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A kinetic model for pure electron plasma compression via the rotating wall technique

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An approximate model for pure electron plasma compression is developed for the case where the rotating wall (RW) electric field couples to the $E \times B$ rotation and axial bounce motion of the electrons. The key assumption in the model is that, throughout the compression, the plasma remains in a slowly evolving thermal equilibrium defined by the plasma temperature and angular momentum. Linearized drift kinetic theory is employed to derive an expression for torque exerted by the RW field on the plasma through coupling to the resonant plasma particles, and averaging is used to find the torque that both compresses and heats the plasma. The evolution equations for the angular velocity and temperature of the plasma include the compression and heating from the torque and cooling from cyclotron radiation.

Keywords: plasma dynamics, plasma confinement, plasma nonlinear phenomena

1. Introduction

Non-neutral plasmas (Malmberg & deGrassie 1975; Davidson 2001) have a wide range of applications in basic science and technology (Davidson 2001; Danielson *et al.* 2015). A recent review (Fajans & Surko 2020) describes the numerous techniques, developed over decades, for the manipulation of antimatter plasmas, with applications to positron emission tomography, material studies, positronium atoms and molecules, and antihydrogen formation.

Confinement of non-neutral plasmas is often realized in Penning–Malmberg (PM) traps (Malmberg & deGrassie 1975). In a PM trap, transverse confinement is provided by an axial magnetic field, while longitudinal confinement and transport are controlled by manipulating the voltages on a series of cylindrical electrodes.

It was shown by O'Neil (1980) that if the canonical angular momentum of the plasma is dominated by the magnetic field contribution, azimuthally symmetric PM traps are subject to a confinement theorem that constrains radial particle transport. In this case, the total

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canonical angular momentum of the plasma is approximately given by

$$P_\theta \sim qB \sum_i r_i^2, \tag{1.1}$$

where q is the elementary charge of the plasma species, B is the axial magnetic field and r_i is the distance of particle i from the trap axis. Azimuthal symmetry requires that the angular momentum is conserved and thus, absent of any external torques, the mean-squared radius of a magnetized non-neutral plasma must be conserved. This confinement theorem predicts very long plasma lifetimes, although in experiments, slow expansion is observed due to non-ideal trap effects. Collisions with residual background gas particles drive radial transport and limit confinement time (deGrassie & Malmberg 1980), in agreement with theory (O'Neil 1980) at high background pressure. However, radial expansion did not follow the expected behaviour as the background pressure was reduced (Driscoll, Fine & Malmberg 1986); this was attributed (Driscoll *et al.* 1986) to azimuthal asymmetries in the magnetic or electric confinement system.

A major advance in reaching long confinement times was achieved with the development (Huang *et al.* 1997) of rotating wall (RW) compression. In the RW scheme, phased voltages are applied to an azimuthally sectored electrode to create a rotating electric field at the plasma. This field exerts a torque which changes the plasma angular momentum and does work on the plasma. We consider here the case where the RW frequency exceeds the plasma rotation frequency, corresponding to a final plasma state which is compressed. The plasma heats and cools during the compression, with the heating provided by the RW doing work on the plasma and the cooling provided by electron cyclotron emission. Rotating wall compression has been developed and exploited to control ion (with buffer gas cooling), electron and positron plasmas (Huang *et al.* 1998; Greaves & Surko 2000; Hollmann, Anderegg & Driscoll 2000; Danielson & Surko 2005, 2006; Greaves & Moxom 2008; Isaac *et al.* 2011). It is now routinely used to both improve confinement and increase plasma densities (Fajans & Surko 2020).

In initial experiments, the coupling of the RW to the plasma was through Trivelpiece–Gould modes (Soga *et al.* 2003). These modes, in turn, couple to the bulk plasma via collisions and Landau damping. An alternative to the Trivelpiece–Gould coupling is known as the strong-drive regime, which is the focus of this paper. In this regime, the plasma rotation frequency almost catches up to the rotating wall frequency. Earlier papers have investigated strong-drive compression (Kiwamoto, Soga & Aoki 2005; Danielson, Surko & O'Neil 2007; van der Werf *et al.* 2012). The single-particle limit was explored by van der Werf *et al.* (2012), a simplified torque-balance model was used to study the relation between the initial and final plasma states by Danielson *et al.* (2007), and Kiwamoto *et al.* (2005) developed a drift kinetic model, similar to what is used here, to evaluate the coupling of the RW to the plasma. In this paper, we use the ansatz that the system can be described at any time by a slowly evolving thermal equilibrium and derive a full set of coupled equations by combining equilibrium thermodynamics with the drift kinetic equation.

The key assumptions of our model are that: (i) the RW torque is generated by coupling to particles that are resonant with the RW wave (i.e. the combined $E \times B$ and axial motion of the particles allows them to remain in phase with the RW wave); (ii) the resonant particle coupling causes a weak perturbation to the Vlasov distribution function; (iii) the plasma evolves slowly, remaining in thermal equilibrium as the density increases and the temperature evolves; (iv) the total potential used to find the torque is calculated self-consistently; and (v) the particles lose energy via cyclotron emission. The assumptions

(i) and (ii) allow us to calculate the evolution equations for the plasma temperature and rotation frequency, and use linear kinetic theory to evaluate the torque generated by resonant coupling. Under these assumptions, we avoid having to explicitly calculate any collisional integrals.

2. Thermal equilibrium model for rotating wall compression

In this section, a complete derivation of our RW model is given. In § 2.1, we derive the equations of motion (EOM) starting from thermodynamic considerations and incorporating Vlasov dynamics. In § 2.2, we find explicit expressions for the energy and angular momentum in terms of thermodynamic state variables (temperature and plasma rotation frequency). In § 2.3, we use drift kinetic theory to derive the torque that the RW imparts to the plasma. We use cgs units throughout and set $k_B = 1$.

