - O_p(T^{-1/2}k_T^{1/2}) \xrightarrow{w} U. \quad \textbf{(16)}$$

We can thus state the following corollary.

COROLLARY 2. *Under Assumptions 1, 3, and 3′,*

$$a_T^{-1}T^{1/2}(\hat{\mu}_T - \mu_T) \xrightarrow{w} U. \quad \textbf{(17)}$$

We now establish a result that applies when $\nu = 0$, i.e., the no drift case.

THEOREM 3. *Under Assumptions 1–3, with $\nu = 0$, we have the following joint convergence results:*

$$\text{(a)}\quad (a_{k_T}^2 k_T^2 \rho_T^{2T})^{-1}\sum_{t=1}^{T}y_{t-1}^2 \xrightarrow{w} \frac{1}{2c}Y^2;$$
$$\text{(b)}\quad (a_{k_T}k_T^{3/2}\rho_T^T)^{-1}\sum_{t=1}^{T}y_{t-1} \xrightarrow{w} \frac{1}{c}Y; \quad \textbf{(18)}$$
$$\text{(c)}\quad (a_{k_T}a_T k_T \rho_T^T)^{-1}\sum_{t=1}^{T}y_{t-1}u_t \xrightarrow{w} XY;$$
$$\text{(d)}\quad a_T^{-1}T^{-1/2}\sum_{t=1}^{T}u_t \xrightarrow{w} U \sim MN(0,\sigma_u^2), \quad \sigma_u^2 = C(1)^2\int_0^1 g(r)^2 dr,$$

U is independent of X and Y.

The following corollary states the limit distribution of the OLS estimator when $\nu = 0$.

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COROLLARY 3. *Under Assumptions 1–3, with $\nu = 0$, we have*

$$a_T^{-1} a_{k_T} k_T \rho_T^T (\hat{\rho}_T - \rho_T) \xrightarrow{w} \frac{2cX}{Y}.$$

Remark 5. In the stationary autoregressive framework analyzed by Xu (2008), the limit distribution of the autoregressive estimates does not depend on the volatility scale a_T, while the limit distribution of the estimate of the deterministic component depends on a_T. In the present context, the limit distribution of both $\hat{\rho}_T$ and $\hat{\mu}_T$ depend on the volatility scale. Moreover, when the deterministic component is large enough, the limit distribution of $\hat{\rho}_T$ depends on the magnitude of the deterministic component as well (Corollary 1).

Remark 6. The limit distribution of $\hat{\rho}_T$ under either $\nu > 0$ or $\nu = 0$ does not require Assumption 3′, i.e., the rate of growth of volatility need not be restricted to be slower than $O(T^{1/2})$. Intuitively, the signal from the explosive component is strong enough that the consistency and limit distribution of $\hat{\rho}_T$ remain unaltered by the growth rate of volatility in the noise component.

Remark 7. The limit distribution of $\hat{\rho}_T$ is non-pivotal regardless of the magnitude of the drift as it depends on the unknown volatility process $g(\cdot)$. In particular, the standard inferential result (4) based on the Cauchy distribution as derived in Phillips and Magdalinos (2007a, 2007b) is no longer valid in the current context and thus the PM interval (5) does not have asymptotically correct coverage. The finite-sample implications of this result are investigated via simulations in Section 5.

The final step in establishing the limit distribution of the t-statistic (9) entails obtaining the limit of the HAC estimator $\hat{\Omega}$ defined in (8). We make the following assumption that governs the behavior of the weight function $w(\cdot)$ and the bandwidth b_T.

Assumption 4. (i) The function $w(\cdot)$ is a continuous and even function with $|w(\cdot)| \le 1$, $w(0) = 1$ and $\int_{-\infty}^{\infty} w^2(x) < \infty$. (ii) The bandwidth satisfies $b_T^{-1} + k_T^{-1/2} b_T \to 0$ as $T \to \infty$.

The following lemma states a key result instrumental in deriving the asymptotic distribution of the long-run variance estimator $\hat{\Omega}$.

LEMMA 1. *Define* $\Phi_T(j) = a_T^{-2} k_T^{-1} \sum_{t=1}^{T-|j|} \rho_T^{-2(T-t)+|j|-2} u_t u_{t+|j|}$. *Under Assumptions 1–3′, as $T \to \infty$, the following result holds:* $\sum_{j=-(T-1)}^{T-1} w(j/b_T)\Phi_T(j) \xrightarrow{w} V_x$.

We can then state the following result regarding the limit behavior of $\hat{\Omega}$.

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THEOREM 4. *Under Assumptions 1–3′, as $T \to \infty$,*

$$T(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \hat{\Omega} \xrightarrow{w} V_x \left[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \right]^2. \tag{19}$$

Remark 8. The limit of $\hat{\Omega}$ is nonstandard and depends on nuisance parameters. In particular, the limit involves the volatility function $g(\cdot)$, the localizing parameter c, and the drift magnitude ν.

Remark 9. The condition $b_T/k_T^{1/2} \to 0$ is stronger than the condition $b_T/T^{1/2} \to 0$ typically adopted to establish the consistency of the long-run variance estimator in the standard stationary framework (e.g., Jansson, 2002). This condition in turn restricts the allowable set of mildly explosive neighborhoods if a data-dependent rule is used to select the bandwidth as in Andrews (1991). For instance, Andrews (1991) shows that using the Quadratic Spectral kernel yields an estimated bandwidth of order $O_p(T^{1/5})$ which, by Assumption 3′, rules out neighborhoods in which $k_T = O(T^{2/5})$. However, as noted by an anonymous referee, it is plausible that this condition is not necessary—intuitively, when $k_T = O(1)$, the signal of y_t becomes stronger relative to $k_T = O(T^{\alpha})$ so that Theorem 4 can be expected to hold even though the condition $b_T/k_T^{1/2} \to 0$ is obviously not satisfied. While relaxing this condition can potentially be pursued by invoking uniformity results as in Andrews, Cheng, and Guggenberger (2020), we believe a separate, detailed treatment of this issue is required and thus leave it as a possible avenue for future research.

Remark 10. The analysis in Phillips (2023) can be used to provide guidance on the range of permissible bandwidths in practice. Allowing for weakly dependent errors, Phillips (2023) develops a consistent estimator $\hat{\alpha}_T$ of α in the MEA framework with $k_T = T^{\alpha}$ (see his Theorem 3.1). The bandwidth condition $b_T/k_T^{1/2} \to 0$ can then be ensured by requiring that $b_T = hT^{\varphi}$ with φ chosen such that $\varphi < \hat{\alpha}_T/2$. Since $T^{\varphi-\alpha/2} \to 0$ if and only if $T^{\varphi-\hat{\alpha}_T/2} \to 0$, this provides an upper bound on the permissible bandwidth rate. While the framework adopted by Phillips (2023) assumes a homoskedastic error structure, we conjecture that his consistency result for $\hat{\alpha}_T$ will continue to hold in the heteroskedastic framework considered in our article. We are grateful to an anonymous referee for his/her suggestion to include this discussion.

Using the preceding results, it is straightforward to show that

$$\begin{aligned}
(a_T^{-1} a_{k_T} \mu_T k_T^{3/2} \rho_T^T)^2 \hat{\Lambda} &= T^{-1} (a_T^{-1} a_{k_T} \mu_T k_T^{3/2} \rho_T^T)^2 Q_T^{-2} \hat{\Omega} \\
&= \left[T(a_{k_T}^2 \mu_T^2 k_T^3 \rho_T^{2T})^{-1} Q_T \right]^{-2} \left[T(a_{k_T} a_T \mu_T k_T^{3/2} \rho_T^T)^{-2} \hat{\Omega} \right] \\
&\xrightarrow{w} \left(\frac{1}{2c} \left[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \right]^2 \right)^{-2} \left(V_x \left[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \right]^2 \right) \\
&= \frac{4c^2 V_x}{\left[\frac{Y}{\nu} + \frac{1}{c} \mathbf{1}(\gamma = 0) \right]^2}.
\end{aligned} \tag{20}$$

Combining the limit (20) with the limit distribution of $\hat{\rho}_T$ in (15) we finally have

$$t_{hac} = \frac{\hat{\rho}_T - \rho_T}{\hat{\Lambda}^{\frac{1}{2}}} = \frac{(a_T^{-1} a_{k_T} \mu_T k_T^{3/2} \rho_T^T)(\hat{\rho}_T - \rho_T)}{\left[(a_T^{-1} a_{k_T} \mu_T k_T^{3/2} \rho_T^T)^2 \hat{\Lambda}\right]^{\frac{1}{2}}}$$
$$\xrightarrow{w} \frac{2cX/(\frac{Y}{\nu} + \frac{1}{c}\mathbf{1}(\gamma = 0))}{\left[4c^2 V_x/(\frac{Y}{\nu} + \frac{1}{c}\mathbf{1}(\gamma = 0))^2\right]^{\frac{1}{2}}} = \frac{X}{V_x} \sim N(0,1). \tag{21}$$

The standard normal limit of t_{hac} is formalized in the following theorem.

