THE JOURNAL OF SYMBOLIC LOGIC
Volume 88, Number 4, December 2023

CONSERVATION THEOREMS ON SEMI-CLASSICAL ARITHMETIC

MAKOTO FUJIWARA AND TAISHI KURAHASHI

Abstract. We systematically study conservation theorems on theories of semi-classical arithmetic, which lie in-between classical arithmetic PA and intuitionistic arithmetic HA. Using a generalized negative translation, we first provide a structured proof of the fact that PA is Π_{k+2}-conservative over HA + Σ_k-LEM where Σ_k-LEM is the axiom scheme of the law-of-excluded-middle restricted to formulas in Σ_k. In addition, we show that this conservation theorem is optimal in the sense that for any semi-classical arithmetic T, if PA is Π_{k+2}-conservative over T, then T proves Σ_k-LEM. In the same manner, we also characterize conservation theorems for other well-studied classes of formulas by fragments of classical axioms or rules. This reveals the entire structure of conservation theorems with respect to the arithmetical hierarchy of classical principles.

§1. Introduction. It is well-known that classical first-order arithmetic PA is Π_2-conservative over intuitionistic first-order arithmetic HA. There are several approaches to prove this fundamental fact. One simple and well-known approach is to apply the negative (or double negation) translation followed by the Friedman A-translation [4]. Another possible approach is to apply a generalized negative translation developed systematically by Ishihara [9, 10]. In fact, the latter is a combination of Gentzen's negative translation and the Friedman A-translation (cf. [10, Section 4]). In [7, Theorem 6.14], the authors showed a conservation result which generalizes the aforementioned conservation result on PA and HA in the context of semi-classical arithmetic (which lies between classical and intuitionistic arithmetic). In fact, the following is an immediate corollary of [7, Theorem 6.14]:

PROPOSITION 1.1. PA *is* Π_{k+2}*-conservative over* HA + Σ_k-LEM *where* Σ_k-LEM *is the axiom scheme of the law-of-excluded-middle restricted to formulas in* Σ_k.[1]

The proof of [7, Theorem 6.14] in that paper is similar to the former approach in the sense of using the Friedman A-translation. However, the proof has somewhat intricate structure in dealing with the Friedman A-translation of the inner part of Kuroda's negative translation. In Section 3, by extending the latter approach from [9, 10] in the context of semi-classical arithmetic, we provide a much more structured proof of [7, Theorem 6.14]. As an advantage of the structured proof, we obtain an

Received July 25, 2021.
2020 *Mathematics Subject Classification.* 03B20, 03F03, 03F30, 03F50.
Key words and phrases. conservation theorems, semi-classical arithmetic, intuitionistic arithmetic.

[1]At the very end of the revision process of the present paper, the authors found that Proposition 1.1 has been shown by Stefano Berardi with some specific use of the generalized negative translation in the following paper: S. BERARDI, A generalization of a conservativity theorem for classical versus intuitionistic arithmetic. *Mathematical Logic Quarterly*, vol. 50 (2004), no. 1, pp. 41–46.

0022-4812/23/8804-0008
DOI:10.1017/jsl.2022.25

extended conservation result for much larger classes of formulas (see Theorem 3.17 and Remark 3.18).

In Section 4, we relate the classes used in Section 3 (which are based on the classes introduced in [9]) to the classes U_k and E_k introduced in Akama et al. [1] for studying the hierarchy of the constructively-meaningful fragments of classical axioms (including the law-of-excluded-middle and the double-negation-elimination). The classes E_k and U_k correspond to classical Σ_k and Π_k respectively in the sense that every formula in E_k (resp. U_k) is equivalent over PA to some formula in Σ_k (resp. Π_k) and vice versa. This investigation reveals that our extended conservation theorem for $\mathsf{HA} + \Sigma_k$-LEM is applicable to all formulas in E_{k+1} (see Corollary 4.3).