2.1. *Thermodynamic equations for compression*

In this model, we consider a cylindrical pure electron plasma in thermal equilibrium, confined to a PM trap with an axial magnetic field $\boldsymbol{B} = B\hat{z}$. The state variables of the plasma are its temperature T and rotational (angular) velocity ω_r. We neglect finite-length effects by modelling a plasma of physical length L_p as infinite and periodic, with periodicity length $\mathcal{L}_p = 2L_p$ (Eggleston & O'Neil 1999). All integrals along the z-axis in this paper are understood to be over one periodic cell, $-L_p \leq z \leq L_p$.

Electron dynamics follows from the single-particle (sp) Hamiltonian

$$\mathcal{H}_{\mathrm{sp}}(r, \boldsymbol{p}; T, \omega_r) = K + q\phi_0\,(r; T, \omega_r)\,, \tag{2.1}$$

with

$$K = \frac{1}{2m_e}\left(p_r^2 + p_z^2\right) + \frac{1}{2m_e r^2}\left(p_\theta - \frac{1}{2}m_e \Omega_c r^2\right)^2, \tag{2.2}$$

where m_e is the electron mass, $q = -e$ is the electron charge, $\Omega_c = qB/m_e c$ is the (signed) cyclotron frequency, c is the speed of light and $\phi_0(r; T, \omega_r)$ is the mean-field electrostatic potential of the plasma. The electrostatic potential depends on the radial density profile, which, given a fixed number of electrons in the trap, is uniquely determined, and in equilibrium, on the plasma temperature and rotation frequency. The explicit dependence on the thermodynamic state variables following the semicolon (e.g. $\phi_0(r; T, \omega_r)$) is a reminder that we are considering equilibrium quantities that evolve by virtue of the plasma transitioning between thermal equilibrium states during compression.

Though we consider only pure electron plasmas in this paper, our analysis naturally extends to positron plasmas. To facilitate this extension, we use a sign convention that readily encompasses both signs of charge. For a magnetic field pointing along $+\hat{z}$, a positively charged plasma column will rotate in the $-\hat{\theta}$ direction under its $E \times B$ drift. This makes the rotational angular velocity negative, $\omega_r < 0$, for charges with $q > 0$. Other angular rotation frequencies follow the same convention. To summarize our sign conventions, for $q > 0$, we have $\Omega_c > 0$, $\omega_r < 0$ and $p_\theta > 0$ (when strongly magnetized), while for $q < 0$, we have $\Omega_c < 0$, $\omega_r > 0$ and $p_\theta < 0$ (when strongly magnetized).

The distribution function of the plasma is (using the above sign conventions)

$$f_0(r, \boldsymbol{p}; T, \omega_r) = \frac{1}{\mathcal{Z}} \exp\left[-\frac{1}{T}\left(\mathcal{H}_{\mathrm{sp}}(r, \boldsymbol{p}; T, \omega_r) - \omega_r p_\theta\right)\right]. \tag{2.3}$$

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The constant $\mathcal{Z}$ is determined by normalizing the distribution function to the total number of electrons in a periodic cell,

$$\frac{1}{\mathcal{L}_p}\int f_0(r,\boldsymbol{p};T,\omega_r)\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p} = \mathcal{N}_e, \tag{2.4}$$

where $\mathcal{N}_e$ is the line density of electrons along the axis.

The equilibrium density $n_0(r;T,\omega_r)$ is given by

$$n_0\,(r;T,\omega_r) = \int f_0(r,\boldsymbol{p};T,\omega_r)\,\mathrm{d}^3\boldsymbol{p} = n_0(0;T,\omega_r)\exp\left(\psi\,(r;T,\omega_r)\right), \tag{2.5}$$

where $n_0(0;T,\omega_r)$ is the on-axis density. The function $\psi(r;T)$ (see Dubin & O'Neil 1999) combines the electrostatic and centrifugal potentials:

$$\psi(r;T,\omega_r) = -\frac{q}{T}\left(\phi_0(r;T,\omega_r) - \phi_0(0;T,\omega_r) - \frac{m_e\omega_r(\varOmega_c+\omega_r)}{2q}r^2\right). \tag{2.6}$$

The Poisson equation can then be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi(r;T,\omega_r)}{\partial r}\right) = \frac{4\pi q^2 n_0(0;T,\omega_r)}{T}\left(\exp\left(\psi(r;T,\omega_r)\right) - 1 - \delta\right), \tag{2.7}$$

with

$$\delta = -\frac{2m_e\omega_r(\varOmega_c+\omega_r)}{4\pi q^2 n_0(0;T,\omega_r)} - 1. \tag{2.8}$$

The rotating wall perturbs the plasma out of equilibrium:

$$\phi_0(r;T,\omega_r) \mapsto \phi_0(r;T,\omega_r) + \delta\phi(t,r,\theta,z), \tag{2.9a}$$

$$f_0(r,\boldsymbol{p};T,\omega_r) \mapsto f_0(r,\boldsymbol{p};T,\omega_r) + \delta f(t,r,\theta,z,\boldsymbol{p}). \tag{2.9b}$$

Our model avoids explicit calculations involving collisional operators even though the time scale for cooling is much longer than the collision time. This is accomplished by assuming that thermalization via collisions occurs sufficiently rapidly so that the plasma evolves through thermal equilibrium states defined by the average energy $\langle E\rangle$ and average angular momentum $\langle p_\theta\rangle$. The temporal evolution of these states is found by matching the same quantities computed for the perturbed plasma after the RW perturbation has been active for a small time Δt.

