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TEMPERED DISTRIBUTIONS SUPPORTED ON A HALF-SPACE OF $\mathbf{R}^N$ AND THEIR FOURIER TRANSFORMS

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Introduction. A fundamental problem in Fourier analysis is to characterize the behaviour of a function (or distribution) whose Fourier transform vanishes in some particular set. Of course, this is, in general, a very difficult question and little seems to be known, except in some special cases. For example, a theorem of Paley-Wiener (Theorem XII in [6]) characterizes exactly the behaviour of the modulus of a function in $L^2(\mathbf{R})$ whose Fourier transform vanishes on a half-line. A similar result is also available on the circle group for L^2-functions whose Fourier coefficients corresponding to negative frequencies are all zero. (see [5], p. 53). Of course, in both cases, the connection with complex analysis is obvious, and that is, in fact, how those results were originally proved. A theorem of Szegö in prediction theory (see [2], [12]), related to the previous ones, is the following: Given a positive integrable function w on the circle group identified with the interval $[0, 2\pi)$, consider the Hilbert space $L^2_w = L^2(w)$ and the closed subspace M formed by the span of the set $\{e^{ik\theta}, k > 0\}$ in L^2_w, then

$$\inf_{P\in M} \|1 + P\|^2_{L^2_w} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\, w(\theta)\, d\theta\right).$$

Again, this theorem was originally proved using methods of complex analysis, which make it difficult to extend to more general situations. In a 1958 paper in the Acta Mathematica ([4]), Helson and Lowdenslager found a method based on Hilbert spaces theory which enable them, among other things, to extend Szegö's theorem to the n-dimensional torus T^n or even any compact abelian group whose dual is ordered (see also [3], [5] and [7]).

Our objective, in this paper, is to prove continuous versions on $\mathbf{R}^n$ of Szegö's theorem (but only when the measure considered is absolutely continuous) and of the Paley-Wiener theorem. We will consider functions, and more generally distributions on $\mathbf{R}^n$ supported on a half-space of $\mathbf{R}^n$, which is a set whose boundary is a $(n-1)$ dimensional hyperplace of $\mathbf{R}^n$. An example of such hyperplane is the set of points in $\mathbf{R}^n$ whose first component is greater then or equal to zero. For simplicity, we will state most of the results in the case of that particular half-space and for $n > 1$. It will become quite clear to the reader how to formulate the theorems in the case of a general half-space or when $n = 1$. As in the

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Helson-Lowdenslager paper, methods of Hilbert spaces theory are an essential ingredient of our proofs and the role played by complex analysis is reduced to a minimum. An important difference, however, is the use of techniques from the theory of distributions ([9]). In particular, convolution equations "restricted" to a half-space play a crucial role in the following exposition.

In the first section of this paper, we introduce some Hilbert spaces of tempered distributions on $\mathbf{R}^n$, whose Fourier transforms belong to a "weighted" L^2-space on $\hat{\mathbf{R}}^n$, for a suitable weight. The existence and uniqueness of the solution of some "formal" convolution equations restricted to a half-space are obtained in those spaces. Some density results are established in Section 2 and are used in Section 3 to find the explicit form of the solution of a particular restricted convolution equation which turns out to be the key ingredient for generalizing both Szegö's theorem and the Paley-Wiener theorem on $\mathbf{R}^n$ (see Theorem 3.5). The continuous version of Szegö's theorem on $\mathbf{R}^n$ is established in section 4 (Theorem 4.3). The question of non-triviality of some spaces of tempered distributions is studied in section 5. In particular, a necessary and sufficient condition on a weight w is given for the existence of a non-trivial distribution in $\mathcal{S}'(\mathbf{R}^n)$ supported in a half-space and whose Fourier transform belongs to the weighted L^2-space corresponding to w (Theorem 5.3). The $\mathbf{R}^n$ version of the Paley-Wiener theorem is established in section 6 (Theorem 6.1) and an application to a problem of "uniqueness" is given (Corollary 6.2).

0. **Notation.** We will denote by $\mathbf{R}^n$ the n-dimensional euclidean space and by $\hat{\mathbf{R}}^n$ its dual group. If $1 \leq p < \infty$, $L^p(\mathbf{R}^n)$ is the usual Lebesgue space of complex-valued measurable functions f on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} |f(x)|^p \, dx < \infty,$$

and $L^\infty(\mathbf{R}^n)$ is the space of complex-valued, essentially bounded, measurable functions on $\mathbf{R}^n$. $L^1_{\text{loc}}(\mathbf{R}^n)$ consists of those measurable functions on $\mathbf{R}^n$ whose restrictions to every compact set of $\mathbf{R}^n$ are integrable. The following spaces from the theory of distributions will be used here (see [9], for the precise definitions). $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space of (complex-valued) infinitely differentiable rapidly decreasing functions on $\mathbf{R}^n$ and $\mathcal{S}'(\mathbf{R}^n)$, the dual of $\mathcal{S}(\mathbf{R}^n)$, is the space of tempered distributions on $\mathbf{R}^n$. If W is an open set of $\mathbf{R}^n$, $C_o^\infty(W)$ denotes the space of infinitely differentiable functions on $\mathbf{R}^n$ with compact support contained in W. $\mathcal{E}'(\mathbf{R}^n)$ is the space of distributions on $\mathbf{R}^n$ having compact support. We will denote by δ_a the Dirac mass concentrated at the point a of $\mathbf{R}^n$, when $a \neq 0$; we will simply use the notation δ for the Dirac mass at the origin. If $\varphi \in \mathcal{S}(\mathbf{R})$ and $\psi \in \mathcal{S}(\mathbf{R}^{n-1})$, their tensor product is the function $\varphi \otimes \psi \in \mathcal{S}(\mathbf{R}^n)$ defined by $(\varphi \otimes \psi)(x) = \varphi(t)\psi(y)$, for $x = (t, y) \in \mathbf{R}^n$. The bracket $< \cdot, \cdot >$ will always represent the duality between distributions and test functions. We define the Fourier transform of $\varphi \in \mathcal{S}(\mathbf{R}^n)$ by

$$\hat{\varphi}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i(\xi, x)} \varphi(x) \, dx, \forall \xi \in \hat{\mathbf{R}}^n.$$

We will use the same notation for the Fourier transform in euclidean spaces of different dimension. This won't create any confusion since the underlying euclidean space will always be specified. If A is a set, will denote by χ_A the characteristic function of A. We will also use the abbreviation "*a. e.*" for "almost everywhere". Finally, if $T \in \mathcal{S}'(\mathbf{R}^n)$, the distribution $\tilde{T} \in \mathcal{S}'(\mathbf{R}^n)$ is defined by

$$\langle \tilde{T}(x), \varphi(x) \rangle = \langle \overline{T(x), \overline{\varphi}(-x)} \rangle, \forall \varphi \in \mathcal{S}(\mathbf{R}^n).$$

1. **Some Hilbert spaces of tempered distributions and related convolution equations.** In this section, we will consider a tempered distribution Q on $\mathbf{R}^n$ whose Fourier transform, $\hat{Q}$, is a positive measurable function having the properties that, for some positive integer m,

$$\hat{Q}(\xi)(1+|\xi|^2)^{-m} \in L^\infty(\hat{\mathbf{R}}^n), \tag{1.1}$$

and

$$[\hat{Q}(\xi)]^{-1}(1+|\xi|^2)^{-m} \in L^1(\hat{\mathbf{R}}^n). \tag{1.2}$$

We can thus think of $\hat{Q}$ as a weight on $\hat{\mathbf{R}}^n$ and to each such weight we associate the following Hilbert space, denoted by H:

$$H = \left\{ u \in \mathcal{S}'(\mathbf{R}^n),\ \hat{u} \in L^1_{\mathrm{loc}}(\hat{\mathbf{R}}^n),\ \int_{\hat{\mathbf{R}}^n} |\hat{u}(\xi)|^2 \hat{Q}(\xi)\, d\xi < \infty \right\}, \tag{1.3}$$

with inner product given by

$$(u, v)_H = \int_{\hat{\mathbf{R}}^n} \hat{u}(\xi)\, \overline{\hat{v}(\xi)}\, \hat{Q}(\xi)\, d\xi, \forall u, v \in H.$$

It is easy to check that all the Hilbert space properties are satisfied by H. In particular, the completeness of H is a consequence of the fact that H is continuously imbedded in $\mathcal{S}'(\mathbf{R}^n)$. Indeed, if $u \in H$, we have, using the Cauchy-Schwarz inequality and (1.2),

$$\int_{\hat{\mathbf{R}}^n} |\hat{u}(\xi)|(1+|\xi|^2)^{-m/2}\, d\xi \leq \left[\int_{\hat{\mathbf{R}}^n} \frac{1}{\hat{Q}(\xi)} \frac{d\xi}{(1+|\xi|^2)^n} \right]^{1/2} \|u\|_H \tag{1.4}$$

LEMMA 1.1. *$C_o^\infty(\mathbf{R}^n)$ is a dense subspace of H.*

PROOF. This result follows easily from the Riesz representation theorem.

By Plancherel's formula, the inner product in H between two functions φ and ψ in $\mathcal{S}(\mathbf{R}^n)$ has the form: $(\varphi, \psi)_H = \langle Q * \varphi, \bar{\psi} \rangle$. This leads us to define the following formal convolutions.

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DEFINITION 1.2. If $u, v \in H$ and $R \in \mathcal{E}'(\mathbf{R}^n)$, we define the convolutions $Q * u$, $Q * u * v$, or more generally, $R * Q * u$, $R * Q * u * v$, by the formulas:

$$[Q * u]^\wedge = \hat{Q}\hat{u}, [Q * u * v]^\wedge = \hat{Q}\hat{u}\hat{v}$$
$$[R * Q * u]^\wedge = \hat{R}\hat{Q}\hat{u}, [R * Q * u * v]^\wedge = \hat{R}\hat{Q}\hat{u}\hat{v}.$$

Since, as it can be easily checked, all the products appearing on the right side of these equations define tempered functions, these formal convolutions are all well defined. It is also clear that such convolution products are automatically commutative and associative. Of course, because of the density of $C_o^\infty(\mathbf{R}^n)$ in H, they can be seen as limits in $\mathcal{S}'(\mathbf{R}^n)$ of "ordinary" convolutions, for which the "exchange" formula holds.

In the following, we will always identify $\mathbf{R}^n$ with $\mathbf{R} \times \mathbf{R}^{n-1}$ and $\hat{\mathbf{R}}^n$ with $\hat{\mathbf{R}} \times \hat{\mathbf{R}}^{n-1}$. If x denotes a point in $\mathbf{R}^n$, we will write $x = (t, y)$, where $t \in \mathbf{R}$ and $y \in \mathbf{R}^{n-1}$. Similarly, if $\xi \in \hat{\mathbf{R}}^n$, $\xi = (\gamma, \eta)$, where $\gamma \in \hat{\mathbf{R}}$ and $\eta \in \hat{\mathbf{R}}^{n-1}$. We will also denote by U the half-space

$$U = \{x = (t, y) \in \mathbf{R}^n, t > 0\} . \tag{1.5}$$

The following subspace of H will play an important role.

DEFINITION 1.3. H^+ is, by definition, the closure in H of the subspace $C_o^\infty(U)$.

Of course, H^+ is itself a Hilbert space with the inner product induced by H.

In the following lemma and the discussion that follows, we will consider a fixed function $\chi_o \in \mathcal{S}(\mathbf{R})$ having the property that

$$\chi_o(t) = e^{-t}, \forall t \geq 0 \tag{1.6}$$

LEMMA 1.4. *Given $\chi_o \in \mathcal{S}(\mathbf{R})$ satisfying (1.6) and $\psi_o \in \mathcal{S}(\mathbf{R}^{n-1})$, there exists a unique element $u_o \in H^+$ such that*

$$Q * u_o = \chi_o \otimes \psi_o \text{ on } U \tag{1.7}$$

(in the sense of distributions). Furthermore, we have, for some $G \in L^1(\hat{\mathbf{R}}^{n-1})$, that

$$\hat{Q}(\xi)|\hat{u}_o(\xi)|^2 = \frac{1}{1 + (2\pi\gamma)^2} \otimes G(\eta), \forall \text{ a.e. } \xi = (\gamma, \eta) \in \hat{\mathbf{R}}^n, \tag{1.8}$$

REMARK 1.5. Of course, when $n = 1$, ψ_o and G are simply constants.