In Sections 5–7, we investigate the entire structure of conservation theorems in the arithmetical hierarchy of classical principles which was systematically studied first in Akama et al. [1] and further extended by the authors recently in [6]. The first motivation of this investigation comes from the observation that for any semi-classical arithmetic T such that PA is Π_{k+2}-conservative over T, T proves Σ_k-LEM (cf. Lemma 5.5). This means that Proposition 1.1 is optimal in the sense that one cannot replace $\mathsf{HA} + \Sigma_k$-LEM by any semi-classical arithmetic which does not prove Σ_k-LEM. Another motivating fact is that for any semi-classical arithmetic T, PA is Π_2-conservative over T if and only if T is closed under Markov's rule for primitive recursive predicate (cf. [14, Section 3.5.1]). Thus the Π_2-conservativity is also characterized by the Σ_1-fragment of the double-negation-elimination rule. Then it is natural to ask whether this can be relativized in the context of semi-classical arithmetic. Motivated by these facts, in Sections 5 and 6, we study the conservation theorems for the well-studied classes (including Π_k, Σ_k, the classes in [1] and their closed variants) and characterize them by fragments of classical axioms or rules. The conservativity for a class of formulas is equivalent to that restricted only to sentences if the class is closed under taking a universal closure. Then the strength of the conservativity e.g., Π_k does not vary even if we restrict them only to sentences. On the other hand, since Σ_k etc. are not closed under taking a universal closure, this is not the case for such classes. We investigate the conservation theorems for classes of formulas in Section 5 and those for sentences in Section 6. Through a lot of delicate arguments in semi-classical arithmetic, we reveal the detailed structure consisting of the conservation theorems and some fragments of logical principles, which are summarized in Section 7. This exhaustive investigation shed light on the close connection between the notion of conservativity and classical axioms and rules in semi-classical arithmetic. For the purpose of future use, we present our characterization results in a generalized form with adding a set X of sentences into the theories in question.

In the end of this paper, as an appendix, we show the relativized soundness theorem of the Friedman A-translation for $\mathsf{HA} + \Sigma_k$-LEM. By this relativized soundness theorem, one may obtain a simple proof of Proposition 1.1 just by imitating the aforementioned Friedman's approach.

§2. Framework. We work with a standard formulation of intuitionistic arithmetic HA described e.g., in [13, Section 1.3], which has function symbols for all primitive recursive functions. Our language contains all the logical constants $\forall, \exists, \to, \wedge, \vee$ and $\bot$. In our proofs, when we use some principle (including induction hypothesis

[I.H.]) which is not available in HA, it will be exhibited explicitly. As regards basic reasoning over intuitionistic first-order logic, we refer the reader to [3, Section 6.2].

Throughout this paper, let k be a natural number (possibly 0). The classes Σ_k and Π_k of HA-formulas are defined as follows:

- Let Σ_0 and Π_0 be the set of all quantifier-free formulas;
- $\Sigma_{k+1} := \{\exists x_1, \dots, x_n\, \varphi \mid \varphi \in \Pi_k\}$;
- $\Pi_{k+1} := \{\forall x_1, \dots, x_n\, \varphi \mid \varphi \in \Sigma_k\}$.

Let $\mathrm{FV}(\varphi)$ denote the set of all free variables in φ. Note that every formula φ in Σ_{k+1} (resp. Π_{k+1}) is equivalent over HA to some formula ψ in Σ_{k+1} (resp. Π_{k+1}) such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$ and ψ is of the form $\exists x \psi'$ (resp. $\forall x \psi'$) where ψ' is Π_k (resp. Σ_k). For convenience, we assume that Σ_m and Π_m denote the empty set for negative integers m.

The classical variant PA of HA is defined as HA + LEM or HA + DNE, where LEM is the axiom scheme of the law-of-excluded-middle $\varphi \vee \neg\varphi$ and DNE is that of the double-negation-elimination $\neg\neg\varphi \to \varphi$. Recall that Σ_k-LEM and Σ_k-DNE are LEM and DNE restricted to formulas in Σ_k (possibly containing free variables) respectively. Similarly, Π_k-LEM and Π_k-DNE are defined for Π_k. We call a theory T such that $\mathrm{HA} \subseteq T \subseteq \mathrm{PA}$ *semi-classical arithmetic*.

Unless otherwise stated, the inclusion between classes of HA-formulas is to be understood modulo equivalences over HA. That is, for classes Γ and Γ' of HA-formulas, $\Gamma \subseteq \Gamma'$ denotes that for all $\varphi \in \Gamma$, there exists $\varphi' \in \Gamma'$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and $\mathrm{HA} \vdash \varphi' \leftrightarrow \varphi$, and $\Gamma = \Gamma'$ denotes $\Gamma \subseteq \Gamma'$ and $\Gamma' \subseteq \Gamma$. In this sense, one may think of Σ_k and Π_k as sub-classes of $\Sigma_{k'}$ and $\Pi_{k'}$ for all $k' > k$ (see [7, Remark 2.5]).