THEOREM 5. *Under Assumptions 1–3′, as $T \to \infty$, we have $t_{hac} \xrightarrow{w} N(0,1)$.*

Remark 11. A pivotal limit of t_{hac} is attained since the limit of the standard error estimate, though nuisance parameter-dependent, is proportional to the same random variable that appears in the limit distribution of the least squares estimate $\hat{\rho}_T$. The cancelation of the non-pivotal terms in the numerator and denominator of the t-statistic effectuates a pivotal limiting distribution.

Remark 12. We only require Assumption 3 instead of Assumption 3′ to derive the limit of t_{hac} and thus conduct inference on ρ_T. For inference on the intercept, however, Assumption 3′ would be needed as in Xu (2008).

4. DEPENDENT WILD BOOTSTRAP

The previous section established the large sample validity of the HAC-based t-statistic in the potential presence of nonstationary volatility as well as weak dependence in the noise component within the MEA framework. In small samples, however, the performance of HAC-based asymptotic confidence intervals may be less than satisfactory as illustrated via Monte Carlo simulations in Section 5. In response to this possibility, we propose an alternative, bootstrap-based approximation to the finite-sample distribution of the t-statistic that can improve upon the asymptotic approximation provided by the standard normal distribution. In particular, as the ensuing Monte Carlo comparison demonstrates, the bootstrap-based interval is shown to achieve improved coverage while controlling average length, relative to existing asymptotic methods as well as the asymptotic interval (9) based on t_{hac}.

The bootstrap procedure we adopt is the so-called dependent wild bootstrap (DWB, henceforth), introduced by Shao (2010). The DWB is designed to simultaneously capture unconditional heteroskedasticity and potential temporal dependence in the errors, and thus is a natural extension of the wild bootstrap developed by Wu (1986) and Liu (1988) for serially uncorrelated errors. While originally proposed for stationary time series by Shao (2010), several recent studies have investigated its applicability in the nonstationary time series setup. For instance, Smeekes and Urbain (2014) study several modified wild bootstrap methods,

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including the DWB, in a multivariate framework and prove its asymptotic validity in testing for unit roots. Rho and Shao (2019) propose the DWB in the unit-root testing context with piecewise locally stationary errors and provide justification for its consistency. Our paper contributes to the DWB literature by further extending its validity to the MEA framework allowing for general and flexible forms of variance and dependence structures in the errors.[4]

The DWB is based on generating a series of random variables $\{\eta_t\}_{t=1}^T$ that are independent of the data in order to capture the heteroskedasticity in the errors. In the original wild bootstrap (Liu, 1988), the $\{\eta_t\}_{t=1}^T$ are independent while in the DWB, the $\{\eta_t\}_{t=1}^T$ are correlated to accommodate temporal dependence in the errors. Specifically, we make the following assumption on $\{\eta_t\}_{t=1}^T$ (Shao, 2010).

Assumption 5. The series $\{\eta_t\}_{t=1}^T$ is drawn independently of the data such that $\mathbb{E}(\eta_t) = 0,\ \mathrm{Var}(\eta_t) = 1,\ \mathrm{Cov}(\eta_s, \eta_t) = K(\frac{s-t}{l_T})$, where $K\colon \mathbb{R} \to [0,1]$ is a symmetric kernel function that satisfies $K(0) = 1, K(x) = 0$ for $x \geq 1$, $\lim_{x\to 0}[1-K(x)]/|x|^q \neq 0$ for some $q \in (0,2]$, and $\int_{-\infty}^{\infty} K(u)e^{-iux}du \geq 0$ for $x \in R$. The quantity l_T is a bandwidth parameter satisfying $l_T = O(T^g)$, $0 < g < 1/3$. Assume that η_t is l_T-dependent and $\mathbb{E}(\eta_t^4) < \infty$.

In practice, the series $\{\eta_t\}_{t=1}^T$ can be obtained by drawing samples from a multivariate normal distribution with zero mean and covariance function $\mathrm{Cov}(\eta_s, \eta_t) = K(\frac{s-t}{l_T})$.[5] Several kernels popular in practice such as the Bartlett kernel (with $q = 1$) and the Parzen and Tukey-Hanning kernels (with $q = 2$) satisfy Assumption 5. Alternative choices for the bandwidth will be explored via simulations in Section 5. In addition to the restriction on the bandwidth l_T as specified in Assumption 5, our theoretical analysis is based on the following additional assumption which is akin to Assumption 3′ in the preceding section.

Assumption 6. The bandwidth l_T satisfies $k_T^{-1/2} l_T \to 0$ as $T \to \infty$.

With the OLS residuals $\hat{u}_t = y_t - \hat{\mu}_T - \hat{\rho}_T y_{t-1}$ at hand, the DWB residuals are simply constructed as $u_t^* = \eta_t \hat{u}_t$. To analyze the properties of the bootstrap samples, we first derive the following invariance principle for $\{u_t^*\}$ which parallels that derived in Theorem 1 for the original errors $\{u_t\}$.

THEOREM 6. *Under Assumptions 1–5, as $T \to \infty$,*

$$X_T^* := a_T^{-1} k_T^{-1/2} \sum_{t=1}^{T} \hat{\rho}_T^{-(T-t)-1} u_t^* \xrightarrow{w}_p X \sim N(0, V_x),$$

[4] Weak dependence in the errors can be alternatively captured using a block bootstrap based approach (e.g., Carlstein, 1986; Kunsch, 1989) or a sieve bootstrap approach (Bühlmann, 1997). Therefore, apart from the DWB analyzed in this paper, heteroskedasticity-robust versions of certain block and sieve bootstrap methods may also be viable in the present context. A comparison of alternative bootstrap approaches within the MEA framework allowing for unconditionally heteroskedastic and weakly dependent errors is a potentially interesting topic for future research.

[5] As illustrated in Example 4.1 of Shao (2010), $\{\eta_t\}_{t=1}^T$ can also be generated from a non-normal distribution.

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$$Y_T^* := a_{k_T}^{-1} k_T^{-1/2} \sum_{t=1}^{T} \hat{\rho}_T^{-t} u_t^* \xrightarrow{w}_p Y \sim N(0, V_y). \quad \textbf{(22)}$$

This theorem reveals that the DWB is able to mimic the unknown heteroskedasticity and temporal dependence in the errors within the MEA framework, thereby generalizing the set of time series to which the procedure is applicable. However, it is not fully transferable from the stationary/unit-root case since the additional Assumption 5 plays a crucial role in ensuring its validity (see Appendix B of the Supplementary Material for details).

We now discuss how to apply the DWB in constructing a bootstrap-based confidence interval for the autoregressive parameter ρ_T. The following algorithm enumerates the steps involved in implementation of the DWB.

Residual-Based DWB Algorithm

1. Generate T bootstrap innovations η_t, $t = 1, \ldots, T$ from a multivariate normal distribution with zero mean and covariance function $\mathrm{Cov}(\eta_s, \eta_t) = K(\frac{s-t}{l_T})$, and construct the DWB residuals $u_t^* = \eta_t \hat{u}_t$, where $\hat{u}_t = y_t - \hat{\mu}_T - \hat{\rho}_T y_{t-1}$, $t = 1, \ldots, T$, are the OLS regression residuals.
2. Construct the bootstrap samples $\{y_t^*, t = 1, \ldots, T\}$, recursively as

$$y_t^* = \hat{\mu}_T + \hat{\rho}_T y_{t-1}^* + u_t^*, \ t = 1, \ldots, T, \quad \textbf{(23)}$$

with $y_0^* = y_0$.
3. Calculate the t_{hac} statistic defined in (9) for the bootstrap data as $t^*_{\hat{\rho}_T, hac} = (\hat{\rho}_T^* - \hat{\rho}_T)/\hat{\Lambda}^{*\frac{1}{2}}$, where $\hat{\rho}_T^*$ is the bootstrap OLS estimate and $\hat{\Lambda}^*$ is the bootstrap analogue of $\hat{\Lambda}$ computed from the estimated bootstrap residuals $\hat{u}_t^* = y_t^* - \hat{\mu}_T^* - \hat{\rho}_T^* y_{t-1}^*$. Specifically, $\hat{\Lambda}^* = T^{-1} Q_T^{*-2} \hat{\Omega}^*$, where $Q_T^* = T^{-1} \sum_{t=1}^T \dot{y}_{t-1}^{*2}$, $\dot{y}_{t-1}^* = y_{t-1}^* - \bar{y}_{-1}^*$, $\bar{y}_{-1}^* = T^{-1} \sum_{t=1}^T y_{t-1}^*$, and $\hat{\Omega}^*$ is computed as in (8),

$$\hat{\Omega}^* = \sum_{j=-(T-1)}^{T-1} w(j/b_T^*) \hat{\Gamma}^*(j), \quad \hat{\Gamma}^*(j) = T^{-1} \sum_{t=1}^{T-|j|} \dot{y}_{t-1}^* \hat{u}_t^* \dot{y}_{t-1+|j|}^* \hat{u}_{t+|j|}^*. \quad \textbf{(24)}$$

4. Repeat steps 1–3 B times to approximate the distribution of the original statistic t_{hac}. Obtain the $\delta/2$ and $(1-\delta/2)$ quantiles from the empirical distribution of $t^*_{\hat{\rho}_T, hac}$, denoted $t^*_{\delta/2}$ and $t^*_{1-\delta/2}$, respectively. Construct the equal-tailed $100(1-\delta)\%$ bootstrap confidence interval as

$$\left(\hat{\rho}_T - \hat{\Lambda}^{\frac{1}{2}} t^*_{1-\delta/2}, \hat{\rho}_T - \hat{\Lambda}^{\frac{1}{2}} t^*_{\delta/2}\right). \quad \textbf{(25)}$$

Remark 13. The bandwidth parameter b_T^* in step 3 is determined in a data-dependent way as b_T in the original statistic (9), albeit based on the bootstrap data $\{y_t^*\}$. It should be noted that using the same b_T as in the original statistic for all bootstrap replications is also a valid procedure. Nevertheless, the results were

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found to be qualitatively similar in simulations to those reported and are available upon request.