PROOF. We define an anti-linear form L on H^+ by $L(v) =< \bar{v}, \chi_o \otimes \psi_o >$, for all $v \in H^+$, where the bracket $< \cdot, \cdot >$ denotes the duality between distributions in $\mathcal{S}'(\mathbf{R}^n)$

and functions in $\mathcal{S}(\mathbf{R}^n)$. Because of the following inequalities, holding for an arbitrary element v in H,

$$
\begin{aligned}
|L(v)| &= |\langle \bar{v}, \chi_o \otimes \psi_o \rangle| = \Big| \int_{\hat{\mathbf{R}}^n} (\chi_o \otimes \psi_o)^\wedge(\xi)\, \overline{\hat{v}(\xi)}\, d\xi \Big| \\
&\leq \Big[\int_{\hat{\mathbf{R}}^n} |\hat{v}(\xi)|^2 \hat{Q}(\xi)\, d\xi \Big]^{1/2} \Big[\int_{\hat{\mathbf{R}}^n} \frac{|(\chi_o \otimes \psi_o)^\wedge(\xi)|^2}{\hat{Q}(\xi)}\, d\xi \Big]^{1/2} \\
&\leq C_o \|v\|_H \Big[\int_{\hat{\mathbf{R}}^n} \frac{1}{\hat{Q}(\xi)(1+|\xi|^2)^m}\, d\xi \Big]^{1/2} \leq C\|v\|_H,
\end{aligned}
$$

it follows that L is a continuous anti-linear form on H^+. Therefore, by the Riesz representation theorem, there exists a unique element $u_o \in H^+$ satisfying for every $\varphi \in C_o^\infty(U)$, $(u_o, \varphi)_H = \langle Q * u_o, \bar{\varphi} \rangle = \langle \chi_o \otimes \psi_o, \bar{\varphi} \rangle$, which yields (1.7). To prove (1.8), we define $v_o \in \mathcal{S}'(\mathbf{R}^n)$ by $v_o = R * u_o$, where $R = \frac{\partial \delta}{\partial t} + \delta$. We have, thus, using the mentioned properties of our convolution product, that

$$
\begin{aligned}
Q * v_o &= Q * R * u_o = R * Q * u_o \\
&= R * (\chi_o \otimes \psi_o) \text{ on } U \quad \text{(by 1.7)} \\
&= 0 \text{ on } U \text{ (by 1.6)}.
\end{aligned}
$$

This shows that the support of $Q * v_o$ is contained in $\mathbf{R}^n \setminus U$ and, since the support of $\tilde{v}_o$ is contained in that same set, it follows that the support of $Q * v_o * \tilde{v}_o$ is also contained in $\mathbf{R}^n \setminus U$. (We used the fact that $Q * v_o * \tilde{v}_o = \lim_{k \to \infty} (Q * v_o) * \big(\frac{\partial \varphi_k}{\partial t} + \varphi_k \big)^\sim$ in $\mathcal{S}'(\mathbf{R}^n)$, if $\{\varphi_k\}_{k \geq 1}$ is any sequence in $C_o^\infty(U)$ converging to u_o in H^+.) Since $[Q * v_o * \tilde{v}_o]^\sim = Q * v_o * \tilde{v}_o$, it follows that $\operatorname{supp} Q * v_o * \tilde{v}_o \subset \{x = (t, y) \in \mathbf{R}^n, t = 0\}$. Using the fact that $Q * v_o * \tilde{v}_o \in \mathcal{S}'(\mathbf{R}^n)$, it is easily seen, using this last inclusion, that

$$
Q * v_o * \tilde{v}_o = \sum_{0 \leq k \leq K} D^k \delta(t) \otimes S_k(y), \tag{1.9}
$$

where $S_k \in \mathcal{S}'(\mathbf{R}^{n-1})$, for all $k \in \{0, \ldots, K\}$.

By taking the Fourier transform of both sides of (1.9), we obtain thus that

$$
\hat{Q}(\xi)|\hat{v}_o(\xi)|^2 = \sum_{0 \leq k \leq K} (2\pi i \gamma)^k \otimes \hat{S}_k(\eta). \tag{1.10}
$$

Since $\hat{v}_o(\gamma) = (1 + 2\pi i \gamma)\, \hat{u}_o(\gamma)$, we have also that

$$
\int_{\hat{\mathbf{R}}^n} \frac{|\hat{v}_o(\xi)|^2 \hat{Q}(\xi)}{1 + (2\pi\gamma)^2} d\xi = \int_{\hat{\mathbf{R}}^n} |\hat{u}_o(\xi)|^2 \hat{Q}(\xi)\, d\xi < \infty.
$$

By looking at the expression for $|\hat{v}_o|^2 \hat{Q}$ in (1.10), we see that $K = 0$ and that $\hat{S}_0 \in L^1(\hat{\mathbf{R}}^{n-1})$, and we obtain thus (1.8) by letting $G = \hat{S}_o$.

The main theorem of section 3 (THEOREM 3.5) gives an explicit expression for the function G in (1.8) in terms of the weight $\hat{Q}$. The following lemmas will be useful for reducing that problem on $\mathbf{R}^n$ to a one dimensional one.

LEMMA 1.6. *Let* $u \in \mathcal{S}'(\mathbf{R}^n)$ *have the property that its Fourier transform,* $\hat{u}$, *is a tempered function, that is, for some* $m \in \mathbf{N}$, $|\hat{u}(\xi)|(1+|\xi|^2)^{-m} \in L^1(\hat{\mathbf{R}}^n)$. *Define, for* a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, *the distribution* $u_\eta \in \mathcal{S}'(\mathbf{R})$ *by* $\hat{u}_\eta(\gamma) = \hat{u}(\gamma, \eta)$, *for* $\gamma \in \hat{\mathbf{R}}$.

Then, the following are equivalent.

(a) $\operatorname{supp} u \subset \{ x = (t, y) \in \mathbf{R}^n, t \geq 0 \}$.

(b) $\forall$ a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, $\operatorname{supp} u_\eta \subset [0, \infty)$.

PROOF. Let us assume that (a) holds. If $\varphi \in C_o^\infty((-\infty, 0))$ and $\psi \in C_o^\infty(\mathbf{R}^{n-1})$, then $\operatorname{supp} \varphi \otimes \psi \subset \mathbf{R}^n \setminus \bar{U}$ and, therefore, we have, by Plancherel and Fubini's theorem, that

$$\int_{\hat{\mathbf{R}}^{n-1}} \left[\int_{\hat{\mathbf{R}}} \hat{u}(\gamma, \eta)\, \overline{\hat{\varphi}(\gamma)}\, d\gamma \right] \overline{\hat{\psi}(\eta)}\, d\eta = 0$$

Since this equality holds for arbitrary $\psi \in C_o^\infty(\mathbf{R}^{n-1})$, we conclude that there exists a set $A = A(\varphi)$ of $(n-1)$ dimensional Lebesgue measure zero, such that

$$\forall \eta \notin A, \int_{\hat{\mathbf{R}}} \hat{u}_\eta(\gamma)\, \overline{\hat{\varphi}(\gamma)}\, d\gamma = 0$$

Because of the existence of a countable set $\{\varphi_k\}_{k \in \mathbf{Z}}$ of function in $C_o^\infty((-\infty, 0))$ which is dense in $C_o^\infty((-\infty, 0))$, the last equality holds except for a set A of $(n-1)$ dimensional Lebesgue measure zero, but which does not depend on φ, yielding (b). Conversely, if (b) holds, then, going back to the computation in the first part of the proof and applying Fubini's theorem, we obtain that

$$\forall \varphi \in C_o^\infty((0, \infty)), \forall \psi \in C_o^\infty(\mathbf{R}^{n-1}), \langle u, \varphi \otimes \psi \rangle = 0,$$

and (a) follows since finite sums of the form $\sum \varphi_k \otimes \psi_k$ where $\varphi_k \in C_o^\infty((-\infty, 0))$ and $\psi_k \in C_o^\infty(\mathbf{R}^{n-1})$ are dense in $C_o^\infty(\mathbf{R}^n \setminus \bar{U})$.

We now return to the problem of finding an explicit expression for G in (1.8). It is clear from (1.8) that $G \geq 0$ a.e. on $\hat{\mathbf{R}}^{n-1}$. The next lemma reduces the problem to a one-dimensional one and shows also that $G = |\hat{\psi}_o|^2 G_o$ where $G_o > 0$ a.e. on $\hat{\mathbf{R}}^{n-1}$. Before stating the lemma, we need to introduce the following family of weights on $\hat{\mathbf{R}}$, depending upon the parameter $\eta \in \hat{\mathbf{R}}^{n-1}$ and defined by $\hat{Q}_\eta(\eta) = \hat{Q}(\gamma, \eta)$, for $\gamma \in \hat{\mathbf{R}}$. It is clear that, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, the weight $\hat{Q}_\eta$ satisfies the conditions (1.1) and (1.2). Thus, for each of those η we can define the Hilbert spaces H_η and H_η^+ corresponding to the weight $\hat{Q}_\eta$. Hence,

$$H_\eta = \left\{ u \in \mathcal{S}'(\mathbf{R}), \hat{u} \in L^1_{\text{loc}}(\hat{\mathbf{R}}), \int_{\hat{\mathbf{R}}} |\hat{u}(\gamma)|^2 \hat{Q}_\eta(\gamma)\, d\gamma < \infty \right\}$$

with the obvious inner product and, of course, H_η^+ is the closure of $C_o^\infty((0, \infty))$ in H_η.

LEMMA 1.7. *Let u_o be the unique solution in H^+ of the equation $Q * u_o = \chi_o \otimes \psi_o$ on U given by Lemma 1.4 and let u_η be the unique solution in H_η^+ of the equation $Q_\eta * u_\eta = \chi_o$ on* $(0, \infty)$, *defined for* a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$. *Then, for* a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, *we have*

$$\hat{u}_o(\gamma, \eta) = \hat{u}_\eta(\gamma)\hat{\psi}_o(\eta), \ \forall \gamma \in \hat{\mathbf{R}}, \tag{1.11}$$

and, furthermore, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$,

$$G(\eta) = 2\|u_\eta\|_{H_\eta}^2 |\hat{\psi}_o(\eta)|^2 \tag{1.12}$$

PROOF. To prove (1.11), it is sufficient to show that if $v_\eta \in \mathcal{S}'(\mathbf{R})$ is defined by $\hat{v}_\eta(\gamma) = \hat{u}_o(\gamma, \eta)$, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, then $v_\eta \in H_\eta^+$ and satisfies $Q_\eta * v_\eta = \hat{\psi}_o(\eta)\chi_o$ on $(0, \infty)$, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$. Since $u_o \in H^+$, it is easily seen using Fubini's theorem, that $v_\eta \in H_\eta^+$, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$. Now, using the definition of u_o, it follows from Lemma 1.6, that, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, and every $\varphi \in C_o^\infty((0, \infty))$,

$$\int_{\hat{\mathbf{R}}} \hat{v}_\eta(\gamma)\overline{\hat{\varphi}(\gamma)}\hat{Q}_\eta(\gamma)\, d\gamma = \int_{\hat{\mathbf{R}}} \left(\hat{\psi}_o(\eta)\right) \hat{\chi}_o(\gamma)\overline{\hat{\varphi}(\gamma)}\, d\gamma.$$

This says that v_η satisfies $Q_\eta * v_\eta = \left(\hat{\psi}_o(\eta)\right)\chi_o$ on $(0, \infty)$, and, thus, (1.11) holds. Now, by Lemma 1.4, we have, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, that

$$\hat{Q}_\eta(\gamma)|\hat{u}_\eta(\gamma)|^2 = \frac{1}{1 + (2\pi\gamma)^2} G_\eta, \tag{1.13}$$

where, for a fixed η, G_η is a positive constant. By integrating on $\hat{\mathbf{R}}$ both sides of (1.13), we obtain that $G_\eta = 2\|u_\eta\|_{H_\eta}^2$ and (1.12) follows immediately from the previous equality, (1.8) and (1.11).

2. **Density Properties.** In order to give an explicit expression for the function G appearing in (1.8), some density properties in various spaces will need to be established. If $u_o \in H^+$ is the unique solution of the equation (1.7) in Lemma 1.4, we define $v_o \in \mathcal{S}'(\mathbf{R}^n)$, by

$$v_o = \left(\frac{\partial \delta}{\partial t} + \delta\right) * u_o. \tag{2.1}$$

The following facts follow immediately from Lemma 1.4 and Lemma 1.8: supp $v_o \subset \bar{U}$, supp $Q * v_o \subset \mathbf{R}^n \setminus U$, and $\hat{Q}(\xi)|\hat{v}_o(\xi)|^2 = G_o(\eta)|\hat{\psi}_o(\eta)|^2$, where $G_o > 0$ a.e. on $\hat{\mathbf{R}}^{n-1}$. We now consider the following closed subspace M of $L^2(\mathbf{R}^n)$ defined as $M = \{f \in L^2(\mathbf{R}^n), \operatorname{supp} f \subset \bar{U}\}$. We define also the distribution $K \in \mathcal{S}'(\mathbf{R}^n)$ by

$$\hat{K}(\xi) = \hat{Q}(\xi)\overline{\hat{v}_o(\xi)}, \ \forall \xi \in \hat{\mathbf{R}}^n. \tag{2.2}$$

The right side of (2.2) defines a tempered function, since by (1.1) and (2.1), we have

$$\int_{\hat{\mathbf{R}}^n} \frac{|\hat{K}(\xi)|^2}{(1+|\xi|^2)^{m+1}} d\xi = \int_{\hat{\mathbf{R}}^n} \frac{\hat{Q}(\xi)^2 |\hat{u}_o(\xi)|^2 (1+(2\pi\gamma)^2) d\xi}{(1+|\xi|^2)^{m+1}}$$
$$\leq C \int_{\hat{\mathbf{R}}^n} \hat{Q}(\xi) |\hat{u}_o(\xi)|^2 d\xi < \infty. \tag{2.3}$$

Since $K = Q * \tilde{v}_o$, it is clear that supp $K \subset \bar{U}$ and, if $\varphi \in C_o^\infty(U)$, it follows from (2.3) that $\varphi * K \in L^2(\mathbf{R}^n)$ and thus $\varphi * K \in M$. The following theorem establishes the density of functions of the form $\varphi * K$, where $\varphi \in C_o^\infty(U)$, in M, when $|\hat{\psi}_o| > 0$.

It is interesting to notice the connection between this result, Beurling's characterization of outer functions in [1] and its generalization by Helson and Lowdenslager (theorem 6 in [4]; see also [3], [5], and [7]).