§3. A relativization of Ishihara's conservation result in semi-classical arithmetic.

In this section, we simulate Ishihara's proof of [9, Theorem 10] in the specific context of semi-classical arithmetic studied in [1, 7] with some additional arguments. We first recall the translation studied in [9]. In the context of the translation, without otherwise stated, we work in the language with an additional predicate symbol \$ of arity 0, which behaves as "place holder" (see [9, 10] for more information). Let $\mathrm{HA}^\$$ denote HA in that language. On the other hand, $\mathrm{HA}^\$ + \Sigma_k$-LEM denotes $\mathrm{HA}^\$$ augmented with Σ_k-LEM for "HA"-formulas.

Definition 3.1 (cf. [9, Definition 3]). Let $\neg_\$\varphi$ denote $\varphi \to \$$. For each formula φ, its \$-translation $\varphi^\$$ is defined inductively by the following clauses:

- For P prime such that $P \not\equiv \bot$, $P^\$:\equiv \neg_\$\neg_\$ P$;
- $\bot^\$:\equiv \$$;
- $(\varphi_1 \circ \varphi_2)^\$:\equiv \varphi_1^\$ \circ \varphi_2^\$$ for $\circ \in \{\wedge, \to\}$;
- $(\varphi_1 \vee \varphi_2)^\$:\equiv \neg_\$\neg_\$\left(\varphi_1^\$ \vee \varphi_2^\$\right)$;
- $(\forall x \varphi)^\$:\equiv \forall x \varphi^\$$;
- $(\exists x \varphi)^\$:\equiv \neg_\$\neg_\$ \exists x \varphi^\$$.

It is straightforward to see $\mathrm{FV}(\varphi) = \mathrm{FV}\left(\varphi^\$\right)$.

Proposition 3.2 (cf. [9, Proposition 4] and [10, Section 4]).

1. *For any* HA-*formula* φ, $\mathrm{HA}^\$ \vdash \neg_\$\neg_\$\varphi^\$ \leftrightarrow \varphi^\$$;

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

2. *For any* HA-*formula* φ *and any set* X *of* HA-*sentences, if* $\mathsf{PA} + X \vdash \varphi$, *then* $\mathsf{HA}^{\$} + X^{\$} \vdash \varphi^{\$}$, *where* $X^{\$} := \{\psi^{\$} \mid \psi \in X\}$.

PROOF. The proofs are routine: One can show (1) by induction on the structure of formulas, and (2) by induction on the length of the proof of φ in $\mathsf{PA} + X$. ⊣

COROLLARY 3.3. *For any* HA-*formulas* φ_1 *and* φ_2, *if* $\mathsf{PA} \vdash \varphi_1 \leftrightarrow \varphi_2$, *then* $\mathsf{HA}^{\$} \vdash \varphi_1^{\$} \leftrightarrow \varphi_2^{\$}$.

PROOF. If PA proves $\varphi_1 \leftrightarrow \varphi_2$, by Proposition 3.2.(2), we have that $\mathsf{HA}^{\$}$ proves $(\varphi_1 \leftrightarrow \varphi_2)^{\$}$, which is in fact $\varphi_1^{\$} \leftrightarrow \varphi_2^{\$}$. ⊣

LEMMA 3.4. *For a quantifier-free formula* φ_{qf} *of* HA, $\mathsf{HA}^{\$}$ *proves* $\varphi_{\mathrm{qf}}^{\$} \leftrightarrow \varphi_{\mathrm{qf}} \vee \$$.

PROOF. By induction on the structure of quantifier-free formulas of HA.

The case of $\perp$: Since $\perp^{\$} \equiv \$$, we have trivially $\mathsf{HA}^{\$} \vdash \perp^{\$} \leftrightarrow \perp \vee \$$.