Finally, the asymptotic validity of the residual-based DWB is formally established in the following theorem.

THEOREM 7. *Under Assumptions 1, 2, 3′, and 4–5, as $T \to \infty$,*

$$t^*_{\hat{\rho}_T, hac} := \frac{\hat{\rho}^*_T - \hat{\rho}_T}{\hat{\Lambda}^{*\frac{1}{2}}} \xrightarrow{w}_p N(0,1). \quad \textbf{(26)}$$

Theorem 7 demonstrates that the residual-based DWB is consistent, i.e., the bootstrap t-statistic has the same first-order limiting distribution as the original test statistic t_{hac}. Thus, the bootstrap statistic achieves (asymptotically) correct size and the associated bootstrap confidence interval (25) achieves (asymptotically) correct coverage.

Remark 14. The validity of the bootstrap algorithm as stated in Theorem 7 requires the stronger Assumption 3′ instead of Assumption 3 that was sufficient to derive the limit distribution of $\hat{\rho}_T$ (Corollaries 1 and 3). The reason is that in constructing the bootstrap samples, we utilize the estimated deterministic component $\hat{\mu}_T$ whose consistency requires Assumption 3′ (Corollary 2).

5. MONTE CARLO SIMULATIONS

This section conducts a set of Monte Carlo experiments designed to assess the finite-sample adequacy of the asymptotic approximations developed in the preceding section as well as provide a numerical comparison of the proposed approach with existing approaches. In particular, we evaluate the relative efficacy of the different procedures via the coverage rates and average effective length (i.e., length conditional on covering the true parameter value) of the resulting confidence intervals. The simulation design is similar to Guo et al. (2019).

The DGP is given by

$$y_t = \mu_T + \rho_T y_{t-1} + u_t, \quad t = 1, ..., T,$$
$$\rho_T = 1 + \frac{c}{T^\alpha}, \quad c = 0.5, \ \alpha \in \{0.5, 0.8\}.$$

Two specifications for the drift are considered: (i) $\mu_T = 0$; (ii) $\mu_T = T^{-\alpha/4}$. For the noise component u_t, we consider the case with no serial correlation ($u_t = e_t$) as well as cases with the following autoregressive (AR) and moving average (MA) structures:

$$u_t = \phi u_{t-1} + \sqrt{1-\phi^2} e_t,$$
$$u_t = \sqrt{1-\theta^2} e_t + \theta e_{t-1}.$$

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The time series for e_t is generated based on the following specifications that include homoskedastic, conditionally heteroskedastic, and unconditionally heteroskedastic cases with $\varepsilon_t \overset{iid}{\sim} N(0,1)$ throughout:

- DGP-0 [constant volatility]: $e_t = \sigma_t \varepsilon_t,\ \sigma_t = 1$.
- DGP-1 [single volatility shift]: $e_t = \sigma_t \varepsilon_t,\ \sigma_t = \mathbf{1}(t \le \tau_1 T) + \sigma \mathbf{1}(t > \tau_1 T),\ \tau_1 = 0.5,\ \sigma = 1/3$.
- DGP-2 [double volatility shift]: $e_t = \sigma_t \varepsilon_t,\ \sigma_t = \mathbf{1}(t \le \tau_1 T) + \sigma \mathbf{1}(\tau_1 T < t \le \tau_2 T) + \mathbf{1}(t > \tau_2 T),\ (\tau_1, \tau_2) = (0.3, 0.7), \sigma = 3$.
- DGP-3 [trending volatility]: $e_t = \sigma_t \varepsilon_t,\ \sigma_t = 1 + 5t/T$.
- DGP-4 [GARCH]: $e_t = \sqrt{h_t}\varepsilon_t,\ \ h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 e_{t-1}^2,\ \ (\beta_0, \beta_1, \beta_2) = (0.01, 0.9, 0.09)$.
- DGP-5,6 [stochastic volatility]: $e_t = v_t \exp(\frac{1}{2}(\omega_0 + \frac{\omega_1}{T^{1/2}} h_t)),\ h_t = (1 - c_1/T) h_{t-1} + \varepsilon_t,\ h_0 = 0,\ (v_t, \varepsilon_t) \overset{iid}{\sim} N(0, \Sigma_{v\varepsilon}),\ \Sigma_{v\varepsilon} = \begin{bmatrix} 1 & \bar{\omega} \\ \bar{\omega} & 1 \end{bmatrix}$. We set $\omega_0 = 0, \omega_1 = 5$, and $c_1 = 0$. DGP-5 and DGP-6 correspond to the cases with $\bar{\omega} = 0$ and $\bar{\omega} = -0.5$, respectively.

DGP-0 is the base case with constant volatility. DGP-1 and DGP-2 exhibit discrete jumps in volatility, while DGP-3 is a case of trending volatility.[6] DGP-4 follows a GARCH specification adopted from Gonçalves and Kilian (2004) which is in turn based on Engle and Ng (1993).[7] The SV specification for DGP-5 and DGP-6 is borrowed from Cavaliere and Taylor (2009). DGP-5 represents a case with no leverage, while DGP-6 allows for leverage via a nonzero correlation between the shocks v_t and ε_t.[8] Note that DGP-6 is ruled out by Assumption 2 so that the results for this case serve as a check on the robustness of the various methods to the violation of this assumption.

Three alternative values for the sample size are considered: $T \in \{50, 100, 200\}$. The nominal level of the confidence intervals is set at 95%. The results for $\alpha = 0.5$ are reported in the main text, while those for $\alpha = 0.8$ are presented in Appendix C of the Supplementary Material. Except for the PM interval, the estimated regression always includes a constant regardless of whether the true drift is zero or not. All experiments are based on 10,000 Monte Carlo replications and 399 bootstrap replications.

Seven alternative methods for the construction of confidence intervals for ρ_T are considered. These include (i) the HAC-based interval based on the statistic (9), denoted "t_{hac}"; (ii) the Phillips and Magdalinos (2007a) interval (5),

[6]The results for DGP-1 with $\sigma = 3$ (not reported) are qualitatively similar to those with $\sigma = 1/3$, while the results for DGP-2 with $\sigma = 1/3$ (not reported) are qualitatively similar to those with $\sigma = 3$. The full set of results is available upon request.

[7]Engle and Ng (1993, p. 1760) consider two different configurations of parameter values: (i) "medium persistence"—$(\beta_0, \beta_1, \beta_2) = (0.05, 0.9, 0.05)$; (ii) "high persistence"—$(\beta_0, \beta_1, \beta_2) = (0.01, 0.9, 0.09)$. We only present results for (ii) since the results for (i) have a similar overall pattern. The latter set of results is available upon request.

[8]Cavaliere and Taylor (2009) also considered other parameter values, namely, $c_1 \in \{10, 20\}$ and $\omega_1 = 10$. For brevity, we do not present these results given that the results reported are fairly representative of these cases.

denoted "PM";[9] (iii): the Guo et al. (2019) interval based on the t -statistic (6), denoted "GSW"; (iv)–(vi): the dependent wild bootstrap with bandwidth l, denoted DWB_l, $l \in \{3, 5, 10\}$; and (vii) the dependent wild bootstrap with bandwidth chosen according to the deterministic rule $l = \lfloor 4.5(T/100)^{1/4} \rfloor$, denoted DWB_r. This rule yields bandwidths of 3,4,5 for $T = 50$, $T = 100$, $T = 200$, respectively. Rho and Shao (2019) propose an alternative rule in the context of unit-root testing: $l = \lfloor 6(T/100)^{1/4} \rfloor$. In our simulations, we found this rule to generate bandwidths that are too large to deliver confidence intervals with adequate coverage. The Quadratic Spectral kernel is used to construct the HAC long-run variance estimate and, following Andrews (1991), a data-dependent bandwidth rule based on an AR(1) approximating model for each element of the vector $z_t \hat{u}_t$ is used (see equation (6.4) of Andrews, 1991). To improve finite-sample performance, we employ prewhitening as suggested by Andrews and Monahan (1992) based on a VAR(1) model for $z_t \hat{u}_t$.[10] The Bartlett kernel is adopted as the kernel function for implementing the dependent wild bootstrap procedure.