THEOREM 2.1. *Let $\psi_o \in \mathcal{S}(\mathbf{R}^{n-1})$ have the property that $|\hat{\psi}_o(\eta)| \neq 0$, for all $\eta \in \hat{\mathbf{R}}^{n-1}$. If u_o is the unique solution in H^+ of the equation (1.7) in Lemma 1.4 and if K is defined by (2.1) and (2.2), then the subspace $\{\varphi * K,\ \varphi \in C_o^\infty(U)\}$ of M is dense in M.*

PROOF. By the Riesz representation theorem, we have to show that, if $g \in M$ is orthogonal (with respect to the L^2 inner product) to every function of the form $\varphi * K$, where $\varphi \in C_o^\infty(U)$, then $g = 0$. So let us assume the existence of such a function. We have thus, using Plancherel's theorem, that

$$\int_{\hat{\mathbf{R}}^n} \hat{g}(\xi)\, \hat{v}_o(\xi)\, \hat{Q}(\xi)\, \overline{\hat{\varphi}(\xi)}\, d\xi = 0, \forall \varphi \in C_o^\infty(U)$$

This implies that supp $g * v_o * Q \subset \mathbf{R}^n \setminus U$, and, using LEMMA 1.6, that, for a. e. $\eta \in \hat{\mathbf{R}}^{n-1}$,

$$\int_{\hat{\mathbf{R}}} \hat{Q}(\gamma,\eta)\, \hat{v}_o(\gamma,\eta)\, \hat{g}(\gamma,\eta)\, \overline{\hat{\rho}(\gamma)}\, d\gamma = 0, \forall \rho \in C_o^\infty((0,\infty)). \tag{2.4}$$

If the weight $\hat{Q}_\eta$ on $\hat{\mathbf{R}}$ and the Hilbert spaces H_η and H_η^+ are defined as in section one, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, we let u_η be the unique solution in H_η^+ of the equation $Q_\eta * u_\eta = \chi_o$ on $(0,\infty)$, and we define $v_\eta \in \mathcal{S}'(\mathbf{R})$ by $v_\eta = u'_\eta + u_\eta$ and g_η by $\hat{g}_\eta(\gamma) = g(\gamma,\eta)$. It is clear, from LEMMA 1.7, and (2.1) that $\hat{v}_o(\gamma,\eta) = \hat{v}_\eta(\gamma)\hat{\psi}_o(\eta)$, and, from LEMMA 1.6 that $g_\eta \in L^2(\mathbf{R})$ and supp $g_\eta \subset [0,\infty)$, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$. Since $|\hat{\psi}_o| > 0$, it follows from (2.4) that

$$\int_{\hat{\mathbf{R}}} \hat{Q}_\eta(\gamma)\, \hat{v}_\eta(\gamma)\, \hat{g}_\eta(\gamma)\, \overline{\hat{\rho}(\gamma)}\, d\gamma = 0, \forall \rho \in C_o^\infty((0,\infty)),$$

and, thus, the theorem will be proved if we can prove it for $n = 1$. So, let $\hat{Q}$ be a weight on $\hat{\mathbf{R}}$ satisfying (1.1) and (1.2), let H and H^+ the Hilbert spaces corresponding to $\hat{Q}$ as defined in section 1, and let $u_o \in H^+$ be the unique solution of $Q * u_o = \chi_o$ on $(0,\infty)$. Let us assume that $g \in L^2(\mathbf{R})$ is supported in $[0,\infty)$ and satisfies

$$\int_{\hat{\mathbf{R}}} \hat{Q}(\gamma)\, \hat{v}_o(\gamma)\, \hat{g}(\gamma)\, \overline{\hat{\rho}(\gamma)}\, d\gamma = 0, \forall \rho \in C_o^\infty((0,\infty)), \tag{2.5}$$

where v_o is defined by $v_o = u_o' + u_o$. We let $h = g * v_o$ and we will show that $h \in H^+$. To see this, we remark that if $\varphi \in C_o^\infty((0,\infty))$, then $\varphi * v_o \in H^+$. Indeed, if $\{\rho_k\}_{k\geq 0}$ is a sequence in $C_o^\infty((0,\infty))$ converging to u_o in H^+ as $k \longrightarrow \infty$, then $\varphi * (\rho_k' + \rho_k)$ converges to $\varphi * v_o$ in H since

$$\begin{aligned}\|\varphi * \left(\rho_k' + \rho_k - v_o\right)\|_H^2 &= \int_{\hat{\mathbf{R}}} \hat{Q}(\gamma) |\hat{\varphi}(\gamma)(1 + 2\pi i\gamma)[\hat{u}_o(\gamma) - \hat{\rho}_k(\gamma)]|^2 d\gamma \\ &\leq C \int_{\hat{\mathbf{R}}} |\hat{u}_o(\gamma) - \hat{\rho}_k(\gamma)|^2 \hat{Q}(\gamma)\, d\gamma \longrightarrow 0, \text{ as } k \longrightarrow \infty.\end{aligned}$$

Now, if $\{\varphi_k\}_{k\geq 0}$ is a sequence in $C_o^\infty((0,\infty))$ converging to g in L^2, then $\varphi_k * v_o \in H^+$ and $\varphi_k * v_o$ converges to $g * v_o$ in H as $k \longrightarrow \infty$, since, using (1.8) in LEMMA 1.4,

$$\int_{\hat{\mathbf{R}}} |\hat{g}(\gamma)\hat{v}_o(\gamma) - \hat{\varphi}_k(\gamma)\hat{v}_o(\gamma)|^2 \hat{Q}(\gamma) d\gamma = G \int_{\hat{\mathbf{R}}} |\hat{g}(\gamma) - \hat{\varphi}_k(\gamma)|^2 d\gamma \longrightarrow 0, k \longrightarrow \infty.$$

This shows that $h \in H^+$. It follows from (2.5) that h is orthogonal in H^+ to $C_o^\infty((0,\infty))$ and thus $h = 0$. Since, by (1.8), $|\hat{v}_o| > 0$ a.e., we conclude that $g = 0$ a.e., which proves the assertion.

The next theorem gives another description of the space H^+ which is going to be very useful later on. Before stating it, we need the following lemma.

LEMMA 2.2. *Suppose that T and S are two distributions in $\mathcal{S}'(\mathbf{R})$ supported in* $[0,\infty)$*, with the properties that, $\hat{T}(\gamma)(1+|\gamma|^2)^{-m} \in L^1(\hat{\mathbf{R}})$, and $\hat{S}(\gamma)(1+|\gamma|^2)^{-m} \in L^\infty(\hat{\mathbf{R}})$, for some integer $m \geq 0$. Then $T * S$ is supported in* $[0,\infty)$*, where this last convolution is formally defined by $(T*S)^\wedge = \hat{T}\hat{S}$.*

PROOF. We consider first the case $m = 0$. We choose a function φ in $C_o^\infty(\mathbf{R})$ with the properties that $\hat{\varphi} \geq 0$ and $\varphi(0) = 1$. We define, for $k \geq 1$, $\varphi_k(t) = \varphi\left(\frac{t}{k}\right)$, $\forall t \in \mathbf{R}$. Obviously, the distribution $(\varphi_k T) * S$ is supported in $[0,\infty)$. (This last convolution is a "usual" convolution between a distribution in $\mathcal{S}'(\mathbf{R})$ and a distribution having compact support). By the "exchange" formula, we have that $[(\varphi_k T) * S]^\wedge = \left(\hat{\varphi}_k * \hat{T}\right)\hat{S}$, and, since $\{\hat{\varphi}_k\}_{k\geq 1}$ is a mollifying sequence, $\hat{\varphi}_k * \hat{T} \longrightarrow \hat{T}$ in $L^1(\hat{\mathbf{R}})$, as $k \longrightarrow \infty$. Thus $\hat{T}\hat{S} = \lim_{k\to\infty}(\hat{\varphi}_k * T)\hat{S}$ in $\mathcal{S}'(\hat{\mathbf{R}})$, which shows that $T * S$ is supported in $[0,\infty)$. If $m \neq 0$, we define f and g by $\hat{f}(\gamma) = \hat{T}(\gamma)(1+2\pi i\gamma)^{-2m}$ and $\hat{g}(\gamma) = \hat{S}(\gamma)(1+2\pi i\gamma)^{-2m}$. Clearly, we have that $\hat{f} \in L^1(\hat{\mathbf{R}})$ and $\hat{g} \in L^\infty(\hat{\mathbf{R}})$. Furthermore, f and g satisfy the differential equations: $(\delta' + \delta)^{*(2m)} * f = T$ and $(\delta' + \delta)^{*(2m)} * g = S$. Now, if $R \in \mathcal{S}'(\mathbf{R})$ satisfies the differential equation $(\delta' + \delta) * R = 0$ on $(-\infty, 0)$, then $R = C\,e^{-t}$ on $(-\infty,0)$ and thus $R = 0$ on $(-\infty,0)$, since it belongs to $\mathcal{S}'(\mathbf{R})$. This shows, using an induction argument, that supp f and supp g are both contained in $[0,\infty)$. Therefore, it follows from the case $m = 0$, that the support of $f * g$ is contained in $[0,\infty)$, and, thus, since $T * S = (\delta' + \delta)^{*(2m)} * (f * g)$, we conclude that supp $(T * S) \subset [0,\infty)$.

We have the following theorem.

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THEOREM 2.3. $H^+ = \{ u \in H, \, supp \, u \subset \bar{U} \}$.

PROOF. Let N be the closed subspace of H defined by $N = \{ u \in H, \text{ supp } u \subset \bar{U} \}$. Clearly, $C_o^\infty(U)$ is contained in N and we have thus to prove that it is dense in N. Let us use the Riesz representation theorem again and assume the existence of an element $v \in N$, orthogonal to $C_o^\infty(U)$ in H, which is equivalent to supp $Q * v \subset \mathbf{R}^n \setminus U$. If the weight Q_η and the Hilbert spaces H_η and H_η^+ are defined as in section 1, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, and if we define $N_\eta = \{ u \in H_\eta, \text{ supp } u \subset [0, \infty) \}$ and v_η by $\hat{v}_\eta(\gamma) = \hat{v}(\gamma, \eta)$, for $\gamma \in \hat{\mathbf{R}}$, it follows, from LEMMA 1.6, that $v_\eta \in N_\eta$, and that

$$\int_{\hat{\mathbf{R}}} \hat{Q}_\eta(\gamma) \, \hat{v}_\eta(\gamma) \, \overline{\hat{\rho}(\gamma)} \, d\gamma = 0, \forall \rho \in C_o^\infty((0, \infty)),$$

for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$. Hence, the proof can be obviously be reduced to the case $n = 1$. So let $\hat{Q}$ be a weight on $\hat{\mathbf{R}}$ satisfying (1.1) and (1.2) and let H and H^+ be the Hilbert spaces corresponding to $\hat{Q}$, as defined in section 1. Let u_o be the unique element in H^+ satisfying $Q * u_o = \chi_o$ on $(0, \infty)$, and let $v_o = u_o' + u_o$. By LEMMA 1.4, we have that

$$\hat{Q}(\gamma) |\hat{v}_o(\gamma)|^2 = G \tag{2.6}$$

where $G > 0$ is a constant. Let $v \in H$ have the property that supp $v \subset [0, \infty)$. If $\rho \in C_o^\infty((0, \infty))$, we compute, using (2.2), (2.6) and Plancherel's theorem,

$$\begin{aligned} \int_{\hat{\mathbf{R}}} |\hat{v}(\gamma) - \hat{\rho}(\gamma)|^2 \hat{Q}(\gamma) \, d\gamma &= \frac{1}{G} \int_{\hat{\mathbf{R}}} |\hat{Q}(\gamma) \overline{\hat{v}_o(\gamma)} \hat{v}(\gamma) - \hat{Q}(\gamma) \overline{\hat{v}_o(\gamma)} \rho(\gamma)|^2 \, d\gamma \\ &= \frac{1}{G} \int_{\mathbf{R}} |Q * \tilde{v}_o * v - Q * \tilde{v}_o * \rho|^2 \, dx \\ &= \frac{1}{G} \int_{\mathbf{R}} |Q * \tilde{v}_o * v - K * \rho|^2 \, dx \end{aligned} \tag{2.7}$$

Because of (1.1) and (2.6), we have that $(Q * \tilde{v}_o)^\wedge(\gamma)(1 + |\gamma|^2)^{-m} \in L^\infty(\hat{\mathbf{R}})$, and from the fact that $v \in H$, it follows from (1.4) that $|\hat{v}(\gamma)|(1 + |\gamma|^2)^{-m} \in L^1(\hat{\mathbf{R}})$. Since supp $(Q * \tilde{v}_o)$ and supp v are both contained in $[0, \infty)$, LEMMA 2.2 shows that supp $Q * \tilde{v}_o * v \subset [0, \infty)$, and, thus, $Q * \tilde{v}_o * v \in M$. By THEOREM 2.1, the last integral in (2.7) can be made arbitrarily small, by choosing ρ appropriately, and the conclusion follows.

3. **An explicit formula for G.** The main goal in this section is to find an explicit expression for the function G, which appear in (1.8) of LEMMA 1.4, in terms of the weight $\hat{Q}$. In view of LEMMA 1.7, it will be sufficient to consider the case $n = 1$. We will thus assume that we are given a weight $\hat{Q}$ on $\hat{\mathbf{R}}$ which satisfies the condition (1.1) and also, for the time being, the condition

$$\int_{\hat{\mathbf{R}}} \frac{1}{\hat{Q}(\gamma)} \, d\gamma < \infty, \tag{3.1}$$

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stronger than (1.2). We will show that in fact, when $n = 1$ and $\psi_o = 1$, the constant G in LEMMA 1.4 is

$$G = \exp\left(-2\int_{\hat{\mathbf{R}}} \frac{\log \hat{Q}(\gamma)}{1+(2\pi\gamma)^2}\,d\gamma\right). \tag{3.2}$$

To obtain this result, our plan is, in a first step, to approximate the weight $\hat{Q}$ by weights which are periodic, and, in a second step, to approximate those periodic weights by positive trigonometric polynomials bounded away from zero. For each of these approximating weights, we will consider the corresponding Hilbert spaces H and H^+ and the corresponding solution in H^+ of equation (1.7). It turns out that an explicit expression for the solution of this equation can be found when the weight is a positive trigonometric polynomial bounded away from zero, and the expression (3.2) for G will follow easily in that case. To obtain the result in the general case, one needs to show that the expression (3.2) for G is preserved when passing to the limit in each of the approximating steps.

Now, if $\hat{Q}$ is a weight on $\hat{\mathbf{R}}$ satisfying (1.1) an (3.1), we associate with $\hat{Q}$ the sequence of weights $\{\hat{Q}_N\}_{N\geq 0}$, defined by

$$1/\hat{Q}_N(\gamma) = \sum_{j\in\mathbf{Z}} \left[1/\hat{Q}(\gamma + j2^N)\right], \forall\gamma \in \hat{\mathbf{R}} \tag{3.3}$$

It is easily checked that the sequence $\{\hat{Q}_N\}_{N\geq 0}$ is an increasing sequence converging a.e. to $\hat{Q}$. Furthermore, $\hat{Q}_N$ is periodic of period 2^N, $1/\hat{Q}_N$ is locally integrable and, using (1.1), $\hat{Q}_N$ is bounded. (We used the fact that the integer m in (1.1) must be strictly greater than 0). It is thus clear that each weight $\hat{Q}_N$ satisfies the condition (1.1) and (1.2) (with $m = 1$) and we can, therefore, consider for each $N \geq 0$, the Hilbert space H_N and H_N^+ associated with $\hat{Q}_N$ as in Section 1. Of course, from THEOREM 2.3, we have that $H_N^+ = \{u \in H_N, \operatorname{supp} u \subset [0,\infty)\}$. We have the following lemmas.