Explicitly, consider the energy and angular momentum functionals

$$\langle E\rangle[\phi,f] = \frac{1}{\mathcal{N}_e\mathcal{L}_p}\int\left[K + \frac{1}{2}q\phi(\boldsymbol{r})\right]f(t,\boldsymbol{r},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}, \tag{2.10a}$$

$$\langle p_\theta\rangle[\phi,f] = \frac{1}{\mathcal{N}_e\mathcal{L}_p}\int p_\theta f(t,\boldsymbol{r},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}, \tag{2.10b}$$

where, since ϕ is the self-field, the factor of 1/2 in the energy is necessary to avoid double-counting. In thermal equilibrium, these functionals become ordinary functions of the state variables:

$$\langle E\rangle[\phi_0(\boldsymbol{r};T,\omega_r),f_0(\boldsymbol{r},\boldsymbol{p};T,\omega_r)] \equiv \langle E\rangle(T,\omega_r), \tag{2.11a}$$

$$\langle p_\theta\rangle[\phi_0(\boldsymbol{r};T,\omega_r),f_0(\boldsymbol{r},\boldsymbol{p};T,\omega_r)] \equiv \langle p_\theta\rangle(T,\omega_r). \tag{2.11b}$$

Our equilibration assumption is

$$\langle E\rangle(T+\Delta T,\omega_r+\Delta\omega_r) = \langle E\rangle[\phi(t_0+\Delta t),f(t_0+\Delta t,\boldsymbol{r},\boldsymbol{p})], \tag{2.12a}$$

$$\langle p_\theta\rangle(T+\Delta T,\omega_r+\Delta\omega_r) = \langle p_\theta\rangle[\phi(t_0+\Delta t),f(t_0+\Delta t,\boldsymbol{r},\boldsymbol{p})], \tag{2.12b}$$

where $f(t_0+\Delta t)$ is the equilibrium distribution at t_0, $f(t_0)\equiv f_0$, evolved with the Vlasov equation to time $t=t_0+\Delta t$ and $\phi(t_0+\Delta t)$ is the corresponding total self-consistent potential. Expanding to linear order and dividing through by Δt gives

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \frac{1}{\mathcal{N}_e\mathcal{L}_p}\int\left\{\left[K+\frac{1}{2}q\phi_0(r;T,\omega_r)\right]\frac{\partial f}{\partial t}+\frac{1}{2}q\frac{\partial\phi}{\partial t}f_0\right\}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}, \tag{2.13a}$$

$$\frac{\mathrm{d}\langle p_\theta\rangle}{\mathrm{d}t} = \frac{1}{\mathcal{N}_e\mathcal{L}_p}\int p_\theta\frac{\partial f}{\partial t}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}, \tag{2.13b}$$

where, on the left-hand side, $\mathrm{d}/\mathrm{d}t = \dot{T}(\partial/\partial T)+\dot{\omega}_r(\partial/\partial\omega_r)$.

Integrals of the form

$$\int g(\boldsymbol{r},\boldsymbol{p})\frac{\partial f}{\partial t}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p} \tag{2.14}$$

can be simplified with the Vlasov equation,

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial\boldsymbol{r}}\cdot\frac{\partial H}{\partial\boldsymbol{p}}+\frac{\partial f}{\partial\boldsymbol{p}}\cdot\frac{\partial H}{\partial\boldsymbol{r}}, \tag{2.15}$$

where

$$H = K+q\phi_0(r;T,\omega)+q\delta\phi \tag{2.16}$$

is the full Hamiltonian. Inserting (2.15) into (2.14) and integrating by parts to pull derivatives off of f yields:

$$\int g(\boldsymbol{r},\boldsymbol{p})\frac{\partial f}{\partial t}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p} = \int f\left(\frac{\partial g}{\partial\boldsymbol{r}}\cdot\frac{\partial H}{\partial\boldsymbol{p}}+g\frac{\partial^2H}{\partial\boldsymbol{r}\cdot\partial\boldsymbol{p}}-\frac{\partial g}{\partial\boldsymbol{p}}\cdot\frac{\partial H}{\partial\boldsymbol{r}}-g\frac{\partial^2H}{\partial\boldsymbol{r}\cdot\partial\boldsymbol{p}}\right)\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}. \tag{2.17}$$

The second derivatives of H cancel and we proceed to evaluate the remaining expressions with $g=p_\theta$ and $g=K+\frac{1}{2}q\phi_0$.

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For $g = p_\theta$, we have

$$\int p_\theta \frac{\partial f}{\partial t} \mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{p} = \int f\left(-\frac{\partial p_\theta}{\partial \boldsymbol{p}}\cdot\frac{\partial H}{\partial \boldsymbol{r}}\right)\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p} = -\int f\frac{\partial H}{\partial\theta}\mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{p} = -\int f\frac{\partial(q\delta\phi)}{\partial\theta}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}. \quad (2.18)$$

For $g = K + \frac{1}{2}q\phi_0$, the terms $\partial K/\partial\boldsymbol{r}\cdot\partial K/\partial\boldsymbol{p}$ cancel. Furthermore, none of the potential terms depend on $\boldsymbol{p}$, so that only remaining terms are

$$\int f\left(\frac{1}{2}\frac{\partial(q\phi_0)}{\partial\boldsymbol{r}}\cdot\frac{\partial K}{\partial\boldsymbol{p}} - \frac{\partial K}{\partial\boldsymbol{p}}\cdot\frac{\partial(q\phi_0+q\delta\phi)}{\partial\boldsymbol{r}}\right)\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}. \quad (2.19)$$

Since ϕ_0 only has radial dependence, we find

$$\int\left(K+\frac{1}{2}q\phi_0\right)\frac{\partial f}{\partial t}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p} = -\int f\left(\frac{p_r}{2m_e}\frac{\partial(q\phi_0)}{\partial r} + \frac{\partial K}{\partial\boldsymbol{p}}\cdot\frac{\partial(q\delta\phi)}{\partial\boldsymbol{r}}\right)\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}. \quad (2.20)$$

To find a closed set of equations, we use linear theory to approximate the evolution of f and $\delta\phi$. With $f = f_0 + \delta f$ in (2.20), terms with only a single perturbed quantity vanish upon azimuthal integration because the equilibrium is azimuthally symmetric and the perturbations are linear combinations of non-zero azimuthal modes. The same argument applies to the $(\partial\phi/\partial t)f_0 = (\partial(\delta\phi)/\partial t)f_0$ term in (2.13*a*), which thus also vanishes. Terms that are the product of two perturbations (i.e. both $\delta\phi$ and δf) can be non-zero after azimuthal integration. There is also one term without any perturbations: $\int f_0((p_r/2m_e)(\partial(q\phi_0)/\partial r))\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}$; this terms vanishes because f_0 is Gaussian in p_r, making the full integrand odd in p_r.