Table 1 presents the empirical coverage rates of the different methods for the case without drift ($\mu_T = 0$). Panels A–C report the results with serially uncorrelated errors, AR errors with $\phi = 0.5$, and MA errors with $\theta = 0.5$, respectively. Consider first the coverage rates based on t_{hac}. With serially uncorrelated errors, the coverage rates of t_{hac} are liberal (i.e., less than the nominal level) regardless of whether the errors are heteroskedastic with the degree of undercoverage being especially severe when the sample size is small. For instance, when $T = 50$, the maximum coverage across all DGPs is only 85%, while coverage is below 80% for five out of the seven DGPs considered, including the constant volatility case. The overall pattern of results with AR errors is similar to that in the serially uncorrelated case although the liberal nature of the confidence intervals is somewhat mitigated in the former case relative to the latter when $T = 50$. When errors are of the MA type, the performance of t_{hac} is considerably improved relative to the other two error structures with coverage exceeding 90% for six of the seven DGPs as long as $T \geq 100$. The t_{hac} intervals in the MA case are notably conservative in the presence of a single volatility shift (DGP-1).

Turning to the PM interval, we find that with constant volatility, its coverage can be quite conservative ($> 98\%$) when $T \leq 100$ regardless of the serial correlation structure but moves closer to the nominal level when $T = 200$. This is not surprising given that the interval is (asymptotically) justified in this case. When the errors are heteroskedastic, the performance of PM depends crucially on the specific form of heteroskedasticity. For DGP-1 and DGP-2 which are

[9] The results for the PM interval are conditioned on those realizations for which $\tilde{\rho}_T > 1$ and increasing the number of Monte Carlo replications till 10,000 estimates satisfying this condition were obtained. Without conditioning, the method often yields poor (liberal) coverage rates, especially when ρ_T is close to (but greater than) unity (e.g., when $\alpha = 0.8$).

[10] We found the prewhitened HAC estimator to deliver considerably more accurate coverage rates relative to its non-prewhitened counterpart. The asymptotic results derived in Sections 3 and 4, however, remain valid for the pre-whitened estimator as well.

TABLE 1. Empirical coverage rate of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.5$, $\mu_T = 0$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: No serial correlation																					
t_{hac}	76.4	87.4	91.4	82.5	90.3	93.1	74.3	85.4	92.2	72.3	83.0	89.4	72.9	84.5	90.9	79.5	88.0	92.1	85.4	90.6	93.7
PM	99.2	98.8	95.8	99.8	100	98.5	99.6	99.8	95.1	93.2	92.9	81.7	95.7	96.0	88.8	94.9	94.9	84.9	91.6	91.8	86.8
GSW	84.2	91.8	94.4	90.7	96.5	99.1	88.2	94.6	98.6	76.0	79.5	82.6	77.1	85.3	89.8	83.5	86.4	87.7	87.1	88.5	88.6
DWB_3	86.9	91.3	92.2	91.1	92.6	93.5	88.8	90.9	92.5	84.5	88.7	91.6	84.9	89.1	92.5	91.7	92.3	93.4	92.7	93.1	94.3
DWB_5	85.1	89.3	90.8	89.7	91.9	92.8	88.4	90.4	92.0	82.9	87.2	90.1	83.4	87.6	91.5	90.5	91.5	92.9	91.8	92.2	93.7
DWB_{10}	81.1	86.2	88.0	86.8	89.9	91.3	86.8	89.5	90.7	80.4	84.2	87.1	80.9	84.8	88.9	88.4	90.0	91.7	89.9	90.7	92.4
DWB_r	88.3	90.9	91.9	91.9	92.7	93.6	89.7	91.0	92.8	85.8	88.6	91.8	86.2	89.1	92.8	92.1	92.3	93.4	93.0	92.7	94.1
Panel B: AR case, $\phi = 0.5$																					
t_{hac}	82.6	88.4	90.1	85.2	91.9	93.6	86.0	90.2	93.0	79.2	83.6	87.6	80.3	85.4	89.4	84.2	89.5	92.2	85.3	90.0	93.0
PM	98.3	98.3	95.3	99.8	99.8	98.5	99.6	99.6	95.4	91.1	90.9	82.0	94.2	94.6	88.8	92.3	92.8	85.5	89.2	89.7	86.7
GSW	87.4	91.2	92.9	90.0	95.6	98.4	96.8	97.9	98.9	86.6	82.9	81.8	84.0	85.7	88.9	86.2	87.5	86.9	84.9	86.0	87.4
DWB_3	86.1	88.2	89.5	89.7	91.2	92.2	89.9	89.9	91.5	84.5	86.4	89.1	84.9	86.4	90.1	90.6	90.7	92.2	90.4	90.9	92.7
DWB_5	84.0	86.9	88.7	89.1	91.0	92.1	88.9	89.6	91.3	81.3	84.7	87.6	82.6	84.9	89.3	89.1	90.0	92.0	89.2	89.9	92.1
DWB_{10}	79.5	83.6	86.5	86.0	89.0	91.1	87.2	88.6	90.2	77.1	81.4	85.1	79.1	81.8	87.1	86.3	88.0	90.9	86.6	88.1	91.0
DWB_r	87.1	88.1	89.6	89.7	91.4	92.1	90.6	90.1	91.5	86.1	86.5	88.8	86.5	86.5	90.3	91.1	90.5	92.1	90.9	90.9	92.7
Panel C: MA case, $\theta = 0.5$																					
t_{hac}	88.1	93.2	95.0	91.7	96.1	97.1	90.1	94.2	96.2	84.0	89.5	93.0	85.6	90.7	93.9	89.6	94.2	96.0	90.1	94.3	96.2
PM	98.8	98.6	95.4	99.9	99.9	98.5	99.5	99.5	95.0	91.7	91.9	80.8	94.8	95.2	88.5	93.4	93.8	85.0	90.5	89.9	86.8
GSW	89.8	92.8	94.3	93.7	97.1	99.2	95.4	97.2	99.1	85.7	81.7	81.7	86.5	87.2	90.5	88.1	88.2	88.1	86.8	87.5	87.4
DWB_3	90.4	92.9	93.5	93.1	94.8	95.5	92.4	94.1	94.9	88.4	91.0	92.6	88.7	91.3	93.8	93.6	94.0	95.3	93.5	94.5	95.7
DWB_5	87.4	90.5	91.3	91.6	93.2	94.1	91.3	92.6	93.6	85.5	88.5	90.3	86.0	88.5	92.1	91.8	92.7	93.8	92.3	92.9	94.2
DWB_{10}	82.9	86.4	87.9	88.2	90.6	92.1	88.8	90.8	91.4	81.5	84.5	86.7	82.4	84.9	89.1	89.1	90.6	92.1	89.6	91.0	92.5
DWB_r	92.4	92.8	93.4	94.2	94.8	95.3	93.9	94.0	94.9	90.6	91.0	92.8	90.8	91.3	93.9	94.6	94.1	95.3	94.6	94.3	95.8

characterized by discrete volatility shifts, the interval continues to be conservative with discernible improvement in coverage observed only for DGP-2 as the sample size increases. In contrast, in the trending volatility case (DGP-3), coverage is at most 82% across the three different error structures when $T = 200$. In fact, coverage declines by at least 9 percentage points as the sample size increases from $T = 100$ to $T = 200$, suggesting the inadequacy of the asymptotic approximation on which the PM interval is based. A deterioration in performance as the sample size increases is also observed for DGPs 4–6, though to varying degrees.

For the GSW interval, the coverage rates are in excess of 90% in the constant volatility case for $T \geq 100$ and gradually approach the nominal level as the sample size increases, consistent with the asymptotic validity of the interval in this case. This is, however, no longer true with time-varying volatility, as exemplified by the results for DGP-1 to DGP-6. When volatility is subject to discrete shifts, the interval becomes more conservative as the sample size increases from $T = 100$ to $T = 200$. For instance, when $T = 200$, the coverage rates for DGP-1/DGP-2 are at least 98%. A similar lack of convergence toward the nominal level is also observed for DGPs 3–6 although in these cases the coverage rates remain notably liberal ($< 90\%$) regardless of the sample size and the serial correlation structure (except when $T = 200$ and errors are of the MA type).

Consider now the coverage rates of the intervals based on the dependent wild bootstrap. Several features of these results are noteworthy. First, coverage performance varies with the bandwidth employed with a smaller bandwidth generally leading to intervals with more accurate coverage. The proposed bandwidth rule exhibits coverage similar to that with the smallest bandwidth. Second, consistent with Theorem 7, the coverage rates typically improve as the sample size increases for each of the DGPs considered. Interestingly, this feature is also observed in the SV case with leverage (DGP-6) despite the fact that this case is not allowed for in the theory. Third, the coverage rates of the bootstrap-based intervals are considerably less sensitive to the nature of serial correlation and the particular form of heteroskedasticity relative to the t_{hac} and PM intervals. Fourth, the performance of the bootstrap-based intervals is particularly impressive when the sample size is small ($T = 50$), where the t_{hac} and PM intervals often suffer from substantial under/over-coverage.