LEMMA 3.1. *Let u_o be the unique solution in H^+ of the equation $Q * u_o = \chi_o$ on $(0,\infty)$, and, for $N \geq 0$, let u_N be the unique solution in H_N^+ of the equation $Q_N * u_N = \chi_o$ on $(0,\infty)$. Then $\lim_{N\to\infty} \|u_N\|_{H_N} = \|u_o\|_H$.*

PROOF. If $M > N \geq 0$, we have the continuous imbeddings $H^+ \subset H_M^+ \subset H_N^+ \subset H_0^+$. The anti-linear form L, defined by $L(u) = \langle \bar{u}, \chi_o\rangle$, is continuous on each of the space H_N^+, for $N \geq 0$ and on H^+ (see LEMMA 1.4). Therefore, the sets A and A_N, for $N \geq 0$, defined as $A = \{u \in H^+, L(u) = 1\}$, and $A_N = \{u \in H_N^+, L(u) = 1\}$, are closed convex sets in H^+ and H_N^+, respectively. In particular, there exists a unique element $v_o \in H^+$ such that $\|v_o\|_H = \inf_{u\in A} \|u\|_H$. Because of the following inequality, holding for arbitrary $u \in A$,

$$1 = |L(u)| = |(u, u_0)_H| \leq \|u\|_H \|u_o\|_H,$$

it follows that $v_o = u_o\|u_o\|_H^{-2}$. Of course, this argument remains valid for each of the weights $\hat{Q}_N$, for $N \geq 0$. Thus, if $v_N \in H_N^+$ is the unique element in A_N with the property that $\|v_N\|_{H_N} = \inf_{u\in A_n}\|u\|_{H_N}$, then, $v_N = u_N\|u_N\|_{H_N}^{-2}$. Because of the inclusions $A \subset A_M \subset A_N$, if $M \geq N \geq 0$ and since the sequence $\{\hat{Q}_N\}_{N\geq 0}$ is increasing, we have the inequalities

$$(3.4) \qquad \|v_N\|_{H_N} \leq \|v_M\|_{H_M} \leq \|v_o\|_H,$$

which show, in particular, that the sequence $\{v_N\}_{N\geq 0}$ is bounded in H_o^+. Therefore, it has a subsequence $\{v_N\}_{N\in\Gamma}$, $\Gamma \subset \mathbf{N}$, which is weakly convergent in H_0^+ to some element $z \in H_0^+$. It is clear that $L(z) = 1$. Now, if $M > 0$, the sequence $\{v_N\}_{\substack{N\geq M\\ N\in\Gamma}}$ is bounded in H_M^+, by (3.4). It has, therefore, a subsequence converging weakly in H_M^+ to some element $z_M \in H_M^+$. Since H_M^+ is continuously imbedded in H_0^+ and since this subsequence converges also weakly to z in H_0^+, we must have that $z_M = z$. It follows thus that $z \in H_M^+$, for all $M \geq 0$, and, using (3.4), that

$$(3.5) \qquad \|z\|_{H_M} \leq \lim_{N\to\infty}\|v_N\|_{H_N} \leq \|v_o\|_H.$$

By Lebesgue's monotone convergence theorem, we deduce that

$$\begin{aligned}\lim_{N\to\infty}\|z\|_{H_N}^2 &= \lim_{N\to\infty}\int_{\hat{\mathbf{R}}}|\hat{z}(\gamma)|^2\hat{Q}_N(\gamma)\,d\gamma \\ &= \int_{\hat{\mathbf{R}}}|\hat{z}(\gamma)|^2\hat{Q}(\gamma)d\gamma < \infty \text{ (by (3.5))}.\end{aligned}$$

This shows that $z \in H$, and that

$$(3.6) \qquad \lim_{N\to\infty}\|z\|_{H_N} = \|z\|_H \leq \|v_o\|_H.$$

Since the support of z is contained in $[0,\infty)$, THEOREM 2.3 implies that $z \in H^+$, and, thus, $z \in A$, since $L(z) = 1$. It follows from (3.6) and from the unicity of the minimizer v_o that $z = v_o$, and, therefore, (3.5) implies that $\lim_{N\to\infty}\|v_N\|_{H_N} = \|v_o\|_H$ which yields the desired result since $\|u_N\|_{H_N} = 1/\|v_N\|_{H_N}$ and $\|u_o\|_H = 1/\|v_o\|_H$.

LEMMA 3.2.

$$(3.7) \qquad \lim_{N\to\infty}\int_{\hat{\mathbf{R}}}\frac{\log\hat{Q}_N(\gamma)}{1+(2\pi\gamma)^2}\,d\gamma = \int_{\hat{\mathbf{R}}}\frac{\log\hat{Q}(\gamma)}{1+(2\pi\gamma)^2}\,d\gamma$$

PROOF. It is clear from (1.1), (3.1) and (3.3) that the functions $\frac{\log\hat{Q}_N(\gamma)}{1+(2\pi\gamma)^2}$ and $\frac{\log\hat{Q}(\gamma)}{1+(2\pi\gamma)^2}$ are absolutely integrable on $\hat{\mathbf{R}}$. Furthermore, we have the inequality

$$|\log\hat{Q}_N| \leq |\log\hat{Q}| + |\log\hat{Q}_0|,$$

for all $N \geq 0$. Thus, (3.7) follows from Lebesgue's dominated convergence theorem, since $\hat{Q}_N$ converges to $\hat{Q}$ a. e. on $\hat{\mathbf{R}}$, as $N \longrightarrow \infty$.

It is now clear from LEMMA 3.1 and LEMMA 3.2 that the expression (3.2) for G will be established for a weight $\hat{Q}$ satisfying (1.1) and (3.1), if we can show that (3.2) holds for the associated periodic weights $\hat{Q}_N$. Let us consider, more generally, a weight $\hat{R}$ on $\hat{\mathbf{R}}$, periodic with period Ω^{-1}, having the properties that $1/\hat{R}$ is locally integrable and that $\hat{R} \in L^\infty(\hat{\mathbf{R}})$. Let ψ be a function in $C_o^\infty(\mathbf{R})$, supported in [-1, 1] such that $\hat{\psi} \geq 0$ and satisfying $\psi(0) = 1$. We let, for $\epsilon > 0$, $\varphi_\epsilon(\gamma) = \epsilon^{-1}\hat{\psi}(\gamma\ \epsilon)^{-1}$, for $\gamma \in \hat{\mathbf{R}}$. Of course, $\{\varphi_\epsilon\}_{\epsilon>0}$ is a regularizing family on $\hat{\mathbf{R}}$ and each of the functions φ_ϵ has the property that $\varphi_\epsilon > 0$ a.e. since it is the Fourier transform of a function with compact support. We define the weight, $\hat{R}_\epsilon$, for $\epsilon > 0$, by $\hat{R}_\epsilon = \hat{R} * \varphi_\epsilon$. It is clear that $\hat{R}_\epsilon \in L^\infty(\hat{\mathbf{R}})$. In fact, since the support of R is contained in the set $\{k\,\Omega, k \in \mathbf{Z}\}$, $\hat{R}_\epsilon$ is a trigonometric polynomial. Furthermore, since, for every $\gamma \in \hat{\mathbf{R}}$, $\hat{R}_\epsilon(\gamma) > 0$ and $\hat{R}_\epsilon$ is periodic, $\hat{R}_\epsilon$ must be bounded away from zero. As we did before, we define Hilbert spaces V, V^+ and V_ϵ, V_ϵ^+ associated with the weights $\hat{R}$ and $\hat{R}_\epsilon$ respectively. We have the following lemmas.

LEMMA 3.3. *Let u_o be the unique solution in V^+ of the equation $R * u_o = \chi_o$ on* $(0, \infty)$, *and let u_ϵ be the unique solution in V_ϵ^+ of the equation $R_\epsilon * u_\epsilon = \chi_o$ on* $(0, \infty)$. *Then* $\lim_{\epsilon \longrightarrow 0^+} \|u_\epsilon\|_{V_\epsilon} = \|u_o\|_V$.

PROOF. We first notice that, for every $\epsilon > 0$, we have the inequality

$$\int_{\hat{\mathbf{R}}} \frac{1}{\hat{R}_\epsilon(\gamma)(1 + (2\pi\gamma)^2)}\, d\gamma \leq C_\Omega \int_{[0,\Omega^{-1}]} \frac{1}{\hat{R}(\gamma)}\, d\gamma \tag{3.8}$$

where C_Ω is a positive constant. Indeed, it follows from Jensen's inequality (see [8], p. 62), that $1/\hat{R}_\epsilon = 1/(\hat{R} * \varphi_\epsilon) \leq (1/\hat{R}) * \varphi_\epsilon$. Therefore, we compute

$$\begin{aligned}
\int_{\hat{\mathbf{R}}} \frac{1}{\hat{R}_\epsilon(\gamma)(1 + (2\pi\gamma)^2)}\, d\gamma
&\leq \int_{\hat{\mathbf{R}}} \left[\left(1/\hat{R}\right) * \varphi_\epsilon\right](\gamma) \frac{1}{1 + (2\pi\gamma)^2}\, d\gamma \\
&= \int_{\hat{\mathbf{R}}} \frac{1}{\hat{R}(\gamma)} \left[\frac{1}{1 + (2\pi\gamma)^2} * \tilde{\varphi}_\epsilon\right](\gamma)\, d\gamma \\
&= \int_{[0,\Omega^{-1}]} \frac{1}{\hat{R}(\gamma)} \left[\left(\sum_{j\in\mathbf{Z}} \frac{1}{1 + [2\pi(\gamma + j\Omega)]^2}\right) * \tilde{\varphi}_\epsilon\right](\gamma)\, d\gamma \\
&\leq C_\Omega \int_{[0,\Omega^{-1}]} \frac{1}{\hat{R}(\gamma)}\, d\gamma,
\end{aligned}$$

where $C_\Omega = \sup_{\gamma\in\hat{\mathbf{R}}} \sum_{j\in\mathbf{Z}} \frac{1}{1+[2\pi(\gamma+j\Omega)]^2}$. Now, since the weight $\hat{R}_\epsilon$ is bounded from above and from below, $V_\epsilon = L^2(\mathbf{R})$ as a set and, thus, if $u \in V_\epsilon^+$, we have, by Plancherel's theorem that

$$\begin{aligned} |(u_\epsilon, u)_{V_\epsilon}| &= |\langle \bar{u}, \chi_o \rangle| = |\int_{\hat{\mathbf{R}}} e^{-t} \bar{u}(t)\, dt| = |\int_{\hat{\mathbf{R}}} \frac{1}{1+2\pi i \gamma} \overline{\hat{u}(\gamma)}\, d\gamma| \\ &\leq \|u\|_{V_\epsilon} \left[\int_{\hat{\mathbf{R}}} \frac{1}{R_\epsilon(\gamma)(1+(2\pi\gamma)^2)}\, d\gamma \right]^{1/2} \\ &\leq \|u\|_{V_\epsilon} \left[C_\Omega \int_{[0,\Omega^{-1}]} \frac{1}{\hat{R}(\gamma)}\, d\gamma \right]^{1/2}, \text{(by 3.8).} \end{aligned}$$

This shows that

$$\|u_\epsilon\|_{V_\epsilon} \leq \left[C_\Omega \int_{[0,\Omega^{-1}]} \frac{1}{\hat{R}(\gamma)}\, d\gamma \right]^{1/2}. \tag{3.9}$$

Furthermore, it follows from (1.8) that $\left(1+(2\pi\gamma)^2\right)\hat{R}_\epsilon |\hat{u}_\epsilon|^2 = 2\|u_\epsilon\|^2_{V_\epsilon}$, and, thus, we have, using (3.9), that

$$\int_{\hat{\mathbf{R}}} |\hat{u}_\epsilon(\gamma)|^2\, d\gamma = 2\|u_\epsilon\|^2_{V_\epsilon} \int_{\hat{\mathbf{R}}} \frac{1}{\hat{R}_\epsilon(\gamma)(1+(2\pi\gamma)^2)}\, d\gamma \leq 2\left[C_\Omega \int_{[0,\Omega^{-1}]} \frac{1}{\hat{R}(\gamma)}\, d\gamma \right]^2$$

Hence, we conclude, using Plancherel's theorem, that the family $\{u_\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^2(\mathbf{R})$. Since $\hat{R} \in L^\infty(\hat{\mathbf{R}})$, the family $\{u_\epsilon\}_{\epsilon>0}$ is also uniformly bounded in V and thus in V^+, since each $u_\epsilon \in V^+$, by THEOREM 2.3. If $\varphi \in C_o^\infty((0,\infty))$, we compute:

$$\begin{aligned} (u_\epsilon, \varphi)_V = \langle R * u_\epsilon, \bar{\varphi} \rangle &= \langle R_\epsilon * u_\epsilon, \bar{\varphi} \rangle + \langle (R - R_\epsilon) * u_\epsilon, \bar{\varphi} \rangle \\ &= (u, \varphi)_V + \langle (R - R_\epsilon) * u_\epsilon, \bar{\varphi} \rangle. \end{aligned}$$

Furthermore, using the boundedness of the family $\{u_\epsilon\}_{\epsilon>0}$ in $L^2(\mathbf{R})$, we have

$$\begin{aligned} |\langle (R - R_\epsilon) * u_\epsilon, \bar{\varphi} \rangle &= |\int_{\hat{\mathbf{R}}} (\hat{R}(\gamma) - \hat{R}_\epsilon(\gamma))\, \hat{u}_\epsilon(\gamma)\, \hat{\varphi}(\gamma)\, d\gamma| \\ &\leq \left[\int_{\hat{\mathbf{R}}} |\hat{u}_\epsilon(\gamma)|^2\, d\gamma \right]^{1/2} \left[\int_{\hat{\mathbf{R}}} |\hat{R}(\gamma) - \hat{R}_\epsilon(\gamma)|^2 |\hat{\varphi}(\gamma)|^2\, d\gamma \right]^{1/2} \\ &\leq C \left[\int_{\hat{\mathbf{R}}} |\hat{R}(\gamma) - (\hat{R} * \varphi_\epsilon)(\gamma)|^2 |\hat{\varphi}(\gamma)|^2 d\gamma \right]^{1/2} \longrightarrow 0, \epsilon \longrightarrow 0^+, \end{aligned}$$

since $\{\varphi_\epsilon\}_{\epsilon>0}$ is a regularizing family. This shows that, for every $\varphi \in C_o^\infty((0,\infty))$, $(u_\epsilon, \varphi)_V$ converges to $(u,\varphi)_V$ as $\epsilon \longrightarrow 0^+$. Hence, using the density of $C_o^\infty(0,\infty)$ in V^+ and the boundedness of the family $\{u_\epsilon\}_{\epsilon>0}$ in V^+, it follows that u_ϵ converges weakly to u in V^+ as $\epsilon \longrightarrow 0^+$. We have thus

$$\|u_\epsilon\|^2_{V_\epsilon} = (u_\epsilon, u_\epsilon)_{V_\epsilon} = \langle \bar{u}_\epsilon, \chi_0 \rangle = (u, u_\epsilon)_V \longrightarrow \|u\|^2_V, \epsilon \longrightarrow 0^+,$$

which proves the lemma.