Putting everything together, we have

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = -\frac{1}{\mathcal{N}_e\mathcal{L}_p}\int\delta f\frac{\partial K}{\partial\boldsymbol{p}}\cdot\frac{\partial(q\delta\phi)}{\partial\boldsymbol{r}}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}, \quad (2.21a)$$

$$\frac{\mathrm{d}\langle p_\theta\rangle}{\mathrm{d}t} = -\frac{1}{\mathcal{N}_e\mathcal{L}_p}\int\delta f\frac{\partial(q\delta\phi)}{\partial\theta}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}. \quad (2.21b)$$

Note that $\partial K/\partial\boldsymbol{p} = \partial\mathcal{H}_{\mathrm{sp}}/\partial\boldsymbol{p} = \dot{\boldsymbol{r}}$ since the potential term has no momentum dependence. Thus, $\partial K/\partial\boldsymbol{p}\cdot\partial(q\delta\phi)/\partial\boldsymbol{r} = \dot{r}(\partial(q\delta\phi)/\partial r) + \dot{z}(\partial(q\delta\phi)/\partial z) + \dot{\theta}(\partial(q\delta\phi)/\partial\theta)$. We neglect the first term because it is proportional to the inverse of the compression time scale ($\dot{r}\partial\delta\phi/\partial r \sim \delta\phi/\tau_{\mathrm{comp}} \ll \dot{\theta}\partial\delta\phi/\partial\theta \sim \omega_r\delta\phi$). We neglect the second term because it describes the heating of particles resonant with the RW wave as their distribution flattens in a process akin to nonlinear Landau damping, which will not occur assuming that thermalization is rapid enough to maintain thermal equilibrium during compression. Since the plasma rotates coherently in equilibrium, we have $\dot{\theta} = \omega_r =$ const. and we can move ω_r outside the integral.

Electrons cool via cyclotron emission and are heated by a thermal reservoir at temperature $T_{\mathrm{res.}}$. We account for this by subtracting the term $(1/\tau_R)(T - T_{\mathrm{res.}})$ in the EOM for $\langle E\rangle$, where $\tau_R = 9m_ec^3/8q^2\Omega_c^2$ is the time scale for cyclotron radiation (Beck, Fajans & Malmberg 1996).

Lastly, we add a phenomenological drag torque τ_{dr} and its corresponding heating or cooling term P_{dr}. If the drag torque can be modelled with a potential perturbation in a manner similar to the RW torque, then $P_{\mathrm{dr}} = \tau_{\mathrm{dr}}\omega_r$, but in more general cases (such as collisions with background gas particles deGrassie & Malmberg 1980), the two quantities are independent.

Our full system of equations is then

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \langle\tau\rangle\omega_r - \frac{1}{\tau_R}(T - T_{\mathrm{res.}}) + P_{\mathrm{dr}}, \tag{2.22a}$$

$$\frac{\mathrm{d}\langle p_\theta\rangle}{\mathrm{d}t} = \langle\tau\rangle + \tau_{\mathrm{dr}}, \tag{2.22b}$$

$$\langle\tau\rangle = -\frac{1}{\mathcal{N}_e\mathcal{L}_p}\int \delta f\frac{\partial(q\delta\phi)}{\partial\theta}\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}, \tag{2.22c}$$

where the last line introduces the average torque acting on the plasma.

We pause here to remark about the signs of various terms. Recall from our sign conventions in § 2.1 that electrons ($q < 0$) have $\omega_r > 0$ and $\langle p_\theta\rangle \sim qB\sum_i r_i^2 < 0$. For the plasma to compress, we need $\langle p_\theta\rangle$ to become less negative (smaller $\sum_i r_i^2$), which requires a positive torque according to (2.22*b*). A positive torque and rotation frequency in turn means the first term of (2.22*a*) is positive, leading to energy gain and heating. This is the expected behaviour since the RW torque does work on the plasma to compress it. For $q > 0$ (e.g. positrons), the signs are reversed: $\omega_r < 0$, $\langle p_\theta\rangle > 0$, $\langle\tau\rangle < 0$, but as expected, the first term of (2.22*a*) remains positive since the RW still does work to compress the plasma.

Next, we rewrite (2.22*a*) and (2.22*b*) using the dimensionless variables

$$D = 1 - \frac{\omega_r}{\omega_{\mathrm{RW}}}, \tag{2.23a}$$

$$\Theta = \frac{T}{m_e v_{\phi,0}^2}, \tag{2.23b}$$

$$\hat{t} = \frac{1}{\tau_R}t, \tag{2.23c}$$

where ω_{RW} is the angular rotation frequency of the rotating wall (the sign of ω_{RW} will depend on the direction of rotation, but to compress electrons, we need $\omega_{\mathrm{RW}} > 0$), $v_{\phi,0} = |\omega_{\mathrm{RW}}|/k_0$ is the phase velocity of the fundamental axial RW mode and

$$k_0 = \frac{2\pi}{\mathcal{L}_p} \tag{2.24}$$

is the wavenumber of the fundamental mode. The variable D is a normalized detuning between the plasma rotation and RW frequencies. Variable D is bounded above by unity so long as angular velocity of the plasma is parallel to that of the RW. Variable D is bounded below by 0 so long as $|\omega_r| < |\omega_{\mathrm{RW}}|$, which is the case analysed here. When $D \approx 0$, the plasma has (practically) fully compressed because $\omega_r \approx \omega_{\mathrm{RW}}$. The variable $\Theta = T/m_e v_{\phi,0}^2$ is the ratio between the thermal energy of an electron and the kinetic energy of an electron that is in-phase with the fundamental RW mode. The variables D and Θ are in bijective correspondence with $\langle\mathcal{H}\rangle$ and $\langle p_\theta\rangle$ as we show in the next section. Hereafter, we take D and Θ to be the state variables.