Table 2 reports the coverage rates when the DGP includes a drift. The performance of t_{hac} in this case generally improves relative to the no drift case by ameliorating the extent of undercoverage especially when the sample size is small. The PM interval, on the other hand, is now seen to be severely conservative in most cases with coverage being as high as 100% for four of the seven volatility specifications considered, including the constant volatility case. Further, the coverage rates do not necessarily approach the nominal level as the sample size increases for any of the volatility structures. This feature can be explained by the fact that when $\nu \neq 0$ in Assumption 2, the Cauchy limit distribution underlying the PM interval

Table 2. Empirical coverage rate of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.5$, $\mu_T = T^{-\alpha/4}$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: No serial correlation																					
t_{hac}	88.9	91.3	92.0	91.9	94.0	94.1	93.2	93.9	93.8	81.9	88.6	91.3	87.2	90.0	92.0	87.4	91.0	93.4	84.2	90.6	93.6
PM	99.8	99.9	100	99.9	100	100	99.9	99.9	100	97.2	98.6	99.3	100	100	100	96.7	97.1	94.5	94.4	94.7	89.3
GSW	94.3	95.0	94.5	98.0	98.8	99.4	98.9	99.6	99.7	83.8	85.3	82.6	90.8	91.0	91.1	89.0	88.7	89.1	86.9	87.6	88.1
DWB_3	90.1	91.7	92.7	91.9	93.5	94.3	87.4	88.3	92.8	86.8	89.7	91.6	89.3	91.1	93.1	91.6	91.8	92.8	91.8	92.0	93.9
DWB_5	88.1	90.3	91.6	90.5	92.6	93.8	86.8	87.3	92.2	84.8	87.4	90.3	87.3	89.3	92.0	90.7	90.9	92.1	91.2	91.6	93.3
DWB_{10}	84.0	87.1	88.9	87.5	90.4	92.6	85.7	85.3	90.6	80.9	84.0	86.9	83.6	86.0	89.6	88.6	89.5	91.1	89.7	90.3	92.6
DWB_r	91.0	91.7	92.6	92.3	93.3	94.2	87.8	88.2	92.8	88.2	89.7	91.6	90.5	91.2	93.4	91.7	91.7	92.8	92.0	92.0	93.9
Panel B: AR case, $\phi = 0.5$																					
t_{hac}	85.7	89.5	91.1	89.3	92.9	93.9	91.6	93.4	93.3	80.1	86.3	88.3	84.1	87.9	89.9	85.6	90.4	92.0	84.8	89.9	92.6
PM	98.8	99.0	99.0	99.1	99.1	98.9	99.4	99.4	98.7	93.3	94.6	93.7	99.8	99.9	100	94.3	95.0	91.9	91.6	92.3	87.2
GSW	90.3	92.3	93.1	93.4	96.3	98.6	98.0	99.2	99.3	86.3	83.5	82.4	86.6	87.6	89.1	87.1	86.8	87.3	85.2	86.6	86.3
DWB_3	86.5	88.7	90.3	88.7	91.0	92.5	87.5	85.8	89.6	84.2	86.5	89.0	86.0	87.7	90.7	90.3	89.9	91.2	89.8	90.7	92.5
DWB_5	84.9	87.7	89.8	87.7	90.7	92.6	86.9	84.8	89.3	81.9	84.2	87.8	84.0	86.5	90.2	89.1	89.3	91.0	88.5	89.8	92.3
DWB_{10}	81.1	84.9	87.9	84.9	89.2	91.8	85.7	83.5	88.3	77.5	80.9	84.9	80.1	83.4	87.9	86.7	87.9	89.8	86.2	88.1	91.3
DWB_r	87.4	88.3	90.7	89.0	91.1	92.7	87.9	85.8	89.7	86.1	86.3	88.9	86.9	87.8	91.1	90.6	90.1	91.2	90.0	90.7	92.6
Panel C: MA case, $\theta = 0.5$																					
t_{hac}	92.7	94.3	95.1	95.2	96.8	97.2	95.7	97.1	97.1	86.9	91.6	93.6	90.0	93.2	94.6	91.6	94.6	96.1	89.5	94.4	96.1
PM	99.4	99.5	99.7	99.4	99.5	99.6	99.7	99.7	99.6	94.6	96.5	96.7	100	100	100	96.0	96.1	93.0	92.8	93.5	88.1
GSW	94.1	95.0	94.6	97.0	98.5	99.2	98.7	99.3	99.5	86.2	84.1	82.8	90.4	90.3	91.4	89.3	88.6	88.2	87.2	87.8	87.2
DWB_3	91.2	93.1	94.4	92.9	95.1	96.0	90.5	89.5	93.7	88.8	91.2	92.8	90.5	92.6	94.6	93.3	93.5	94.7	93.2	93.9	95.7
DWB_5	88.7	90.9	92.3	91.4	93.7	94.9	89.1	87.6	92.3	85.8	88.1	90.3	87.8	90.2	92.6	91.9	92.1	93.1	91.9	92.6	94.3
DWB_{10}	84.5	87.2	89.2	88.4	91.2	93.2	87.3	85.4	90.1	81.5	84.2	86.7	83.8	86.3	89.6	89.6	90.1	91.4	89.6	90.7	92.8
DWB_r	93.0	93.1	94.4	94.1	95.1	96.1	91.4	89.8	93.7	91.0	91.2	92.8	92.5	92.6	94.5	94.4	93.8	94.7	94.1	94.0	95.7

is no longer valid.[11] The coverage rates of the GSW interval remain inadequate when volatility is time-varying although some improvements may be noted for DGP-3 and DGP-4 in the serially uncorrelated case. In contrast, the bootstrap-based intervals are much more stable with coverage rates bearing a very similar pattern to those in the no drift case. In summary, the coverage results in Tables 1 and 2 make a favorable case for employing the dependent wild bootstrap based on a small bandwidth or the recommended bandwidth rule compared to the asymptotic approaches.

Table 3 presents the average effective lengths of the confidence intervals, normalized with respect to the t_{hac} interval. Thus, a ratio smaller (larger) than one indicates an interval with average effective length shorter (longer) than the t_{hac} interval. The results reveal the following notable patterns. First, the PM interval can be discernibly longer than the other intervals in cases where the errors have constant conditional variance or involve discrete shifts in volatility. In contrast, it typically delivers the shortest average length in the trending and nondeterministic volatility cases (DGPs 3–6) when $T = 200$. Thus, as with coverage, the length of the PM interval can be quite sensitive to the underlying volatility specification. Second, the length of the GSW interval depends to a considerable extent on both the serial correlation and volatility structures driving the true DGP. For instance, this interval is the shortest on average relative to the other intervals for DGPs 3–6 when $T \leq 100$ but always longer than the interval based on t_{hac} for DGP-1 and DGP-2. In the constant volatility case, the GSW interval is longer than the HAC-based interval with serially uncorrelated errors but shorter than the same in the serially correlated scenarios. Third, for the bootstrap-based intervals, the average length is generally shorter, the larger the bandwidth employed. The average length based on the bandwidth rule DWB_r typically lies between the average lengths for the smallest and largest bandwidths considered. Fourth, the length improvements offered by the bandwidth rule over the asymptotic methods are primarily concentrated in situations where the errors are serially correlated and volatility is subject to discrete shifts. Fifth, the performance of the rule-based bandwidth is substantially more stable across the different volatility specifications relative to the asymptotic procedures, a feature also previously observed for the coverage rates. Table 4 reports the corresponding length results in the drift case. These results paint a qualitatively similar overall picture as the results in the no drift case.

In summary, the Monte Carlo results indicate that while employing a relatively smaller bandwidth leads to more accurate coverage, it also leads to longer average effective lengths. The recommended bandwidth rule offers a reasonable approach to addressing the coverage-length trade-off by delivering relatively short intervals while retaining adequate coverage properties. Additional Monte Carlo results

[11] When $\nu = \infty$, $\tilde{\rho}_T$ converges to ρ_T at rate $\mu_T k_T^{3/2} \rho_T^T$, which is faster than the rate $k_T \rho_T^T$ in Phillips and Magdalinos (2007a) and $\tilde{\rho}_T$ is asymptotically normal (see Fei, 2018; Liu and Peng, 2019). When $\nu \in (0, \infty)$, the limit distribution is mixed normal (see Guo et al., 2019).