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LEMMA 3.4. *Let $\hat{R}$ and $\hat{R}_\epsilon$, $\epsilon > 0$, be the weights in Lemma 3.3. Then we have*

$$\lim_{\epsilon\to 0^+}\int_{\hat{\mathbf{R}}}\frac{|\log \hat{R}_\epsilon(\gamma)-\log \hat{R}(\gamma)|}{1+(2\pi\gamma)^2}\,d\gamma = 0$$

PROOF. Using the notation $(1/\hat{R})_\epsilon = (1/\hat{R}) * \varphi_\epsilon$, we compute

$$\begin{aligned}
\int_{\hat{\mathbf{R}}}&\frac{|\log \hat{R}_\epsilon - \log \hat{R}|}{1+(2\pi\gamma)^2}\,d\gamma \\
&= \int_{\hat{\mathbf{R}}}\frac{|\log\left[\hat{R}_\epsilon(1/\hat{R})_\epsilon\right]+\log\left[1/\hat{R}\right]-\log\left[(1/\hat{R})_\epsilon\right]|\,d\gamma}{1+(2\pi\gamma)^2} \\
&\leq \int_{\hat{\mathbf{R}}}\frac{|\log\left[\hat{R}_\epsilon(1/\hat{R})_\epsilon\right]|d\gamma}{1+(2\pi\gamma)^2}+\int_{\hat{\mathbf{R}}}\frac{|\log\left[1/\hat{R}\right]-\log\left[(1/\hat{R})_\epsilon\right]|\,d\gamma}{1+(2\pi\gamma)^2} \\
&= A_\epsilon + B_\epsilon.
\end{aligned}$$

If $M = \|\hat{R}\|_{L^\infty}$, we have $(1/\hat{R})_\epsilon \geq M^{-1}$, and, therefore, if we let $G_\Omega(\gamma) = \sum_{j\in\mathbf{Z}}\left[1+(2\pi(\gamma+j\Omega^{-1}))^2\right]^{-1}$, we obtain that

$$\begin{aligned}
B_\epsilon &\leq M\int_{\hat{\mathbf{R}}}\frac{|1/\hat{R}-(1/\hat{R})_\epsilon|}{1+(2\pi\gamma)^2}d\gamma = M\int_{[0,\Omega^{-1}]}|1/\hat{R}-(1/\hat{R})_\epsilon|G_\Omega\,d\gamma \\
&\leq C\int_{[0,\Omega^{-1}]}|1/\hat{R}-(1/\hat{R})_\epsilon|\,d\gamma \longrightarrow 0,\ \epsilon\longrightarrow 0^+,
\end{aligned}$$

since $\{\varphi_\epsilon\}_{\epsilon>0}$ is a regularizing family. By Jensen's inequality we have that $\hat{R}_\epsilon(1/\hat{R})_\epsilon \geq 1$. Hence, using now Jensen's inequality with the probability measure $2/1+(2\pi\gamma)^2$, we obtain that

$$A_\epsilon = \frac{1}{2}\int_{\hat{\mathbf{R}}}\frac{\log\left[\hat{R}_\epsilon(1/\hat{R})_\epsilon\right]2}{1+(2\pi\gamma)^2}d\gamma \leq \frac{1}{2}\log\int_{\hat{\mathbf{R}}}\frac{\hat{R}_\epsilon(1/\hat{R})_\epsilon 2\,d\gamma}{1+(2\pi\gamma)^2}.$$

The integral on the right side of this inequality converges to 1 as $\epsilon\longrightarrow 0^+$. Indeed,

$$\begin{aligned}
|\int_{\hat{\mathbf{R}}}\frac{\hat{R}_\epsilon(1/\hat{R})_\epsilon-1}{(1+2\pi\gamma)^2}\,d\gamma| &\leq |\int_{\hat{\mathbf{R}}}\frac{(\hat{R}_\epsilon-\hat{R})\,d\gamma}{\hat{R}(1+(2\pi\gamma)^2}| + \int_{\hat{\mathbf{R}}}\frac{|(1/\hat{R})_\epsilon - 1/\hat{R}|\,\hat{R}_\epsilon\,d\gamma}{1+(2\pi\gamma)^2} \\
&= C_\epsilon + D_\epsilon\,, \qquad (3.10)
\end{aligned}$$

Each term of (3.10) goes to 0 as $\epsilon\longrightarrow 0^+$. Indeed, $h = 1/[\hat{R}(1+(2\pi\gamma)^2]\epsilon\, L^1(\hat{\mathbf{R}})$ and thus

$$C_\epsilon = |\int_{\hat{\mathbf{R}}}\hat{R}\,[h*\tilde{\varphi}_\epsilon - h]\,d\gamma| \leq \|\hat{R}\|_{L^\infty}\int_{\hat{\mathbf{R}}}|h*\tilde{\varphi}_\epsilon - h|d\gamma \longrightarrow 0, \epsilon\longrightarrow 0^+,$$

since $\{\tilde{\varphi}_\epsilon\}$ is also a regularizing family. Moreover,

$$|D_\epsilon| \leq C\|\hat{R}\|_\infty\int_{[0,\Omega^{-1}]}|(1/\hat{R})_\epsilon - 1/\hat{R}|d\gamma \longrightarrow 0, \epsilon\longrightarrow 0^+.$$

This shows that $\lim_{\epsilon\to 0^+} A_\epsilon = 0$ and concludes the proof.

We now have all the ingredients to give an explicit expression for the function G appearing in (1.8).

THEOREM 3.5. *Let $\hat{Q}$ be a weight on $\hat{\mathbf{R}}^n$ satisfying (1.1) and (1.2) and consider the Hilbert space H^+ associated to $\hat{Q}$ as in* DEFINITION 1.2. *If $\chi_o \in \mathcal{S}(\mathbf{R})$ satisfies (1.6) and if $\psi_o \in \mathcal{S}(\hat{\mathbf{R}}^{n-1})$, then the unique solution u_o in H^+ of the equation $Q * u_o = \chi_o \otimes \psi_o$ on U satisfies, for* a.e. $\xi = (\gamma, \eta) \in \hat{\mathbf{R}}^n$,

$$\hat{Q}(\xi)|\hat{u}_o(\xi)|^2 = \frac{|\hat{\psi}_o(\eta)|^2}{1+(2\pi\gamma)^2}\exp\left(-2\int_{\hat{\mathbf{R}}}\frac{\log\hat{Q}(\gamma,\eta)\,d\gamma}{1+(2\pi\gamma)^2}\right) \tag{3.11}$$

PROOF. As mentioned before, by LEMMA 1.7, we need only to consider the case $n = 1$. We will assume first that $\hat{Q}$ is a weight on $\hat{\mathbf{R}}$ which satisfies (1.1) and (3.1). We can thus approximate first $\hat{Q}$ by the sequence $\{\hat{Q}_N\}$ defined in (3.3) and then approximate each of the weights $\hat{Q}_N$ by the weights $\hat{R}_{N,\epsilon} = \hat{Q}_N * \varphi_\epsilon$, which are positive trigonometric polynomials bounded from below. It is clear from LEMMA 3.1, LEMMA 3.2, LEMMA 3.3 and LEMMA 3.4 that we need only to prove (3.11) for a weight $\hat{R}$ on $\hat{\mathbf{R}}$ which is a positive trigonometric polynomial bounded away from 0, of the form

$$\hat{R}(\gamma) = \sum_{|k|\le K} C_k e^{2\pi i k\Omega\gamma}, \forall\gamma \in \hat{\mathbf{R}}.$$

A theorem of L. Fejér and F. Riesz (see [3], p. 20) states that any such polynomial has the form.

$$\hat{R}(\gamma) = C\prod_{k=0}^{K}|1-a_k e^{-2\pi i\Omega\gamma}|^2, \gamma \in \hat{\mathbf{R}},$$

where $C > 0$ and $|a_k| < 1$, for all $k \in \{0, \dots, K\}$. We will first compute the expression on the right side of (3.2) in that case. We will use the following known Fourier series representation

$$\log\left(|1-a\,e^{-2\pi i\Omega\gamma}|^{-2}\right) = \sum_{k=1}^{\infty}\frac{a^k}{k}e^{-2\pi ik\Omega\gamma} + \sum_{k=1}^{\infty}\frac{\bar{a}^k}{k}e^{2\pi ik\Omega\gamma}, \tag{3.12}$$

if $a \in \mathbf{C}$, $|a| < 1$. The function G_Ω defined by

$$G_\Omega(\gamma) = \sum_{j\in\mathbf{Z}}\frac{1}{1+(2\pi(\gamma+j\Omega^{-1}))^2}, \forall\gamma \in \hat{\mathbf{R}},$$

is periodic with period Ω^{-1} and its k^{th} Fourier coefficient is

$$\Omega\int_{[0,\Omega^{-1}]}G_\Omega(\gamma)e^{-2\pi i\Omega k\gamma}\,d\gamma = \Omega\int_{\hat{\mathbf{R}}}\frac{1}{1+(2\pi\gamma)^2}e^{-2\pi i\Omega k\gamma}\,d\gamma = \frac{\Omega}{2}e^{-\Omega|k|}, \tag{3.13}$$

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since $\left[\frac{e^{-|t|}}{2}\right]^{\wedge}(\gamma) = \frac{1}{1+(2\pi\gamma)^2}$, for $\gamma \in \hat{\mathbf{R}}$. We obtain thus, from (3.12) and (3.13), using Parseval's identity, that, if $|a| < 1$,

$$\begin{aligned}\Omega\int_{\hat{\mathbf{R}}} \frac{\log\left(|1-a\,e^{-2\pi i\Omega\gamma}|^{-2}\right)}{1+(2\pi\gamma)^2}\,d\gamma &= \Omega\int_{[0,\Omega^{-1}]} \log\left(|1-a\,e^{-2\pi i\Omega\gamma}|^{-2}\right)\overline{G_\Omega(\gamma)}\,d\gamma \\ &= \frac{\Omega}{2}\left[\sum_{k\geq 1}\frac{a^k}{k}e^{-k\Omega} + \sum_{k\geq 1}\frac{(\bar{a})^k}{k}e^{-k\Omega}\right] \\ &= \frac{\Omega}{2}\log\left(|1-a\,e^{-\Omega}|^{-2}\right),\end{aligned}$$

and, therefore, that

$$-\int_{\hat{\mathbf{R}}}\frac{\log\hat{R}(\gamma)}{1+(2\pi\gamma)^2}d\gamma = \frac{1}{2}\log\left(C^{-1}\prod_{k=0}^{K}|1-a_k e^{-\Omega}|^{-2}\right) \tag{3.14}$$

Now, let us define u_o by

$$\hat{u}_o(\gamma) = \frac{C^{-1}}{1+2\pi i\gamma}\prod_{k=0}^{K}\frac{\left(1-a_k e^{-2\pi i\Omega\gamma}\right)^{-1}}{\left(1-\bar{a}_k e^{-\Omega}\right)}. \tag{3.15}$$

We have thus that

$$\hat{R}(\gamma)|\hat{u}_o(\gamma)|^2 = \frac{C^{-1}}{1+(2\pi\gamma)^2}\prod_{k=0}^{K}|1-a_k e^{-\Omega}|^{-2}, \tag{3.16}$$

and, clearly, $u_o \in H$. Since, for all $\gamma \in \hat{\mathbf{R}}$, we have

$$\frac{1}{1+2\pi i\gamma} = \left[e^{-t}\chi_{(0,\infty)}\right]^{\wedge}(\gamma) \text{ and } \left(1-a_k e^{-2\pi i\Omega\gamma}\right)^{-1} = \left[\sum_{j\geq 0}(a_k)^j\delta_{j\Omega}\right]^{\wedge}(\gamma),$$

it follows easily from (3.15) that supp $u_o \subset [0,\infty)$, and, thus, $u_o \in H^+$, by THEOREM 2.3. We will show that u_o satisfies the equation

$$R * u_o = \chi_o = e^{-t} \text{ on } (0,\infty). \tag{3.17}$$

Indeed, we have the identity

$$(R*u_o)^{\wedge}(\gamma) = \hat{R}(\gamma)\hat{u}_o(\gamma) = \frac{1}{1+2\pi i\gamma}\prod_{k=0}^{K}\frac{\left(1-\bar{a}_k e^{2\pi i\Omega\gamma}\right)}{\left(1-\bar{a}_k e^{-\Omega}\right)}$$

and, thus,

$$R * u_o = \left[\prod_{k=0}^{K}\left(1 - \bar{a}_k e^{-\Omega}\right)^{-1}\right] [(\delta - \bar{a}_0\delta_{-\Omega}) * \cdots * (\delta - \bar{a}_K\delta_{-\Omega}) * e^{-t}\chi_{(0,\infty)}\ ].$$