In terms of these dimensionless variables, our EOM governing the energy and angular momentum dynamics of the plasma are

$$\begin{bmatrix} \dot{\Theta} \\ \dot{D} \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} (1-D)\hat{\tau}(D,\Theta) - (\Theta - \Theta_{res.}) + \hat{P}_{dr} \\ \hat{\tau} + \hat{\tau}_{dr} \end{bmatrix}, \tag{2.25}$$

where

$$\Theta_{res.} = \frac{T_{res.}}{m_e v_{\phi,0}^2}, \tag{2.26}$$

$$\hat{P}_{dr} = \frac{\tau_R}{m_e v_{\phi,0}^2} P_{dr}, \tag{2.27}$$

$$\hat{\tau} = \left(\frac{\tau_R \omega_{RW}}{m_e v_{\phi,0}^2}\right) \langle \tau \rangle, \tag{2.28}$$

with the last line defining the dimensionless torque (and a similar relation holds between $\hat{\tau}_{dr}$ and τ_{dr}). The partial derivative matrix needed to change variables is

$$\mathcal{M} = \begin{bmatrix} \dfrac{\partial \hat{E}}{\partial \Theta} & \dfrac{\partial \hat{E}}{\partial D} \\ \dfrac{\partial \hat{p}_\theta}{\partial \Theta} & \dfrac{\partial \hat{p}_\theta}{\partial D} \end{bmatrix}, \tag{2.29}$$

where

$$\hat{E} = \frac{\langle E \rangle}{m_e v_{\phi,0}^2}, \tag{2.30}$$

$$\hat{p}_\theta = \frac{\langle p_\theta \rangle \omega_{RW}}{m_e v_{\phi,0}^2}. \tag{2.31}$$

To arrive at a closed set of equations, we (i) express $\hat{E}$, $\hat{p}_\theta$ and their partial derivatives in terms of Θ and D, and (ii) explicitly evaluate the perturbations $\delta\phi$ and δf and substitute them into the torque (2.22*c*).

2.2. *Expressions for the energy and angular momentum*

We begin with the expression for energy in equilibrium:

$$\langle E \rangle (T, \omega_r) = \frac{1}{\mathcal{N}_e \mathcal{L}_p} \int \left[K + \frac{1}{2} q \phi_0(r) \right] f_0(r, \boldsymbol{p}) \, \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{p}. \tag{2.32}$$

Completing the square on p_θ in the exponent of f_0, (2.3) allows us to perform the momentum integrals as simple Gaussian expectations. We take advantage of the usual Gaussian expectation results $\langle p_r^2 \rangle_G / 2m_e = \langle p_z^2 \rangle_G / 2m_e = \langle (p_\theta - \langle p_\theta \rangle)^2 \rangle_G / 2m_e r^2 = T/2$ and $\langle p_\theta \rangle_G = \frac{1}{2} m_e (\Omega_c + 2\omega_r) r^2$, where the G subscript reminds us that we average over

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a Gaussian distribution here. Simplifying the resulting expression leaves

$$\langle E\rangle = \frac{3T}{2} + \frac{\displaystyle\int \left(\frac{1}{2}m_e\omega_r^2 r^2 + \frac{1}{2}q\phi_0\right) \mathrm{e}^{\psi}\, r\,\mathrm{d}r}{\displaystyle\int \mathrm{e}^{\psi}\, r\,\mathrm{d}r}, \tag{2.33}$$

where ψ was introduced in (2.6), and we performed the trivial θ and z integrals.

We now introduce the Debye length, normalized radial coordinate, ratio of plasma rotation to cyclotron frequency and a convenient variable $\varDelta$:

$$\lambda_D^2 = \frac{T}{4\pi q^2 n_0(r=0; T, \omega_r)}, \tag{2.34a}$$

$$\rho \equiv r/\lambda_D, \tag{2.34b}$$

$$\chi \equiv \omega_r/\varOmega_c, \quad |\chi| \ll 1, \tag{2.34c}$$

$$\varDelta \equiv \frac{2\omega_r(\varOmega_c + \omega_r)m_e\lambda_D^2}{T} = -(1+\delta). \tag{2.34d}$$

The normalization $\int n_0\, \mathrm{d}^3\boldsymbol{r} = N_e \implies 2\pi n_0(r=0)\int \mathrm{e}^{\psi}\, r\,\mathrm{d}r = \mathcal{N}_e$ can be rewritten with the help of the Debye length and normalized radial coordinate

$$\int \mathrm{e}^{\psi}\,\rho\,\mathrm{d}\rho = 2q^2\mathcal{N}_e/T. \tag{2.35}$$

Substituting all of these in (2.33) and exchanging the remaining ϕ_0 in favour of ψ gives

$$\langle E\rangle = \frac{3T}{2} + \frac{q\phi_0(r=0; T, \omega_r)}{2} + \frac{T^2}{4q^2\mathcal{N}_e}\left(-\int \psi\,\mathrm{e}^{\psi}\,\rho\,\mathrm{d}\rho + \frac{\varDelta}{4}\left[\frac{1+3\chi}{1+\chi}\right]\int \rho^3\,\mathrm{e}^{\psi}\,\mathrm{d}\rho\right). \tag{2.36}$$

To calculate the potential at the centre of the plasma column, note that the electric field can be written as

$$E(r) = \frac{4\pi q}{r}\int_0^r n_0(r')r'\,\mathrm{d}r'. \tag{2.37}$$

Setting $\phi_0(R_W) = 0$ (conducting wall at the boundary), we have

$$\phi_0(r=0; T, \omega_r) = \int_0^{R_w} \frac{4\pi q}{r}\int_0^r n_0(r')r'\,\mathrm{d}r'\,\mathrm{d}r. \tag{2.38}$$

This can be rewritten in terms of ψ and ρ as

$$\frac{q\phi_0(r=0; T, \omega_r)}{2} = \frac{T}{2}\int_0^{\rho_w}\frac{\mathrm{d}\rho}{\rho}\int_0^{\rho} \exp(\psi(\rho'))\rho'\,\mathrm{d}\rho', \tag{2.39}$$

where $\rho_w = R_w/\lambda_D$. This completes our expression for the energy in terms of T and ω_r (and hence in terms of $\varTheta$ and D). The remaining implicit dependence of ψ on T can be uncovered by searching for δ such that the solution of the Poisson equation (2.7) satisfies the normalization $\int \mathrm{e}^{\psi}\,\rho\,\mathrm{d}\rho = 2q^2\mathcal{N}_e/T$.