TABLE 3. Effective interval length ratio (benchmark: t_{hac}) of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.5$, $\mu_T = 0$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: No serial correlation																					
PM	2.62	2.06	1.47	3.23	3.01	2.49	2.52	2.11	1.38	1.42	1.04	0.49	1.68	1.31	0.71	1.27	0.88	0.41	1.06	1.13	0.58
GSW	1.07	1.02	1.09	1.12	1.22	1.39	1.26	1.40	1.54	0.98	0.81	0.70	0.92	0.90	0.90	0.83	0.71	0.58	0.52	0.71	0.56
DWB_3	1.28	1.18	1.11	1.16	1.01	0.94	1.13	1.09	1.17	1.33	1.24	1.19	1.30	1.22	1.16	1.57	1.27	1.20	2.54	1.37	1.22
DWB_5	1.25	1.15	1.09	1.16	1.00	0.93	1.13	1.08	1.17	1.27	1.20	1.15	1.26	1.18	1.13	1.54	1.20	1.14	2.48	1.28	1.15
DWB_{10}	1.21	1.12	1.03	1.20	1.00	0.93	1.19	1.12	1.17	1.21	1.14	1.08	1.20	1.12	1.07	1.50	1.09	1.06	2.41	1.16	1.06
DWB_r	1.31	1.18	1.11	1.18	1.01	0.95	1.15	1.09	1.16	1.35	1.24	1.19	1.33	1.21	1.15	1.60	1.27	1.20	2.33	1.36	1.22
Panel B: AR case, $\phi = 0.5$																					
PM	1.90	1.62	1.22	2.67	2.32	1.92	2.02	1.53	1.03	0.47	0.73	0.45	1.15	0.89	0.56	0.85	0.70	0.29	1.33	0.98	0.48
GSW	0.99	0.92	0.97	1.09	1.05	1.07	1.43	1.40	1.35	0.44	0.80	0.78	0.80	0.74	0.77	0.62	0.68	0.50	0.70	0.64	0.54
DWB_3	1.24	1.02	1.01	0.97	0.83	0.77	0.92	0.85	0.90	1.23	1.09	1.04	1.23	1.05	1.03	1.37	1.15	1.05	1.66	1.42	1.11
DWB_5	1.25	1.00	1.00	0.96	0.81	0.78	0.91	0.82	0.88	1.21	1.05	1.03	1.20	1.02	1.02	1.12	1.10	1.02	1.54	1.49	1.08
DWB_{10}	1.33	0.97	0.96	0.96	0.79	0.75	0.93	0.83	0.89	1.15	0.98	0.98	1.10	0.95	0.97	0.94	1.02	0.97	1.39	1.35	1.02
DWB_r	1.20	1.01	1.01	0.99	0.83	0.77	0.93	0.86	0.90	1.25	1.09	1.04	1.22	1.05	1.02	1.34	1.14	1.03	1.68	1.48	1.11
Panel C: MA case, $\theta = 0.5$																					
PM	1.76	1.49	1.09	2.51	2.20	1.62	1.89	1.35	0.92	0.64	0.67	0.39	0.88	0.82	0.55	0.43	0.59	0.31	1.17	0.76	0.37
GSW	0.92	0.87	0.84	1.07	1.05	1.02	1.29	1.12	1.21	0.58	0.71	0.65	0.62	0.67	0.69	0.32	0.56	0.49	0.61	0.51	0.41
DWB_3	1.10	0.99	0.99	0.96	0.81	0.82	0.91	0.84	0.88	1.23	1.08	1.03	1.18	1.03	1.01	1.54	1.13	1.02	1.58	1.27	1.11
DWB_5	1.04	0.92	0.94	0.94	0.77	0.79	0.89	0.79	0.83	1.17	0.99	0.97	1.10	0.96	0.95	1.47	1.06	0.96	1.46	1.22	1.07
DWB_{10}	0.95	0.85	0.85	0.92	0.74	0.74	0.88	0.77	0.80	1.07	0.78	0.88	1.00	0.88	0.87	1.35	0.96	0.87	1.32	1.08	0.99
DWB_r	1.15	0.99	0.99	1.00	0.81	0.82	0.94	0.84	0.87	1.26	1.02	1.03	1.21	1.03	1.00	1.59	1.13	1.01	1.66	1.26	1.11

TABLE 4. Effective interval length ratio (benchmark: t_{hac}) of various inferential methods for $\rho = 1 + c/T^{\alpha}$, 95% nominal rate, $T = 50, 100, 200$, $c = 0.5$, $\alpha = 0.5$, $\mu_T = T^{-\alpha/4}$.

DGP	0			1			2			3			4			5			6		
T	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200	50	100	200
Panel A: No serial correlation																					
PM	11.5	16.0	20.1	18.3	27.2	38.5	6.55	10.7	14.9	2.16	2.42	3.16	11.0	15.2	19.1	1.88	1.59	1.23	1.33	1.31	0.71
GSW	1.11	1.06	1.03	1.31	1.33	1.49	1.58	1.58	1.62	0.96	0.81	0.78	0.96	0.93	0.93	0.80	0.74	0.67	0.72	0.82	0.72
DWB_3	1.14	1.08	1.06	1.04	1.02	1.01	0.99	0.95	1.01	1.39	1.19	1.11	1.22	1.12	1.08	1.68	1.33	1.14	2.20	1.65	1.20
DWB_5	1.07	1.03	1.02	0.99	0.99	0.99	0.98	0.92	0.98	1.38	1.14	1.07	1.13	1.06	1.03	1.62	1.26	1.11	2.06	1.52	1.14
DWB_{10}	0.95	0.95	0.96	0.92	0.95	0.95	1.00	0.87	0.94	1.37	1.05	0.98	1.00	0.96	0.96	1.51	1.18	1.04	1.94	1.45	1.10
DWB_r	1.18	1.08	1.06	1.07	1.02	1.01	1.01	0.95	1.01	1.38	1.19	1.11	1.27	1.12	1.08	1.71	1.32	1.15	2.14	1.56	1.20
Panel B: AR case, $\phi = 0.5$																					
PM	4.48	5.62	6.95	8.09	10.1	11.7	2.84	3.80	4.92	0.72	0.99	0.99	3.33	7.16	10.3	1.08	1.06	0.69	1.47	0.93	0.43
GSW	0.99	0.93	1.03	1.19	1.13	1.16	1.45	1.56	1.45	0.53	0.75	0.74	0.51	0.76	0.87	0.62	0.68	0.61	0.91	0.70	0.57
DWB_3	1.16	1.06	1.02	1.07	0.99	0.96	0.95	0.90	0.92	1.25	1.10	1.09	1.30	1.10	1.05	1.52	1.47	1.06	1.59	1.29	1.11
DWB_5	1.12	1.05	1.02	1.06	0.98	0.98	0.95	0.88	0.93	1.22	1.08	1.06	1.24	1.08	1.04	1.40	1.48	1.02	1.55	1.23	1.05
DWB_{10}	1.04	1.00	0.98	1.04	0.98	0.96	0.99	0.86	0.92	1.12	1.03	1.01	1.33	1.00	0.99	1.29	1.34	0.94	1.39	1.15	1.00
DWB_r	1.17	1.07	1.02	1.10	0.99	0.96	0.95	0.90	0.92	1.26	1.11	1.08	1.32	1.10	1.05	1.54	1.49	1.05	1.65	1.28	1.10
Panel C: MA case, $\theta = 0.5$																					
PM	5.33	7.71	10.3	9.33	13.0	20.0	3.27	5.05	7.13	1.12	1.18	1.39	4.99	8.66	11.6	0.48	0.39	0.73	0.93	0.77	0.42
GSW	0.89	0.84	0.87	1.13	1.04	1.40	1.35	1.38	1.40	0.75	0.67	0.67	0.63	0.75	0.78	0.25	0.22	0.55	0.58	0.54	0.50
DWB_3	1.14	1.05	1.00	1.03	0.98	0.96	0.92	0.88	0.91	1.29	1.12	1.04	1.21	1.08	1.02	1.67	1.18	1.12	1.66	1.31	1.07
DWB_5	1.06	0.97	0.95	0.96	0.92	0.92	0.90	0.83	0.84	1.24	1.05	0.98	1.11	1.02	0.96	1.55	1.11	1.02	1.62	1.18	0.99
DWB_{10}	0.96	0.88	0.87	0.89	0.84	0.88	0.90	0.78	0.80	1.12	0.93	0.89	0.96	0.90	0.87	1.40	1.01	0.97	1.52	1.09	0.90
DWB_r	1.16	1.05	1.00	1.07	0.99	0.96	0.96	0.88	0.89	1.31	1.11	1.05	1.26	1.08	1.02	1.68	1.17	1.08	1.69	1.26	1.06

presented in Appendix C of the Supplementary Material for the case $\alpha = 0.8$ further confirm the effectiveness of the proposed procedure for conducting inference within the MEA framework.

6. EMPIRICAL APPLICATIONS

This section illustrates the proposed methodology on two sets of time series. The first application revisits the empirical application in GSW where the degree of explosiveness of 10 major stock market indices in the pre-2008 financial exuberance period is studied. In particular, our analysis accounts for the potential nonstationary volatility pattern in the stock indices and highlights the difference between our results and those in GSW which are based on assuming stationary volatility. The second application investigates the extent of explosive behavior in three monthly U.S. home price indices during the 2002–2006 housing bubble, and the details are suppressed into Appendix D of the Supplementary Material.

6.1. Stock Market Indices

GSW employ a two-step testing strategy to identify the degree of explosiveness in 10 major stock indices over the period leading up to the 2008 financial crisis. The first step entails a pretest for detecting whether the time series is explosive, and the second step uses their proposed method to construct a confidence interval if the pretest signals the presence of explosive behavior. They find limited evidence of explosive behavior with most series either only mildly explosive, or not explosive at all. To facilitate comparison with their results, our analysis employs the same dataset as GSW.[12] The data are weekly, and the sample size is $T = 100$ for all series. Specifically, the data are collected in a way that end at the (pre-selected) highest point in the pre-2008 financial crisis period and then span 100 periods before that highest point. All of the series peaked at some point during 2007–2008, thereby making the whole sample approximately span from 2005 to 2008. A plot of the indices is displayed in Figure 1.