Now, it is easily seen, by induction on K, that the right side of the previous expression coincides with e^{-t} on $(0,\infty)$ which proves (3.17). It follows then from (3.14) and (3.16) that (3.11) holds and the theorem is thus proved in the case where $1/\hat{Q}$ is integrable. If $\hat{Q}$ satisfies the weaker condition (1.2), we can write $\hat{Q}$ as $\hat{Q}(\gamma) = \hat{Q}_1(\gamma)\left(1 + (2\pi\gamma)^2\right)^{-m}$, for some $m \geq 0$, where $\hat{Q}_1$ satisfies (1.1) and $1/\hat{Q}_1$ is integrable. Let u_1 be the unique solution in the space H^+ corresponding to $\hat{Q}_1$ of the equation $Q_1 * u_1 = e^{-t}$ on $(0,\infty)$. We will show that the unique solution u_o in the space H^+ corresponding to $\hat{Q}$ of the equation $Q * u_o = e^{-t}$ on $(0,\infty)$ can be defined by

$$\hat{u}_o(\gamma) = 2^m(1 + 2\pi i\gamma)^m\hat{u}_1(\gamma) \tag{3.18}$$

Indeed, if u_o is defined by (3.18), we have that

$$(Q * u_o)^\wedge(\gamma) = (Q_1 * u_1)^\wedge(\gamma)(1 - 2\pi i\gamma)^{-m}2^m,\ \gamma \in \hat{\mathbf{R}}.$$

This shows that $Q * u_o$ satisfies the equation

$$2^{-m}\left(\delta - \delta'\right)^{*(m)} * (Q * u_o) = e^{-t} \text{ on } (0,\infty). \tag{3.19}$$

Now, if $S \in \mathcal{S}'(\mathbf{R})$ satisfies the differential equation $2^{-1}\left(\delta - \delta'\right) * S = e^{-t}$ on $(0,\infty)$, then $S = e^{-t}$ on $(0,\infty)$, since any other solution of that equation would be of the form $e^{-t} + c\,e^t$, $c \neq 0$ (on $(0,\infty)$), which would prevent S from being in $\mathcal{S}'(\mathbf{R})$. This shows, by an induction argument using (3.19), that $Q * u_o = e^{-t}$ on $(0,\infty)$. It is also quite clear from (3.18) that u_o is supported in $[0,\infty)$ and thus u_o belongs to the space H^+ corresponding to $\hat{Q}$. In order to show that (3.11) holds for u_o, we notice that for the weight $\hat{R} = 1 + (2\pi\gamma)^2$, the unique solution in the space H^+ corresponding to $\hat{R}$ of the equation $R * v = v - v'' = e^{-t}$ on $(0,\infty)$ is given by

$$\hat{v}(\gamma) = \frac{1}{2(1 + 2\pi i\gamma)^2},\ \gamma \in \hat{\mathbf{R}}, \tag{3.20}$$

as it can be easily checked. Since $1/\hat{R}$ is integrable, we deduce from the first part of the proof and (3.20), that

$$\hat{R}(\gamma)|\hat{v}(\gamma)|^2 = \frac{1}{\left(1 + (2\pi\gamma)^2\right)}\exp\left(-2\int_{\hat{\mathbf{R}}}\frac{\log(1 + (2\pi\gamma)^2)}{1 + (2\pi\gamma)^2}\,d\gamma\right) = \frac{1}{4(1 + (2\pi\gamma)^2)},$$

which shows that, for every $m \geq 0$,

$$\frac{1}{4^m} = \exp\left(-2\int_{\hat{\mathbf{R}}}\frac{\log\left[\left(1 + (2\pi\gamma)^2\right)^m\right]}{1 + (2\pi\gamma)^2}d\gamma\right). \tag{3.21}$$

The desired result follows now easily from (3.18) and (3.21).

COROLLARY 3.6. *Let L be the anti-linear form defined on H^+ by $L(v) = \langle \bar{v}, \chi_o \otimes \psi_o \rangle$, for all $v \in H^+$, and let $A \subset H^+$ be defined by $A = \{ v \in H^+, L(v) = 1 \}$. Then,*

$$(3.22) \qquad \inf_{v \in A} \|v\|_H = \left[\frac{1}{2} \int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(-2 \int_{\hat{\mathbf{R}}} \frac{\log \hat{Q}(\gamma,\eta)\, d\gamma}{1 + (2\pi\gamma)^2} \right) d\eta \right]^{-1/2}.$$

PROOF. It is clear that $\inf_{v\in A} \|v\|_H = \left[\sup_{\|v\|_H = 1} |L(v)| \right]^{-1} = \|u_o\|_H^{-1}$, and the norm of u_o in H can be computed by integrating both sides of (3.11) over $\hat{\mathbf{R}}^n$, yielding (3.22).

4. **A generalization on $\mathbf{R}^n$ of Szegö's theorem.** . The purpose of this section is to extend to $\mathbf{R}^n$ Szegö's theorem ([12]; [2], p. 44). In order to do so, we will need the following lemma.

LEMMA 4.1. *Let $\hat{Q}$ be a weight on $\hat{\mathbf{R}}^n$ which satisfies (1.1) and (1.2) and consider the Hilbert spaces H and H^+ associated with $\hat{Q}$ defined in section 1. Let P be the closed subspace of H defined by $P = \{ v \in H, Q * v = 0 \text{ on } U \}$, and let P_0 be the subspace of P defined by $P_0 = \{ v \in H, Q * v = \varphi, \text{ where } \varphi \in C_o^\infty(\mathbf{R}^n) \text{ and } \operatorname{supp} \varphi \subset \mathbf{R}^n \setminus \bar{U} \}$. Then, P_o is dense in P.*

PROOF. We use the Riesz representation theorem again. Suppose $u \in P$ is orthogonal (in H) to P_o. We have thus $(u, v)_H = 0$, for all $v \in P_o$, or, equivalently,

$$\int_{\hat{\mathbf{R}}^n} \hat{u}(\xi)\, \overline{\hat{\varphi}(\xi)}\, d\xi = 0,$$

for every $\varphi \in C_o^\infty(\mathbf{R}^n)$ for which $\operatorname{supp} \varphi \subset \mathbf{R}^n \setminus \bar{U}$. This shows that $\operatorname{supp} u \subset \bar{U}$ and, thus, that $u \in H^+$, by THEOREM 2.3. Since $u \in P$, we have that $Q * u = 0$ on U, and, therefore, u is orthogonal to $C_o^\infty(U)$, which shows that $u = 0$.

The following lemma is a generalization of Szegö's theorem for a weight $w = 1/\hat{Q}$ on $\hat{\mathbf{R}}^n$, where $\hat{Q}$ satisfies (1.1) and (1.2). We will use the notation V for the open set

$$V = \left\{ x = (t, y) \in \hat{\mathbf{R}}^n, t < 0 \right\}.$$

LEMMA 4.2. *Let $\hat{Q}$ be a weight on $\hat{\mathbf{R}}^n$ satisfying (1.1) and (1.2) and let H and H^+ the Hilbert spaces associated with $\hat{Q}$ as defined in section 1. Let $\psi_o \in \mathcal{S}(\hat{\mathbf{R}}^{n-1})$ and let $B \subset H$ be defined by $B = \{ v \in H, Q * v = \chi_o \otimes \psi_o \text{ on } U \}$. Then,*

$$\begin{aligned} \inf_{v \in B} \|v\|_H^2 &= \inf_{\varphi \in C_o^\infty(V)} \int_{\hat{\mathbf{R}}^n} |\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) - \hat{\varphi}(\xi)|^2 \frac{1}{\hat{Q}(\xi)}\, d\xi \\ &= \frac{1}{2} \int_{\hat{\mathbf{R}}^{n-1}} \exp\left(-2 \int_{\hat{\mathbf{R}}} \frac{\log \hat{Q}(\gamma,\eta)}{1 + (2\pi\gamma)^2}\, d\gamma \right) |\hat{\psi}_o(\eta)|^2\, d\eta \end{aligned}$$

PROOF. The set B is a closed, convex set in H and therefore, there exists a unique element v_o in H having the property that $\|v_o\|_H = \inf_{v\in B} \|v\|_H$. Since B consists of all those elements in H whose orthogonal projection to H^+ coincide with v_o, we must have that $\|u_o\|_H \leq \|v\|_H$, for all $v \in B$ and thus $v_o = u_o$. Now, since $Q * u_o = \chi_o \otimes \psi_o + S$, where $S \in \mathcal{S}'(\mathbf{R}^n)$ and supp $S \subset \mathbf{R}^n \setminus U$, it follows that

$$\hat{u}_o(\xi) = \frac{\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) + \hat{S}(\xi)}{\hat{Q}(\xi)}, \forall \text{ a. e.} \xi \in \hat{\mathbf{R}}^n. \tag{4.1}$$

Let g be defined by $\hat{g}(\xi) = \hat{S}(\xi)/\hat{Q}(\xi)$, for $\xi \in \hat{\mathbf{R}}^n$. It is clear, from (4.1), that $g \in H$, and, furthermore, since $Q*g = S$, $g \in P$. By LEMMA 4.1, g is the limit in H of a sequence $\{g_k\}_{k\geq 1}$ where $Q * g_k = \varphi_k \in C_o^\infty(V)$, and, thus, u_o is the limit in H of the sequence $\{u_k\}_{k\geq 1}$ where $\hat{u}_k(\xi) = [\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) + \hat{\varphi}_k(\xi)]/\hat{Q}(\xi)$. Therefore, we have that

$$\begin{aligned} \|u_o\|_H^2 &= \lim_{k\to\infty} \int_{\hat{\mathbf{R}}^n} |\hat{u}_k(\xi)|^2 \hat{Q}(\xi)\, d\xi \\ &= \lim_{k\to\infty} \int_{\hat{\mathbf{R}}^n} |\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) + \hat{\varphi}_k(\xi)|^2 \frac{1}{\hat{Q}(\xi)}\, d\xi \\ &= \inf_{\varphi \in C_o^\infty(V)} \int_{\hat{\mathbf{R}}^n} |\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) + \hat{\varphi}(\xi)|^2 \frac{1}{\hat{Q}(\xi)}\, d\xi, \end{aligned}$$

and the result follows from THEOREM 3.5.

We can now state our extension of Szëgo's theorem, which holds for weights on $\hat{\mathbf{R}}^n$ which are "tempered functions".

THEOREM 4.3. *Let $w \geq 0$ be a weight on $\hat{\mathbf{R}}^n$ satisfying*

$$\int_{\hat{\mathbf{R}}^n} \frac{w(\xi)}{(1 + |\xi|^2)^m}\, d\xi < \infty, \tag{4.2}$$

for some $m \in \mathbf{N}$, let $\chi_o \in \mathcal{S}(\mathbf{R})$ satisfy (1.6) and let $\psi_o \in \mathcal{S}(\mathbf{R}^{n-1})$. Then,

$$\begin{aligned} &\inf_{\varphi \in C_o^\infty(V)} \int_{\hat{\mathbf{R}}^n} |\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) - \hat{\varphi}(\xi)|^2 w(\xi)\, d\xi \\ &\qquad = \frac{1}{2} \int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(2 \int_{\hat{\mathbf{R}}} \frac{\log w(\gamma, \eta)}{1 + (2\pi\gamma)^2}\, d\gamma\right) d\eta. \end{aligned}$$

(we use the convention that $\exp(-\infty) = 0$.)

PROOF. For a weight w satisfying (4.2), we define

$$\Phi(w) = \inf_{\varphi \in C_o^\infty(V)} \int_{\hat{\mathbf{R}}^n} |\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) - \hat{\varphi}(\xi)|^2 w(\xi)\, d\xi.$$

We choose a decreasing sequence of weights $\{w_N\}_{N\geq 0}$ converging a.e. to w and having the property that, for all $N \geq 0$, $\hat{Q}_N = 1/w_N$ satisfies (1.1) and (1.2) (with constant m depending upon N). Obviously, we have, for all $N \geq 0$, that

$$\Phi(w) \leq \Phi(w_N) = \frac{1}{2}\int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(2\int_{\hat{\mathbf{R}}} \frac{\log\, w_N(\gamma,\eta)d\gamma}{1+(2\pi\gamma)^2}\right)\, d\eta \tag{4.3}$$

Therefore, by letting $N \longrightarrow \infty$ in (4.3) and using Lebesgue's monotone convergence theorem, it follows that

$$\Phi(w) \leq \frac{1}{2}\int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(2\int_{\hat{\mathbf{R}}} \frac{\log\, w(\gamma,\eta)\, d\gamma}{1+(2\pi\gamma)^2}\right)\, d\eta \tag{4.4}$$

On the other hand, if we had a strict inequality in (4.4), there would exist $\varphi \in C_o^\infty(V)$ such that

$$\begin{aligned}\int_{\hat{\mathbf{R}}^n} |\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) - \hat{\varphi}(\xi)|^2 w(\xi)\, d\xi \\ < \frac{1}{2}\int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(2\int_{\hat{\mathbf{R}}^n} \frac{\log\, w(\gamma,\eta)}{1+(2\pi\gamma)^2}\, d\gamma\right)\, d\eta,\end{aligned}$$

and thus, if N is large enough, we would have that

$$\begin{aligned}\int_{\hat{\mathbf{R}}^n} |\hat{\chi}_o(\gamma)\hat{\psi}_o(\eta) - \hat{\varphi}(\xi)|^2 w_N(\xi)\, d\xi \\ < \frac{1}{2}\int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(\int_{\hat{\mathbf{R}}} \frac{\log\, w_N(\gamma,\eta)}{1+(2\pi\gamma)^2}\, d\gamma\right)\, d\eta,\end{aligned}$$

which contradicts Lemma 4.2. We have thus an equality in (4.4) and the result follows.