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The calculation of the angular momentum proceeds in a similar manner. We can take advantage of our previous result for the expectation of p_θ under Gaussian integration:

$$\langle p_\theta \rangle = \frac{\displaystyle\int \frac{1}{2} m_e(\Omega_c + 2\omega_r) r^2 \, \mathrm{e}^{\psi} \, r \, \mathrm{d}r}{\displaystyle\int \mathrm{e}^{\psi} \, r \, \mathrm{d}r}. \tag{2.40}$$

Substituting our dimensionless variables gives

$$\langle p_\theta \rangle = \frac{T^2 \varDelta}{8q^2 \mathcal{N}_e} \frac{1 + 2\chi}{\Omega_c \chi (1+\chi)} \int \rho^3 \, \mathrm{e}^{\psi} \, \mathrm{d}\rho. \tag{2.41}$$

Our final expressions for the dimensionless energy and angular momentum are thus

$$\begin{aligned} \hat{E} &= \frac{3}{2}\Theta + \frac{1}{2}\Theta \int_0^{\rho_w} \frac{\mathrm{d}\rho}{\rho} \int_0^{\rho} \exp(\psi(\rho'))\rho' \, \mathrm{d}\rho' \\ &\quad + \frac{\alpha \Theta^2}{4} \left(- \int \psi \, \mathrm{e}^{\psi} \rho \, \mathrm{d}\rho + \frac{\varDelta}{4} \left[\frac{1+3\chi}{1+\chi} \right] \int \rho^3 \, \mathrm{e}^{\psi} \, \mathrm{d}\rho \right), \end{aligned} \tag{2.42a}$$

$$\hat{p} = \frac{\alpha \Theta^2}{8} \frac{\varDelta}{1 - D} \frac{1+2\chi}{1+\chi} \int \rho^3 \, \mathrm{e}^{\psi} \, \mathrm{d}\rho, \tag{2.42b}$$

where $\alpha \equiv m v_{\phi,0}^2 / q^2 \mathcal{N}_e$. As noted above, $\varDelta$ and Θ are implicitly linked by the Poisson equation. Equation (2.42) can be simplified further using the assumption $|\chi| \ll 1$, as is typically the case in experiments.

Despite the seeming complexity of the expressions (2.42) in dimensionless variables, the underlying physical quantities are straightforward averages over the plasma. So the average energy $\langle E \rangle$ (see (2.33)) is a sum of the average thermal energy $3T/2$, the average rotational kinetic energy $\langle \frac{1}{2} m_e \omega_r r^2 \rangle$ and the self-energy $\langle \frac{1}{2} q \phi_0 \rangle$. The average angular momentum $\langle p_\theta \rangle$ (see (2.40)) is a sum of the kinetic angular momentum $\langle m_e \omega_r r^2 \rangle$ and the magnetic field correction $(qB/2c)\langle r^2 \rangle$. The evolution is driven by the average torque $\langle \tau \rangle$, which is just the average of $\boldsymbol{r} \times \boldsymbol{F} = -q\boldsymbol{r} \times \boldsymbol{\nabla}\phi = -q(\partial \phi / \partial \theta)\hat{z}$.

We now turn to the final piece of our model: the derivation of the torque.

2.3. *Derivation of the rotating wall torque*

The RW torque is found by first solving for δf and $\delta \phi$, and then inserting them into (2.22*c*). In the previous section, the Vlasov distribution on the full six-dimensional phase space was for the thermodynamic equations. In this section, we take advantage of the small Larmor radius and use the guiding centre approximation. Our distribution will thus be over a four-dimensional phase space (r, θ, z, v_z), with the first three variables denoting the position of the guiding centre and the last being the guiding centre velocity along the magnetic field. See Lifshitz & Pitaevski (1981) for details on transitioning from the particle to the guiding centre phase space. From now on, all variables are assumed to refer to the guiding centre. The full distribution is

$$f(t, r, \theta, z, v_z) = f_0(r, v_z; \Theta, D) + \delta f(t, r, \theta, z, v_z). \tag{2.43}$$

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The equilibrium distribution function, $f_0(r, v_z)$, can be written as the equilibrium density, $n_0(r; \Theta, D)$, multiplied by a Maxwellian velocity distribution,

$$f_0(r, v_z; \Theta, D) = n_0\,(r; \Theta, D)\,\frac{1}{\sqrt{2\pi v_{th}^2}}\exp\left[-\frac{1}{2}\left(\frac{v_z}{v_{th}}\right)^2\right], \tag{2.44}$$

with thermal velocity $v_{th} = v_{\phi,0}\sqrt{\Theta} = \sqrt{T/m_e}$.

The distribution function, $f(t, r, \theta, z, v_z)$, evolves according to the drift kinetic equation (DKE) (Dubin & O'Neil 1999; Lifshitz & Pitaevski 1981),

$$\frac{\partial f}{\partial t} + v_z\frac{\partial f}{\partial z} + \frac{c}{|\boldsymbol{B}|^2}\,(\boldsymbol{E}\times\boldsymbol{B})\cdot\nabla_\perp f - \frac{q}{m_e}\frac{\partial\phi}{\partial z}\frac{\partial f}{\partial v_z} = \epsilon\delta f, \tag{2.45}$$

where ϵ is an effective collision frequency, δf is the perturbation of the distribution function, and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the total electric and magnetic fields. Note that $\boldsymbol{E} = -\nabla\phi(t, r, \theta, z)$ includes the RW potential.