We start with an assessment of the time series behavior of volatility in these series to justify the plausibility of allowing for unconditional heteroskedasticity. Following Cavaliere and Taylor (2007a), Figure 2 plots the estimated variance profile, defined by $\widehat{\mathrm{VP}}(s) = \left(\sum_{t=1}^{\lfloor sT \rfloor} \hat{e}_t^2 + (sT - \lfloor sT \rfloor)\hat{e}_{\lfloor sT \rfloor + 1}^2\right) / \sum_{t=1}^{T} \hat{e}_t^2$, $0 \leq s \leq 1$, as well as the volatility estimates $\hat{\sigma}_t^2$, obtained by fitting a nonparametric regression to the squared residuals $\hat{e}_t^2$ as suggested by Xu and Phillips (2008)[13]. Specifically, to adjust for potential serial correlation in the errors, the residuals $\hat{e}_t$ are obtained by fitting an autoregressive model to the series with lags determined by BIC with

[12]The data come from Wind Economic Database and consist of 10 countries/districts, namely, USA, Brazil, China, Hong Kong, Australia, France, Germany, Italy, Egypt, and Nigeria, which are representatives of the world stock markets in different continents: America, Asia-Pacific, Europe, and Africa. See Guo et al. (2019) for further details.

[13]To make the estimated volatility curves comparable across the different time series, we plot the estimated volatility ratio over $t = 1, \ldots, T$: $\hat{\sigma}_t^2 / \bar{\hat{\sigma}}^2$, where $\bar{\hat{\sigma}}^2 = \sum_{t=1}^{T} \hat{\sigma}_t^2 / T$.

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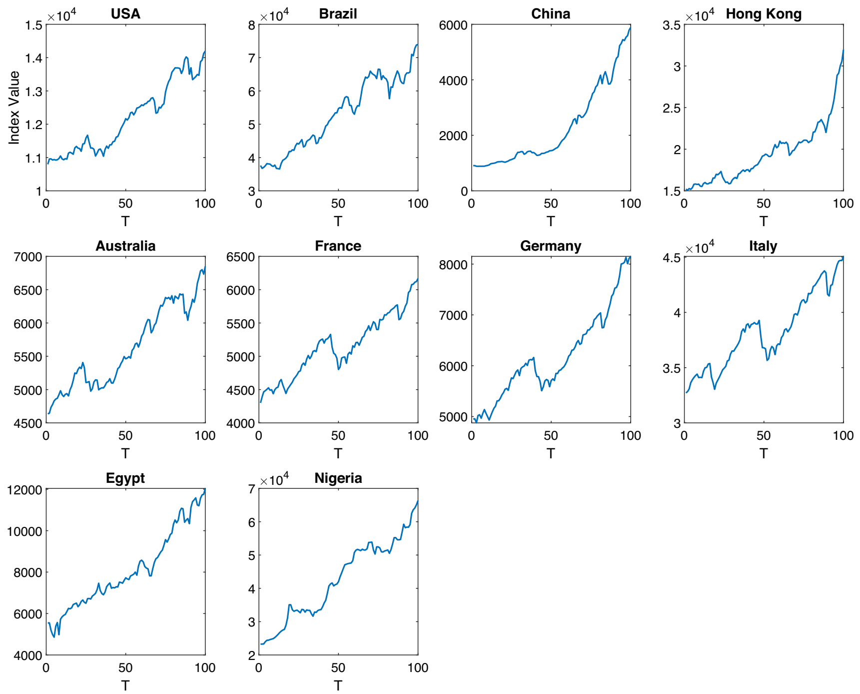

FIGURE 1. Plots of 10 stock indices during 2005–2008.

a maximum of six lags. For the nonparametric estimates, a Gaussian kernel is used with the bandwidth chosen by cross-validation, searching over bandwidths $h_i = c_i T^{-0.4}, i = 1, \ldots, 4$, with $\{c_1, \ldots, c_4\} = \{0.25, 0.4, 0.6, 0.75\}$. As observed from Figure 2, the estimated variance profile of several series, especially USA, Brazil, China, and Hong Kong, deviates substantially from the 45° line, which represents the constant volatility scenario. The corresponding nonparametric estimates of the volatility clearly depict the underlying nonstationary evolution of the sample volatility paths, indicating smooth trending changes for USA, Brazil, China, and Hong Kong, and possibly single/multiple shifts for the remaining countries.

In addition to visualizing the sample volatility paths, we also conduct formal diagnostic tests for the stationarity of unconditional volatility proposed by Cavaliere and Taylor (2007b). They present four test statistics, $\mathcal{H}_{KS}$, $\mathcal{H}_R$, $\mathcal{H}_{CVM}$, $\mathcal{H}_{AD}$, and derive their asymptotic distributions under the stationarity null from which the relevant critical values of the tests are obtained. In implementing these tests, the squared residuals $\hat{e}_t^2$ are used in constructing the stationary volatility test statistics. A long-run variance estimator based on the Bartlett kernel with lag truncation parameter 4 is employed.[14] Panel A of Table 5 presents the testing

[14]To conserve space, we omit the details pertaining to the construction of the test statistics and refer the interested reader to Cavaliere and Taylor (2007b).

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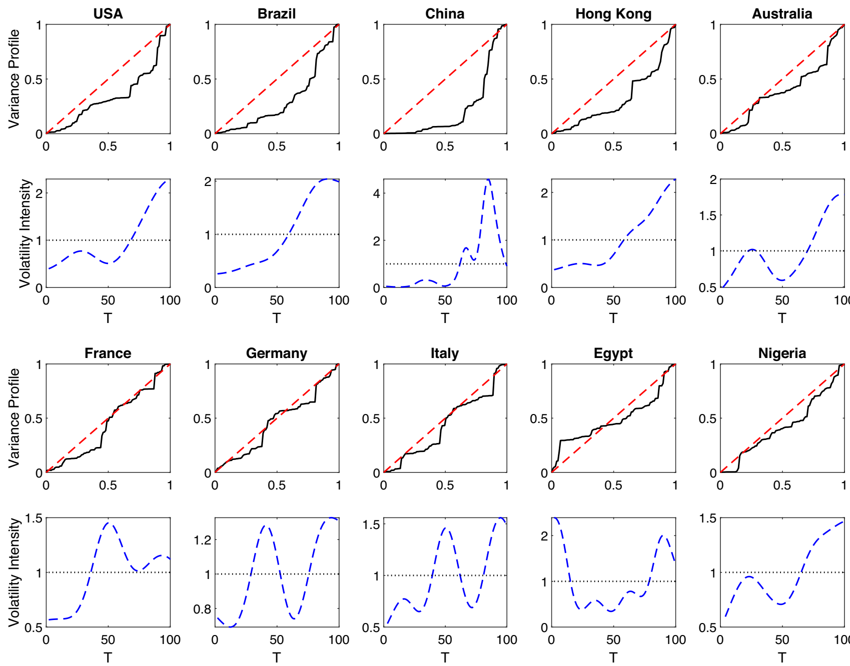

FIGURE 2. Variance profiles and nonparametric volatility estimates of 10 stock indices.

results along with the 10%, 5%, and 1% critical values. It is clear that the first four series, namely, USA, Brazil, China, and Hong Kong, show evidence of significant nonstationary volatility from a majority of the tests. As noted in the simulation evidence in Cavaliere and Taylor (2007b), when volatility exhibits trending behavior or a single abrupt break, $\mathcal{H}_{AD}$ and $\mathcal{H}_{CVM}$ usually have the highest finite-sample power out of the four tests, while the $\mathcal{H}_R$ test is the least powerful. In contrast, the $\mathcal{H}_R$ test is the most powerful in the presence of multiple discrete volatility breaks. Considering these facts together with the visual evidence presented in Figure 2, we believe a smooth trending variation of the volatility is more likely to prevail in these four series, as opposed to single/multiple discrete volatility break(s).

We now turn to the two-step testing strategy adopted by GSW. Their first step involves a pretest for explosiveness using the right-tailed augmented Dickey–Fuller (RADF) and the supremum augmented Dickey–Fuller (SADF) tests proposed by Phillips et al. (2011; PWY henceforth) and Phillips et al. (2015; PSY henceforth), both of which assume stationary volatility. In contrast, drawing upon recent developments in the literature, we employ two set of tests proposed in Harvey et al. (2019, 2020) which allow for nonstationary volatility. In the first set of tests, Harvey et al. (2019) modify the RADF statistic of PWY using a

Table 5. Empirical results table for 10 stock indices, consisting of three panels. Panel A: Nonstationary volatility tests (Cavaliere and Taylor, 2007b) and their critical values (C.V.). Panel B: Explosiveness estimates and p-values of bubble tests allowing for nonstationary volatility (Harvey, Leybourne, and Zu, 2019, 2020). Panel C: AR(1) estimates and 95% confidence intervals (presented for $(\rho_T - 1) \times 100$) of various methods.