5. **Non-triviality of some weighted spaces.** In this section, we will consider more general weighted spaces than those studied in sections 1, 2 and 3. In particular, those weights are not assumed to satisfy (1.1). We will only assume that the weights w considered are positive measurable function on $\hat{\mathbf{R}}^n$ taking on, possibly, the value ∞ and having the property that, for some $m \in \mathbf{N}$,

$$\int_{\hat{\mathbf{R}}^n} \frac{1}{w(\xi)(1+|\xi|^2)^m}\, d\xi < \infty. \tag{5.1}$$

Since the condition (1.1) is not satisfied in general, the space $C_o^\infty(\mathbf{R}^n)$ is not necessarily dense in the space H corresponding to w and defined by

$$H = \{\, u \in \mathcal{S}'(\mathbf{R}^n)\,,\, \hat{u} \in L^1_{loc}(\hat{\mathbf{R}}^n),\ \int_{\hat{\mathbf{R}}^n} |\hat{u}(\xi)|^2 w(\xi)\, d\xi < \infty \,\}\,. \tag{5.2}$$

In fact, H might not contain any non-trivial function with compact support. We define the corresponding space H^+ by

$$H^+ = \left\{ u \in H,\ \mathrm{supp}\ u \subset \bar{U} \right\}. \tag{5.3}$$

Thus, if w satisfies the condition (1.1), this definition coincide with DEFINITION 1.3, by THEOREM 2.3. We remark that the space H (and also H^+) is continuously imbedded in $\mathcal{S}'(\mathbf{R}^n)$ since, the inequality (1.4) still applies. Our goal, in this section, is to characterize those weights w which give rise to non-trivial spaces H^+. We will need the following lemmas.

LEMMA 5.1. *Let $w \geq 0$ be a measurable function on $\hat{\mathbf{R}}$ satisfying (5.1). If $u \in H^+$, where H^+ is defined in (5.3), we define $F(z) = \langle u(t), e^{-zt}\rho_o(t)\rangle$, for any complex number z, with Re $z > 0$, where $\rho_o \in C^\infty(\mathbf{R})$ has the property that ρ_o is identically 1 on a neighborhood of $[0,\infty)$ and $\rho_o(t) = 0$, if $t < t_o$, for some $t_o < 0$. Then, F is holomorphic in the set $\{z \in \mathbf{C}$, Re $z > 0\}$. Furthermore, if $F(1) = DF(1) = \cdots = D^kF(1) = 0$, the distribution $v \in \mathcal{S}'(\mathbf{R})$ defined by $\hat{v}(\gamma) = \hat{u}(\gamma)/(-1+2\pi i\gamma)^{k+1}$, for $\gamma \in \hat{\mathbf{R}}$, belongs to H^+, and if G is defined by $G(z) = \langle v(t), e^{-zt}\rho_o(t)\rangle$, for all $z \in \mathbf{C}$, with Re $z > 0$, then $F(z) = (z-1)^{k+1}G(z)$. In particular, $(D^{(k+1)}F)(1) = (k+1)!G(1)$.*

PROOF. By induction, the case $k > 0$, follows immediately from the case $k = 0$. If $k = 0$, we have that

$$F(1) = \langle u(t), e^{-t}\rho_o(t)\rangle = 0. \tag{5.4}$$

Let $v \in \mathcal{S}'(\mathbf{R})$ be defined as above. Obviously, $v \in H$. We will show that, in fact, $v \in H^+$. Indeed, if $\varphi \in C_o^\infty(\mathbf{R})$ is supported in $(-\infty, -a)$, where $a > 0$, we have that

$$\langle v, \varphi\rangle = \langle u * [e^t\chi_{(-\infty,0)}], \varphi\rangle = \langle u, [e^{-t}\chi_{(0,\infty)}] * \varphi\rangle = 0,$$

since the function $[e^{-t}\chi_{(0,\infty)}] * \varphi = c\, e^{-t}$ on the interval $[-a, \infty)$, and u satisfies (5.4). This shows that the support of v is contained in $[0,\infty)$ and thus $v \in H^+$. We compute, for $z \in \mathbf{C}$, Re $z > 0$, using the definition of v:

$$\begin{aligned} F(z) = \langle u(t), e^{-zt}\rho_o(t)\rangle &= \langle -v(t) + v'(t), e^{-zt}\rho_o(t)\rangle \\ &= \langle v(t), ze^{-zt}\rho_o(t) - e^{-zt}\rho_o'(t) - e^{-zt}\rho_o(t)\rangle \\ &= \langle v(t), (z-1)e^{-zt}\rho_o(t)\rangle = (z-1)G(z). \end{aligned}$$

Since F is clearly holomorphic on the set $\{z \in \mathbf{C}$, Re $z > 0\}$, the proof is thus completed.

Now, given a weight w on $\hat{\mathbf{R}}^n$ which satisfies (5.1), we define, for $\eta \in \hat{\mathbf{R}}^{n-1}$, the one-dimensional weights w_η, by $w_\eta(\gamma) = w(\gamma, \eta)$, for $\gamma \in \hat{\mathbf{R}}$. Since, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, the weight w_η satisfies (5.1) on $\hat{\mathbf{R}}$, by Fubini's theorem, we can define the corresponding spaces H_η and H_η^+ for the weight w_η, using (5.2) and (5.3). We need the following lemma.

LEMMA 5.2. *Suppose that the space H^+ corresponding to a weight $w \geq 0$ on $\hat{\mathbf{R}}^n$ satisfying (5.1) is non-trivial. If $\varphi_o \in \mathcal{S}(\mathbf{R}^{n-1})$ has the property that*

$$\{\eta \in \hat{\mathbf{R}}^{n-1},\ \hat{\varphi}_o(\eta) = 0\} \subset \{\eta \in \hat{\mathbf{R}}^{n-1},\ H_\eta^+ \text{ is trivial}\},$$

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then, there exists $v \in H^+$ such that $\langle \bar{v}, \chi_o \otimes \psi_o \rangle \neq 0$, where χ_o satisfies (1.6).

PROOF. Let $u \in H^+$ be such that $u \not\equiv 0$. We define, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, $u_\eta \in H^+_\eta$ by $\hat{u}_\eta(\gamma) = \hat{u}(\gamma, \eta)$, for $\gamma \in \hat{\mathbf{R}}$, (see Lemma 1.6), and the function F_η, holomorphic in $\{z \in \mathbf{C}, \operatorname{Re} z > 0\}$ by $F_\eta(z) = \langle u_\eta(t), e^{-zt}\rho_o(t)\rangle$. We notice that the set

$$A = \{\eta \in \hat{\mathbf{R}}^{n-1}, F_\eta \equiv 0\}$$

has non-zero $(n-1)$ dimensional Lebesgue measure. Indeed, if for some $\eta \in \hat{\mathbf{R}}^{n-1}$, $F_\eta \equiv 0$, then $\langle u_\eta(t), e^{-t}e^{-2\pi i\gamma t}\rho_o(t)\rangle = 0$, for $\gamma \in \hat{\mathbf{R}}$. But, this implies that $\left(u_\eta e^{-t}\rho_o\right)^\wedge \equiv 0$ and thus that $u_\eta \equiv 0$. Since, $u \not\equiv 0$, the set $\{\eta \in \hat{\mathbf{R}}^{n-1}, u_\eta \equiv 0\}$ cannot have zero $(n-1)$ dimensional Lebesgue measure, and thus $|A| > 0$. Now, we can write $A = \bigcup_{k \geqq 0} A_k$, where $A_k = \{\eta \in \hat{\mathbf{R}}^{n-1}, F_\eta^{(k)}(1) \neq 0, F_\eta^{(j)}(1) = 0, \text{ if } 0 \leq j < k\}$. Therefore, there exists $k_o \in \mathbf{N}$, such that $|A_{k_o}| > 0$. If $\eta \in A_{k_o}$, we define $v_\eta \in H_\eta$ by $\hat{v}_\eta(\gamma) = \hat{u}_\eta(\gamma)/(-1 + 2\pi i\gamma)^{k_o}$, for $\gamma \in \hat{\mathbf{R}}$. By Lemma 5.1, we have, in fact, that $v_\eta \in H^+_\eta$, if $\eta \in A_{k_o}$, and also that

$$\langle v_\eta(t), e^{-t}\rho_o(t)\rangle = \langle v_\eta, \chi_o\rangle \neq 0. \tag{5.5}$$

We remark also that, $\hat{\psi}_o(\eta) \neq 0$ if $\eta \in A_{k_o}$, since, by (5.5), H^+_η is non-trivial.
We define $v \in H$, by $\hat{v}(\gamma, \eta) = \hat{v}_\eta(\gamma) g(\eta)$ for $\eta \in A_{k_o}$ and 0 for $\eta \notin A_{k_o}$, where

$$\forall \eta \in A_{k_o},\ g(\eta) = \frac{\langle v_\eta, \chi_o\rangle \overline{\hat{\psi}_o(\eta)}}{|\langle v_\eta, \chi_o\rangle \hat{\psi}_o(\eta)|}.$$

It follows, from LEMMA 1.6, that $v \in H^+$, and, furthermore,

$$\begin{aligned}
\langle \bar{v}, \chi_o \otimes \psi_o\rangle &= \int_{A_{k_o}} \int_{\hat{\mathbf{R}}} \overline{\hat{v}_\eta(\gamma)}\, \hat{\chi}_o(\gamma)\, d\gamma\ g(\eta)\, \hat{\psi}_o(\eta)\, d\eta \\
&= \int_{A_{k_o}} \overline{\langle v_\eta, \chi_o\rangle} g(\eta)\, \hat{\psi}_o(\eta)\, d\eta (\text{since } \overline{\langle v_\eta, \chi_o\rangle} = \langle \overline{v_\eta}, \chi_o\rangle) \\
&= \int_{A_{k_o}} |\langle v_\eta, \chi_o\rangle|\, |\hat{\psi}_o(\eta)|\, d\eta > 0,
\end{aligned}$$

which proves the lemma.

The following theorem provides a necessary and sufficient condition for the non-triviality of H^+.

THEOREM 5.3. *Let $w \geq 0$ be a weight on $\hat{\mathbf{R}}^n$ satisfying (5.1) and let H^+ be the Hilbert space associated with w, defined in (5.3). Then H^+ is non-trivial if and only if,*

$$\left| \left\{ \eta \in \hat{\mathbf{R}}^{n-1}, \int_{\hat{\mathbf{R}}} \frac{\log\ w(\gamma, \eta)}{1+\gamma^2} d\gamma < \infty \right\} \right| > 0, \tag{5.6}$$

where, $|\cdot|$ denotes the $(n-1)$-dimensional Lebesgue measure if $n > 1$. When $n = 1$, (5.6) has to be replaced by

$$\int_{\hat{\mathbf{R}}} \frac{\log\ w(\gamma)}{1+\gamma^2} d\gamma < \infty.$$

PROOF. Let $\psi_o \in \mathcal{S}(\mathbf{R}^{n-1})$ be such that $\hat{\psi}_o(\eta) \neq 0, \forall \eta \in \hat{\mathbf{R}}^{n-1}$, and let us choose an increasing sequence of weights $\{ w_k \}_{k \geq 0}$, each satisfying (1.1) and (1.2), converging to w a.e. on $\hat{\mathbf{R}}^n$. For each $k \geq 0$ we define, the Hilbert spaces H_k and H_k^+, corresponding to the weights w_k (as in (5.2) and (5.3)), and the sets A_k by $A_k = \{ v \in H_k^+, \langle \bar{v}, \chi_o \otimes \psi_o \rangle = 1 \}$. It follows from COROLLARY 3.6 that

$$C_k = \inf_{v \in A_k} \| v \|_{H_k}^2$$

$$(5.7) \qquad = \left[\frac{1}{2} \int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(-2 \int_{\hat{\mathbf{R}}} \frac{\log\, w_k(\gamma, \eta)}{1 + (2\pi\gamma)^2} d\gamma \right) \, d\eta \right]^{-1}$$

If H^+ is non-trivial, LEMMA 5.2 shows that the set $A = \{ v \in H^+, \langle \bar{v}, \chi_o \otimes \psi_o \rangle = 1 \}$ is non-empty. Since this set is closed and convex in H^+, there exists a unique element $v_o \in H^+$ such that $\| v_o \|_H = \inf_{v \in A} \| v \|_H$. It is clear that, for every $k \geq 0$, $v_o \in A_k$, and thus,

$$\forall k \geq 0, C_k = \inf_{v \in A_k} \| v \|_{H_k}^2 \leq \| v_o \|_{H_k}^2 \leq \| v_o \|_H^2.$$

The sequence $\{ C_k \}_{k \geq 0}$ is thus bounded, and, therefore, it follows, by letting $k \longrightarrow \infty$ in (5.7) and using Lebesgue's monotone convergence theorem, that

$$\int_{\hat{\mathbf{R}}^{n-1}} |\hat{\psi}_o(\eta)|^2 \exp\left(-2 \int_{\hat{\mathbf{R}}} \frac{\log\, w(\gamma, \eta)}{1 + (2\pi\gamma)^2} \, d\gamma \right) \, d\eta > 0,$$

which is equivalent to (5.6). Conversely, if (5.6) holds, then, by (5.7), the sequence $\{ C_k \}_{k \geq 0}$ is bounded. Let v_k be the unique element in A_k having the property that

$$\| v_k \|_{H_k} = \inf_{v \in A_k} \| v \|_{H_k} = C_k.$$

The sequence $\{ v_k \}_{k \geq 0}$ is bounded in H_0^+ since $\| v_k \|_{H_0} \leq \| v_k \|_{H_k} \leq C$, for $k \geq 0$. It has, thus, a subsequence $\{ v_k \}_{k \in \Gamma}$, $\Gamma \subset \mathbf{N}$, converging weakly in H_0 to some $v \in H_0^+$, as $k \longrightarrow \infty$. By an argument used in the proof of LEMMA 3.1, one can show that, in fact, $v \in H_k^+$, for every $k \geq 0$ and also that $\| v \|_{H_k} \leq C$. Hence, $v \in H$, by Lebesgue's monotone convergence theorem. It is clear that $v \in H^+$, since $v \in H_0^+$, and, $v \neq 0$, since

$$\langle \bar{v}, \chi_o \otimes \psi_o \rangle = \lim_{k \longrightarrow \infty, k \in \Gamma} \langle \bar{v}_k, \chi_o \otimes \psi_o \rangle = 1.$$

This shows that H^+ is non-trivial and concludes the proof.

COROLLARY 5.4. *Let $w \geq 0$ be a weight on $\hat{\mathbf{R}}^n$ satisfying (5.1) and let H^+ be defined by (5.3). If $\Omega \subset \hat{\mathbf{R}}^{n-1}$ is defined by*

$$\Omega = \left\{ \eta \in \hat{\mathbf{R}}^{n-1}, \int_{\hat{\mathbf{R}}} \frac{\log\, w(\gamma, \eta)}{1 + \gamma^2} \, d\gamma = \infty \right\}$$

and, if $u \in H^+$, then $\hat{u} = 0$ a.e. on the set $\hat{\mathbf{R}} \times \Omega$.