We solve the linearized DKE together with the linearized Poisson equation

$$\nabla^2\delta\phi(t, r, \theta, z) = -4\pi q\delta n(t, r, \theta, z), \tag{2.46}$$

where

$$\delta n = \int \delta f\,\mathrm{d}v_z, \tag{2.47}$$

by expanding in Fourier modes:

$$\delta f(t, r, \theta, z, v_z) = \sum_{m,l,s} a_{m,l,s}\delta f_{m,l,s}(r, v_z)\exp\left(\mathrm{i}\left(mk_0z + l\theta - st\omega_{\mathrm{RW}}\right)\right), \tag{2.48a}$$

$$\delta n(t, r, \theta, z) = \sum_{m,l,s} a_{m,l,s}\delta n_{m,l,s}(r)\exp\left(\mathrm{i}\left(mk_0z + l\theta - st\omega_{\mathrm{RW}}\right)\right), \tag{2.48b}$$

$$\delta\phi(t, r, \theta, z) = \sum_{m,l,s} a_{m,l,s}\delta\phi_{m,l,s}(r)\exp\left(\mathrm{i}\left(mk_0z + l\theta - st\omega_{\mathrm{RW}}\right)\right). \tag{2.48c}$$

The mode functions $\delta\phi_{m,l,s}(r)$ satisfy the boundary condition $\delta\phi_{m,l,s}(R_w) = 1$. For convenience, we chose the same expansion coefficients $a_{m,l,s}$ for all three of $\delta\phi$, δf and δn. The mode functions $\delta f_{m,l,s}$ and $\delta n_{m,l,s}$ are then uniquely determined by $\delta\phi_{m,l,s}$ as we show below. The coefficients $a_{m,l,s}$ are determined by requiring that the RW potential at the trap radius satisfies

$$\delta\phi(R_w) = V(z)\cos\left(\theta - \omega_{\mathrm{RW}}t\right), \tag{2.49}$$

where $V(z) = \Phi_0$ on the RW electrodes and $V(z) = 0$ elsewhere. We picked this form to simplify our calculations, although one could modify our model to include the sectored nature of RW apparatuses. As such, only the $l = s = \pm 1$ modes are non-zero in our model. The coefficients, $a_{m,l,s}$, are thus given by the usual relation $a_{m,\pm1,\pm1} = (1/2\mathcal{L}_p)\int\cos(mk_0z)V(z)\,\mathrm{d}z$. Because $V(z)$ is a step function, the coefficients are

$$a_{m,\pm1,\pm1} = \Phi_0\frac{\sin\left(mk_0\mathcal{L}_e/2\right)}{mk_0\mathcal{L}_p}, \tag{2.50}$$

where $\mathcal{L}_e = 2L_e$ is twice the physical electrode length.

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Returning to the DKE, we insert the Fourier decompositions and linearize to find

$$\delta f_{m,l,s}(r, v_z) = \frac{q}{m_e v_{\mathrm{th}}^2}\left(\frac{mk_0 v_z}{s\omega_{\mathrm{RW}} - l\omega_{E\times B} - mk_0 v_z - \mathrm{i}\epsilon}\right) f_0(r, v_z; \Theta, D)\delta\phi_{m,l,s}(r). \tag{2.51}$$

In (2.51), we dropped the diamagnetic drift, $(m_e v_{\mathrm{th}}^2/q)(lc/rB)(\partial \ln n_0(r; \Theta, D)/\partial r)$, which is one of the terms arising from $\nabla_\perp f(t, r, \theta, z, v_z)$ in (2.45). This term is small for the regime of interest here (see, for example, Section IIB of Dubin & O'Neil 1999). In the absence of any diamagnetic drift,

$$\omega_r = \omega_{E\times B} = \frac{v_{E\times B}}{r} = \frac{|\boldsymbol{E}\times\boldsymbol{B}|}{B^2 r}c, \tag{2.52}$$

and we use these interchangeably in the rest of the paper.

The density perturbation, $\delta n_{m,l,s}(r)$, is obtained by integrating $\delta f_{m,l,s}(r, v_z)$ over the axial velocity v_z. Using the plasma dispersion function Z (Fried & Conte 2015), we have

$$\delta n_{m,l,s} = -\frac{q}{T} n_0\,(r; T, \omega_r)\,\delta\phi_{m,l,s}\left[1 + \frac{v_{\mathrm{res}}}{v_{\mathrm{th}}\sqrt{2}} Z\left(\frac{v_{\mathrm{res}}}{v_{\mathrm{th}}\sqrt{2}}\right)\right], \tag{2.53}$$

where the resonance between the rotating wall and the combined electron rotational and axial motion is given by

$$v_{\mathrm{res}} = \frac{s\omega_{\mathrm{RW}} - l\omega_{E\times B}}{mk_0}. \tag{2.54}$$

Inserting this density perturbation into the linearized Poisson equation (2.46) yields an equation for $\delta\phi_{m,l,s}$:

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \left(\frac{l^2}{r^2} + (mk_0)^2 + \frac{4\pi q^2 n_0\,(r; \Theta, D)}{T}\right.\right.$$
$$\left.\left.\times\left[1 + \frac{v_{\mathrm{res}}}{v_{\mathrm{th}}\sqrt{2}} Z\left(\frac{v_{\mathrm{res}}}{v_{\mathrm{th}}\sqrt{2}}\right)\right]\right)\right\}\delta\phi_{m,l,s} = 0. \tag{2.55}$$

This is a homogeneous Bessel-like equation with a source term proportional to density manifesting a Debye screening effect.