	Panel A: Heteroskedasticity tests				Panel B: Explosiveness estimates and p-values						
Country	$\mathcal{H}_{KS}$	$\mathcal{H}_R$	$\mathcal{H}_{CVM}$	$\mathcal{H}_{AD}$	$\hat{\rho}_T$	$\mathcal{U}$	supBZ	supDF	uPSY	PSY	sPSY
USA	1.382	1.328*	0.541**	2.965**	1.003	0.100	0.072	0.293	0.000	0.607	0.000
Brazil	1.560	1.487**	0.870***	4.223***	1.001	0.070	0.034	0.210	0.006	0.555	0.006
China	1.717*	1.634***	0.993***	4.744***	1.025	0.022	0.002	0.022	0.000	0.016	0.000
Hong Kong	1.368	1.368**	0.728**	3.753**	1.051	0.004	0.006	0.004	0.000	0.038	0.000
Australia	1.024	0.993	0.280	1.582	1.000	0.062	0.052	0.493	0.086	0.497	0.080
France	1.264	1.055	0.174	0.890	0.998	0.060	0.054	0.505	0.000	0.766	0.000
Germany	0.968	0.747	0.098	0.523	1.012	0.004	0.002	0.054	0.000	0.190	0.000
Italy	0.879	0.716	0.108	0.660	0.998	0.112	0.106	0.461	0.000	0.673	0.000
Egypt	1.660*	0.914	0.190	1.387	1.007	0.012	0.008	0.044	0.010	0.337	0.010
Nigeria	1.070	0.941	0.205	1.190	1.004	0.024	0.030	0.014	0.000	0.020	0.000
C.V. (10%)	1.620	1.230	0.347	1.933							
C.V. (5%)	1.750	1.360	0.461	2.492							
C.V. (1%)	2.010	1.630	0.743	3.850							

Panel C: AR(1) estimates and 95% confidence intervals (presented for $(\rho_T - 1) \times 100$)

Country	$\hat{\rho}_T$	t_{hac}	PM	GSW	DWB_3	DWB_5	DWB_{10}	DWB_r
USA	1.003	[0.1, 3.1]	[0.1, 5.3]	[0.1, 2.8]	[0.1, 3.9]	[0.1, 3.8]	[0.1, 3.5]	[0.1, 3.7]
Brazil	1.001	[0.1, 2.4]	[0.1, 2.4]	[0.1, 2.6]	[0.1, 3.3]	[0.1, 3.3]	[0.1, 3.1]	[0.1, 3.4]
China	1.025	[0.8, 4.2]	[0.1, 7.9]	[1.1, 3.9]	[0.7, 4.5]	[0.8, 4.4]	[1.2, 4.3]	[1.0, 4.4]
Hong Kong	1.051	[2.1, 8.1]	[4.2, 6.0]	[2.6, 7.6]	[2.2, 8.8]	[2.2, 9.2]	[2.0, 9.3]	[2.0, 9.0]
Australia	1.000	–	–	–	–	–	–	–
France	0.998	–	–	–	–	–	–	–
Germany	1.012	[0.1, 3.1]	[0.1, 10.4]	[0.1, 3.3]	[0.1, 3.5]	[0.1, 3.6]	[0.1, 3.8]	[0.1, 3.5]
Italy	0.998	–	–	–	–	–	–	–
Egypt	1.007	[0.1, 3.4]	[0.1, 9.4]	[0.1, 3.0]	[0.1, 4.1]	[0.1, 4.0]	[0.1, 3.4]	[0.1, 3.9]
Nigeria	1.004	[0.1, 2.5]	[0.1, 6.8]	[0.1, 2.3]	[0.1, 2.7]	[0.1, 2.4]	[0.1, 2.3]	[0.1, 2.6]

Note: * denotes 10%, ** denotes 5%, and *** denotes 1% significance levels for the above tests.

weighted least squares-based variant (supBZ) which is borrowed from Boswijk and Zu (2018). To further increase power, they propose a union of rejections test ($\mathcal{U}$) that combines the original PWY test statistic (supDF) and supBZ. In the second set of tests, Harvey et al. (2020) suggest a sign-based version (sPSY) of the PSY test for multiple bubbles and for the same reason also advocate a union of

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rejections test (uPSY) which consists of sPSY and the original PSY test. Following these studies, we present the bootstrap p-values of these six tests in Panel B of Table 5. It is evident that both the union tests reject the unit-root null at the 10% significance level for all series except USA and Italy for the $\mathcal{U}$ test. As noted by Harvey et al. (2020), the first set of tests—$\mathcal{U}$, supBZ, and supDF tests, which build on the PWY testing approach—are usually less powerful than the second set of tests based on the PSY testing principle. Interestingly, we also observe a similar phenomenon here: for each case except Australia, uPSY test has a lower p-value than $\mathcal{U}$. Based on the overall pattern found in Panel B of Table 5 and the superior power performance of uPSY test revealed in Harvey et al. (2020), all of the 10 series are deemed to be explosive. However, to construct a meaningful confidence interval in the second step, we conclude a series to be explosive if and only if both of the following two conditions are satisfied—the pretest must reject the unit-root null and the point estimate of the degree of explosiveness must exceed unity, i.e., $\hat{\rho}_T > 1$. Taking into account these conditions, we exclude Australia, France, and Italy from the second step analysis since their estimates $\hat{\rho}_T \leq 1$[15].

The second step entails constructing HAC-based confidence intervals for the parameter ρ_T that governs the degree of explosiveness. To this end, we follow the approach in GSW of testing over a certain grid of values, $H_0 : \rho \in \{1.001, 1.002, \ldots, 1.500\}$. In practice, this corresponds to constructing a $100(1-\delta)\%$ confidence interval $[\hat{\rho}_L, \hat{\rho}_U]$ such that

$$\hat{\rho}_L = \max\{1.001, \hat{\rho}_T - D_{1-\delta/2} \times \hat{\sigma}(\hat{\rho}_T)\}, \quad \hat{\rho}_U = \hat{\rho}_T - D_{\delta/2} \times \hat{\sigma}(\hat{\rho}_T), \tag{27}$$

where $D_{\delta/2}, D_{1-\delta/2}$ represents the $\delta/2$ and $1-\delta/2$ percentiles of the approximating distribution D, and $\hat{\sigma}(\hat{\rho}_T)$ is an estimate of the standard deviation of $\hat{\rho}_T$. In our case, D is either the standard normal distribution or the DWB distribution and $\hat{\sigma}(\hat{\rho}_T)$ is the HAC estimate as defined in (9). Panel C of Table 5 presents the confidence intervals constructed by the HAC-based approach as well as the other methods compared in the simulations. Overall, our proposed DWB approach is supportive of the hypothesis in GSW that most series are mildly explosive ($\rho_T \in [1.004, 1.04]$). However, the DWB-based confidence intervals are in general a bit wider than those of GSW with a larger upper bound for the former, which is consistent with the preceding simulation evidence that GSW usually under-covers in most of the nonstationary volatility cases considered. Hence, the GSW interval tends to understate both the sampling uncertainty associated with the point estimate of ρ_T as well as the degree of explosiveness driving the time series. Moreover, for the four series which showed considerable evidence of time-varying volatility in the foregoing analysis, the difference between the GSW and DWB$_r$ methods is more prominent than for countries such as Germany and Nigeria which do not show

[15] It is worth noting that in GSW, the time series for Australia, France, and Italy are also categorized as nonexplosive but for a different reason—due to failing to reject their bubble detection tests that do not allow for nonstationary variance. In contrast, after adjusting for potential nonstationarity in the variance, the three series show significant evidence of explosive behavior, although the p-values for Australia (0.086) and Italy (0.112) are the largest in the uPSY and $\mathcal{U}$ tests, respectively.

significant time variation in volatility. This pattern suggests that when volatility is not time-varying, our DWB-based t_{hac} method suffers little efficiency loss, further highlighting the advantages of using our proposed procedure relative to those that do not control for unconditional heteroskedasticity. Finally, the PM intervals are in general too wide and thus not particularly informative, again consistent with the simulation evidence that PM intervals tend to over-cover under nonstationary volatility.

In summary, we find that four out of the seven explosive stock market series analyzed in GSW show strong evidence of time-varying volatility. For these explosive series, our DWB-based method produces wider confidence intervals than GSW, while the two methods provide very similar intervals for series that do not exhibit time variation in volatility as indicated by the tests for stationary volatility. These patterns are consistent with the simulation results in Section 5 and confirm the effectiveness of our proposed method in constructing confidence intervals for the degree of explosiveness in time series analysis.

6.2. U.S. Housing Price Indices

The details of this application can be found in Appendix D of the Supplementary Material.

7. CONCLUSION

The recent upsurge of interest in the MEA framework has been spurred by its ability to provide a simple yet effective tool for modeling the presence of asset market bubbles. The development of this framework has been followed by a plethora of theoretical and empirical studies that have sought to generalize the original framework or apply it to study the time series evolution of several price indices that may potentially be subject to explosive behavior. This paper considers the problem of constructing asymptotically justified confidence intervals for the autoregressive parameter that represents the degree of explosiveness. Existing approaches typically employed in empirical practice are valid only under the assumption of conditional homoskedasticity/heteroskedasticity, notwithstanding extensive empirical evidence against the same for a wide range of important economic and financial time series. Our framework allows the noise component to be unconditionally heteroskedastic and sufficiently general to subsume a variety of volatility specifications common in the literature. We propose a dependent wild bootstrap-t procedure for inference that is shown to provide an improved approximation to the finite-sample distribution of the t-statistic relative to asymptotic methods. Given that the t-statistic is asymptotically pivotal, it is possible that the bootstrap offers asymptotic refinements (Hall, 1992). A theoretical investigation of this possibility is outside the scope of the present paper but a potentially fruitful avenue for future research.

SUPPLEMENTARY MATERIAL

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