PROOF. Consider the weight w_1 on $\hat{\mathbf{R}}^n$ defined by $w_1(\xi) = w(\xi)$ for $\xi \in \hat{\mathbf{R}} \times \Omega$, and $w_1(\xi) = e^{|\gamma|}$ for $\xi \notin \hat{\mathbf{R}} \times \Omega$. If v is defined by $\hat{v}(\xi) = \hat{u}(\xi)\chi_\Omega(\eta)$, for $\xi = (\gamma, \eta) \in \hat{\mathbf{R}}^n$, it follows from LEMMA 1.6 that supp $v \subset \bar{U}$, and, furthermore, v belongs to the weighted L^2-space corresponding to the weight w_1. Thus, by THEOREM 5.3, we obtain that $\hat{v} = 0$ a.e., or, equivalently, that $\hat{u} = 0$ a.e. on the set $\hat{\mathbf{R}} \times \Omega$.

6. **A generalization on $\mathbf{R}^n$ of a theorem of Paley-Wiener.** The purpose of this section is to generalize to $\mathbf{R}^n$ a theorem of Paley-Wiener (Theorem XII in [6]) on the real line, which characterizes the behaviour of the modulus of a function in $L^2(\hat{\mathbf{R}})$ which is the Fourier transform of a function in $L^2(\mathbf{R})$ supported on the half-line $[0, \infty)$. More precisely, it states that if $\Phi \geq 0$ is a function in $L^2(\hat{\mathbf{R}})$ such that $\Phi \not\equiv 0$, then $\Phi = |\hat{f}|$, where $f \in L^2(\mathbf{R})$ and f is supported on $[0, \infty)$ if and only if

$$\int_{\hat{\mathbf{R}}} \frac{\log \Phi(\gamma)}{1 + \gamma^2} d\gamma > -\infty$$

In the following, we will characterize the behaviour of the modulus of the Fourier transform of functions or distributions supported on a half-space of $\mathbf{R}^n$, which is a set whose boundary is a hyperplane of $\mathbf{R}^n$ (we will assure, for convenience, that the origin belongs to that hyperplane). In the particular case where this half-space is the set $A = \bar{U}$, we notice, that if $f \in L^2(\mathbf{R})$ is supported on the half-line $[0, \infty)$ and if g is any function in $L^2(\mathbf{R}^{n-1})$, then the function $F \in L^2(\mathbf{R}^n)$, defined by $F(x) = f(t)g(y)$, for $x = (t, y) \in \mathbf{R}^n$, is supported in A and it is easy to check, using the Paley-Wiener theorem above, that it satisfies, for a.e. $\eta \in \hat{\mathbf{R}}^{n-1}$, that $\hat{F}(\gamma, \eta) = 0$, for a.e. $\gamma \in \hat{\mathbf{R}}$ or

$$\int_{\hat{\mathbf{R}}} \frac{\log |\hat{F}(\gamma, \eta)|}{1 + \gamma^2} d\gamma > -\infty.$$

It turns out that this last property characterizes the behaviour of the modulus of the Fourier transform of any function in $L^2(\mathbf{R}^n)$ supported in A. The proof that we will give is valid for more general classes of functions on $\hat{\mathbf{R}}^n$ and it also offers an alternative, in the case $n = 1$, for the proof given in [6] which was based mainly on complex analysis. We introduce the following notations. If $\nu \in \hat{\mathbf{R}}^n$ has length 1, we define the hyperplane M_ν by $M_\nu = \{\sigma \in \hat{\mathbf{R}}^n, (\sigma, \nu) = 0\}$ where $(\cdot, \cdot)$ denotes the usual scalar product on $\hat{\mathbf{R}}^n$. We will identify M_ν with $\hat{\mathbf{R}}^{n-1}$ and we will denote by dm_ν the $(n-1)$-dimensional Lebesgue measure on M_ν. We have the following theorem.

THEOREM 6.1. *Let $\nu \in \hat{\mathbf{R}}^n$ have length 1 and let $\Phi \geq 0$ be a measurable function on $\hat{\mathbf{R}}^n$ which satisfies*

$$\int_{\hat{\mathbf{R}}^n} \frac{\Phi(\xi)}{(1 + |\xi|^2)^m} d\xi < \infty$$

for some $m \in \mathbf{N}$. Then the following are equivalent.

(a) *There exists a distribution* $F \in \mathcal{S}'(\mathbf{R}^n)$ *supported in the set* $\{ x \in \mathbf{R}^n, (x, \nu) \geq 0 \}$ *which satisfies* $\Phi = |\hat{F}|^2$ a.e. *on* $\hat{\mathbf{R}}^n$.

(b) *Except for a set of* dm_ν *measure zero, each* $\sigma \in M_\nu$ *satisfies:*

$$\int_{\hat{\mathbf{R}}} \frac{\log \Phi(\sigma + t\nu)}{1 + t^2}\, dt > -\infty$$

or for a.e. $t \in \mathbf{R}$, $\Phi(\sigma + t\nu) = 0$.

PROOF. By considering a rotation of $\hat{\mathbf{R}}^n$ which brings the vector ν to the vector (1,0,...,0) and using the fact that the Fourier transformation commutes with rotations, it is easily seen that the proof can be reduced to the case where $\nu = (1, 0, \ldots, 0)$. To prove the implication $(a) \Rightarrow (b)$, it is sufficient, by LEMMA 1.6, to consider the case $n = 1$. Let us thus assume the existence of a distribution $F \in \mathcal{S}'(\mathbf{R})$, supported in $[0, \infty)$ which satisfies $|\hat{F}|^2 = \Phi$ a.e. on $\hat{\mathbf{R}}$. We define the weight w on $\hat{\mathbf{R}}$ by $w(\gamma) = (\Phi(\gamma)(1 + |\gamma|^2))^{-1}$, for $\gamma \in \hat{\mathbf{R}}$, where $1/0 = \infty$, by definition. It is clear that $\hat{F}$ belongs to the weighted L^2-space corresponding to w and thus, if $F \not\equiv 0$, we must have that

$$\int_{\hat{\mathbf{R}}} \frac{\log\ w(\gamma)}{1 + \gamma^2}\, d\gamma < \infty,$$

by THEOREM 5.3, which is equivalent to (b). Conversely, let us assume that (b) holds. We choose, then, an increasing sequence $\{\hat{Q}_N\}_{N \geq 0}$ of weights on $\hat{\mathbf{R}}^n$ satisfying (1.1) and (1.2) (for some $m \in \mathbf{N}$ which depends upon N), with the property that $1/\hat{Q}_N$ converges to Φ a.e. on $\hat{\mathbf{R}}^n$ as $N \longrightarrow \infty$. We let H_N and H_N^+ be the Hilbert spaces associated with the weight $\hat{Q}_N$ (as in section 1). We let $\psi_o \in \mathcal{S}(\mathbf{R}^{n-1})$ have the property that $|\hat{\psi}_o(\eta)| > 0$, for every $\eta \in \hat{\mathbf{R}}^{n-1}$, and we consider the unique solution $u_N \in H_N^+$ of the equation $Q_N * u_N = \chi_o \otimes \psi_o$ on U. Then, by THEOREM 3.5, we obtain the following identity, valid for a.e. $\xi \in \hat{\mathbf{R}}^n$

$$\hat{Q}_N(\xi)|\hat{u}_N(\xi)|^2 = \frac{1}{1 + (2\pi\gamma)^2} |\hat{\psi}_o(\eta)|^2 \exp\left(-2 \int_{\hat{\mathbf{R}}} \frac{\log \hat{Q}_N(\gamma, \eta)}{1 + (2\pi\gamma)^2}\, d\gamma\right). \tag{6.1}$$

On the other hand, since the sequence $\{ \|u_N\|_{H_N} \}_{N \geq 0}$ is decreasing, the sequence $\{ u_N \}_{N \geq 0}$ must be bounded in H_o^+ and, thus, we can assume that it is weakly convergent in H_o^+ to some element u, by passing to a subsequence if necessary. By the same argument, the sequence $\{ u_N \}_{N \geq M}$ is bounded in H_M^+, for each $M \geq 0$ and has therefore a subsequence $\{ u_N \}_{N \in \Gamma_M}$ converging weakly in H_M^+ to some element v_M. Since H_M^+ is continuously imbedded in H_o^+, we have $v_M = u$, for all $M \geq 0$. Hence, $u \in H_M^+$, for all $M \geq 0$, and furthermore

$$\|u\|_{H_M} \leq \lim_{N \in \Gamma_M} \inf \|u_N\|_{H_M} \leq \lim_{N \longrightarrow \infty} \|u_N\|_{H_N}. \tag{6.2}$$

Let us show that u_N converges to u in H_o^+, as $n \longrightarrow \infty$. Indeed, we have

$$(6.3)\qquad \|u-u_N\|^2_{H_o} \le \|u-u_N\|^2_{H_N} = \|u\|^2_{H_N} - 2Re\,[(u_N\,,u)_{H_N}\,] + (u_N,u_N)_{H_N}.$$

Since $(u_N, u)_{H_N} = \langle \bar{u}, \chi_o \otimes \psi_o \rangle$ and, by the continuous embedding of H_o^+ in $\mathcal{S}'(\mathbf{R})$,

$$\lim_{N\to\infty} (u_N, u_N)_{H_N} = \lim_{N\to\infty} \langle \bar{u}_N, \chi_o \otimes \psi_o \rangle = \langle \bar{u}, \chi_o \otimes \psi_o \rangle,$$

it follows from (6.2) and (6.3) that $\lim_{N\to\infty} \|u-u_N\|_{H_o} \le 0$ and thus that

$$\lim_{N\to\infty} \|u-u_N\|_{H_o} = 0$$

. This shows, in particular, using the estimate (1.4) that $\hat{u}_N$ converges to $\hat{u}$ in $L^1_{loc}(\hat{\mathbf{R}}^n)$, as $N \longrightarrow \infty$. Since, from (6.1), $\hat{u}_N$ satisfies a.e. on $\hat{\mathbf{R}}^n$

$$(6.4)\qquad |\hat{u}_N(\xi)|^2 = \frac{1}{1+(2\pi\gamma)^2}|\hat{\psi}_o(\eta)|^2 \frac{1}{\hat{Q}_N(\xi)} \exp\left(-2\int_{\hat{\mathbf{R}}} \frac{\log \hat{Q}_N(\gamma,\eta)}{1+(2\pi\gamma)^2} d\gamma\right),$$

it follows, by letting $N \longrightarrow \infty$ in (6.4), that

$$(6.5)\qquad |\hat{u}(\xi)|^2 = \frac{1}{1+(2\pi\gamma)^2}|\hat{\psi}_o(\eta)|^2 \Phi(\xi)\chi_\Omega(\eta) \exp\left(2\int_{\hat{\mathbf{R}}} \frac{\log\ \Phi(\gamma,\eta)}{1+(2\pi\gamma)^2} d\gamma\right),$$

where $\Omega = \left\{\eta \in \hat{\mathbf{R}}^{n-1},\ \int_{\hat{\mathbf{R}}} \frac{\log\ \Phi(\gamma,\eta)}{1+(2\pi\gamma)^2} d\gamma > -\infty\,\right\}$. If we let, for $\eta \in \hat{\mathbf{R}}^{n-1}$,

$$H(\eta) = \left[|\hat{\psi}_o(\eta)|^2 \exp\left(\int_{\hat{\mathbf{R}}} \frac{\log\ \Phi(\gamma,\eta)}{1+(2\pi\gamma)^2} d\gamma\right)\right]^{-1} \chi_\Omega(\eta),$$

and $\hat{F}(\xi) = (1+2\pi i\gamma)\hat{u}(\xi)H(\eta)$, for $\xi \in \hat{\mathbf{R}}^n$, we see, from LEMMA 1.6, that supp $F \subset \bar{U}$ and since, from the assumption (b), $\Phi = 0$ a.e. on $\mathbf{R} \times \Omega^c$, it follows from (6.5) that $|\hat{F}|^2 = \Phi$ a.e. on $\hat{\mathbf{R}}^n$ which concludes the proof.

It is clear from the previous theorem that, when $n > 1$, the Fourier transform of a function supported on a half-space of the form $\{\,x \in \mathbf{R}^n,\ (x,\nu) \ge 0\,\}$ can certainly vanish on a set of positive measure. However, modulo a set of measure zero, the set where it vanishes must be of the form $\left\{\xi \in \hat{\mathbf{R}}^n,\ \xi = \sigma + t\nu,\ t \in \mathbf{R}, \sigma \in A\,\right\}$ where $A \subset M_\nu$. In particular, we have the following corollary which extend some results of H.S. Shapiro (see [10] and [11]) about functions and distributions with spectral gaps.

COROLLARY 6.2. *Suppose that $\nu_1,\dots,\nu_n$ are n linearly independent vectors in $\hat{\mathbf{R}}^n$ and that $F \in \mathcal{S}'(\mathbf{R}^n)$ is supported in the set $\bigcap_{i=1}^n \{\,x \in \mathbf{R}^n,\ (x,\nu_i) \ge 0\,\}$. Then,*

(a) If $\hat{F}$ satisfies

$$\int_{\hat{\mathbf{R}}^n} \frac{|\hat{F}(\xi)|^2}{(1+|\xi|^2)^m} d\xi < \infty,$$

for some $m \in \mathbf{N}$ and if $\hat{F}$ vanishes on a set of positive measure, $\hat{F} = 0$ a.e. on $\hat{\mathbf{R}}^n$.

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(b) If $\hat{F}$ vanishes on some open ball on $\hat{\mathbf{R}}^n$, F is the zero distribution.

PROOF. (a) follows immediately from the remark above and (b) follows from (a) by considering the sequence $\left\{\hat{F} * \varphi_n\right\}_{n\geq 0}$ where $\{\varphi_n\}_{n\geq 0}$ is a "regularizing" sequence on $\hat{\mathbf{R}}^n$.

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