The solution to this equation together with the relationship between $\delta n_{m,l,s}$ and $\delta\phi_{m,l,s}$ is sufficient to evaluate the torque, but further simplifications can be made. Writing

$$\delta n_{m,l,s}(r) = b_{m,l,s}(r)\delta\phi_{m,l,s(r)}, \tag{2.56}$$

where $b_{m,l,s}$ is defined by (2.53), the torque (2.22c) becomes

$$\begin{aligned}\langle\tau\rangle &= -\frac{q}{\mathcal{L}_p\mathcal{N}_e}\int \delta n \frac{\partial\delta\phi(t, r, \theta, z)}{\partial\theta} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\ &= -\frac{2\pi q}{\mathcal{N}_e}\sum_{m,l,s} il \int b^*_{m,l,s}(r)|a_{m,l,s}\delta\phi_{m,l,s}(r)|^2 r\,\mathrm{d}r \\ &= -\frac{8\pi q}{\mathcal{N}_e}\sum_{m>0}|a_{m,1,1}|^2\int \Im\left[b_{m,1,1}\right]|\delta\phi_{m,1,1}(r)|^2 r\,\mathrm{d}r.\end{aligned} \tag{2.57}$$

In going from the second to the third line of (2.57), we summed over $s = l = \pm 1$, restricted the wavenumber sum to positive m and used the fact that $\Im[b_{m,1,1}] = -\Im[b_{m,-1,-1}]$ (see below).

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To explicitly evaluate $\Im[b_{m,1,1}]$, express the plasma dispersion function using the Sokhotski–Plemelj formula

$$\delta n_{m,l,s}(r) = P\int \frac{g(r, v_z)\delta\phi_{m,l,s}(r)}{v_z - \left(\dfrac{s\omega_{\mathrm{RW}} - \ell\omega_{E\times B}}{mk_0}\right)}\mathrm{d}v_z - \mathrm{i}\pi\delta\phi_{m,l,s}g\left(r, v_z = \frac{s\omega_{\mathrm{RW}} - \ell\omega_{E\times B}}{mk_0}\right), \tag{2.58}$$

where P denotes the Cauchy principal value and

$$g(r, v_z) = \frac{q}{m_e}\frac{\partial f_0}{\partial v_z} = -\frac{qv_z}{m_e v_{th}^2} f_0(r, v_z; \varTheta, D). \tag{2.59}$$

We can now read off $b_{m,l,s}(r)$ from (2.58):

$$b_{m,l,s} = P\int \frac{g(r, v_z)(r)}{v_z - \left(\dfrac{s\omega_{\mathrm{RW}} - \ell\omega_{E\times B}}{mk_0}\right)}\mathrm{d}v_z - \mathrm{i}\pi g\left(r, v_z = \frac{s\omega_{\mathrm{RW}} - \ell\omega_{E\times B}}{mk_0}\right), \tag{2.60}$$

from which we see

$$\Im[b_{m,1,1}] = -\pi g\left(r, v_z = \frac{\omega_{\mathrm{RW}} - \omega_{E\times B}}{mk_0}\right), \tag{2.61}$$

$$\Im[b_{m,-1,-1}] = -\pi g\left(r, v_z = -\frac{\omega_{\mathrm{RW}} - \omega_{E\times B}}{mk_0}\right). \tag{2.62}$$

Since $g(r, v_z)$ is an odd function of v_z, we have $\Im[b_{m,1,1}] = -\Im[b_{m,-1,-1}]$ as stated above.

Recalling that $f_0(\boldsymbol{r}, \boldsymbol{v}; \varTheta, D)$ has the Maxwellian velocity distribution given by (2.44), we obtain the desired expression for the dimensionless torque

$$\hat{\tau}\,(D, \varTheta) = -4\eta\left(\frac{D}{\varTheta^{3/2}}\right)\sum_{m>0}\hat{\tau}_m, \tag{2.63a}$$

$$\hat{\tau}_m = \frac{1}{m}\exp\left[-\frac{1}{2m^2}\frac{D^2}{\varTheta}\right]\left|\frac{a_{m,1,1}}{\varPhi_0}\right|^2\langle\delta\phi_{m,1,1}^2\rangle, \tag{2.63b}$$

where we have defined the dimensionless torque strength

$$\eta = \sqrt{\frac{\pi}{2}}\left(\frac{q\varPhi_0}{m_e v_{\phi,0}^2}\right)^2 \tau_R\omega_{\mathrm{RW}}, \tag{2.64}$$

and introduced the notation

$$\langle\delta\phi_{m,1,1}^2\rangle = \frac{2\pi}{\mathcal{N}_e}\int n_0(r)|\delta\phi_{m,1,1}|^2 r\,\mathrm{d}r = \frac{\displaystyle\int n_0(r)|\delta\phi_{m,1,1}|^2 r\,\mathrm{d}r}{\displaystyle\int n_0(r) r\,\mathrm{d}r}. \tag{2.65}$$

Now it is straightforward to substitute the solution of the Poisson equation (2.55) into the torque. This completes our model.

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3. Discussion and conclusion

We have developed a model for RW compression of a pure electron plasma under the assumption the plasma slowly evolves through states of thermal equilibrium. This allowed us to avoid explicitly tracking non-equilibrium dynamics and collisions, yielding tractable equations suitable for numerical study. We made several additional approximations along the way. By utilizing linearized Vlasov theory in our derivations, we have assumed that any nonlinear Vlasov effects will be muted by re-equilibration due to collisions. We have neglected finite-length effects under the assumption that the axial motion will not be significantly impacted, which limits our analysis to long plasma columns. The validity of our approximations can be tested through detailed numerical studies and experimental comparisons, which we leave for future work.

One aspect of our model that can already be validated against experiments without numerical simulations is the lack of compression beyond $\omega_r = \omega_{\mathrm{RW}}$. That is, as the plasma rotation starts catching up to the rotating wall ($\omega_r \rightarrow \omega_{\mathrm{RW}}$), our torque (2.63*a*) vanishes since $D \rightarrow 0$. This sets a fundamental limit on our compression at a fixed RW frequency, even in the absence of drag torques, and has been verified experimentally (Danielson & Surko 2006). While this suggests our model is capable of matching the final state of plasma experiments, it remains to validate the details of the time evolution and parameter dependencies predicted by the model.

Assuming our model proves sufficiently accurate for experimental purposes, its relative simplicity lends itself to further analytical approximations and numerical solutions. This, in turn, could allow for compression optimization and real-time guidance to experimentalists. We leave such considerations to future work.

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Declaration of interests

The authors report no conflict of interest.

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