The case of that φ_{qf} is a prime formula but $\perp$: It is trivial that $\mathsf{HA}^{\$}$ proves $\varphi_{\mathrm{qf}} \vee \$ \to \neg_{\$}\neg_{\$}\varphi_{\mathrm{qf}}$. On the other hand, since $\mathsf{HA}^{\$}$ proves $\varphi_{\mathrm{qf}} \vee \neg\varphi_{\mathrm{qf}}$ and $\neg_{\$}\neg_{\$}\varphi_{\mathrm{qf}} \wedge \neg\varphi_{\mathrm{qf}} \to \$$, we also have that $\mathsf{HA}^{\$}$ proves $\neg_{\$}\neg_{\$}\varphi_{\mathrm{qf}} \to \varphi_{\mathrm{qf}} \vee \$$.

The case of $\varphi_{\mathrm{qf}} \equiv \varphi_1 \wedge \varphi_2$: We have that $\mathsf{HA}^{\$}$ proves

$$(\varphi_1 \wedge \varphi_2)^{\$} \leftrightarrow \varphi_1^{\$} \wedge \varphi_2^{\$} \underset{\text{I.H.}}{\longleftrightarrow} (\varphi_1 \vee \$) \wedge (\varphi_2 \vee \$) \leftrightarrow (\varphi_1 \wedge \varphi_2) \vee \$.$$

The case of $\varphi_{\mathrm{qf}} \equiv \varphi_1 \vee \varphi_2$: Since φ_1 and φ_2 are decidable in HA (note that they are quantifier-free), we have that $\mathsf{HA}^{\$}$ proves $(\varphi_1 \vee \varphi_2) \vee (\neg\varphi_1 \wedge \neg\varphi_2)$. In the latter case of the disjunction, we have $\neg_{\$}(\varphi_1 \vee \varphi_2 \vee \$)$. Thus $\mathsf{HA}^{\$}$ proves

$$\neg_{\$}\neg_{\$}(\varphi_1 \vee \varphi_2 \vee \$) \to (\varphi_1 \vee \varphi_2) \vee \$.$$

On the other hand, $\mathsf{HA}^{\$}$ also proves

$$(\varphi_1 \vee \varphi_2) \vee \$ \to \neg_{\$}\neg_{\$}(\varphi_1 \vee \varphi_2 \vee \$).$$

Thus $\mathsf{HA}^{\$}$ proves

$$(\varphi_1 \vee \varphi_2)^{\$} \equiv \neg_{\$}\neg_{\$}(\varphi_1^{\$} \vee \varphi_2^{\$}) \underset{\text{I.H.}}{\longleftrightarrow} \neg_{\$}\neg_{\$}(\varphi_1 \vee \varphi_2 \vee \$) \leftrightarrow (\varphi_1 \vee \varphi_2) \vee \$.$$

The case of $\varphi_{\mathrm{qf}} \equiv \varphi_1 \to \varphi_2$: Assume $\varphi_1 \vee \$ \to \varphi_2 \vee \$$. Then we have

$$\varphi_1 \to \varphi_2 \vee \$. \tag{1}$$

Since φ_1 and φ_2 are decidable in HA (note that they are quantifier-free), we have that $\mathsf{HA}^{\$}$ proves $(\varphi_2 \vee \neg\varphi_1) \vee (\varphi_1 \wedge \neg\varphi_2)$. In the former case, we have $\varphi_1 \to \varphi_2$. In the latter case, by (1), we have \$. Thus $\mathsf{HA}^{\$}$ proves

$$(\varphi_1 \vee \$ \to \varphi_2 \vee \$) \to (\varphi_1 \to \varphi_2) \vee \$.$$

On the other hand, $\mathsf{HA}^{\$}$ also proves

$$(\varphi_1 \to \varphi_2) \vee \$ \to (\varphi_1 \vee \$ \to \varphi_2 \vee \$).$$

Thus $\mathsf{HA}^{\$}$ proves

$$(\varphi_1 \to \varphi_1) \vee \$ \leftrightarrow (\varphi_1 \vee \$ \to \varphi_2 \vee \$) \underset{\text{I.H.}}{\longleftrightarrow} \left(\varphi_1^{\$} \to \varphi_2^{\$}\right) \equiv (\varphi_1 \to \varphi_2)^{\$}.$$ ⊣

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

The following lemma is the key for our generalized conservation result:

LEMMA 3.5. *For a formula* φ *of* HA, *the following hold*:

1. *If* $\varphi \in \Pi_k$, $\mathsf{HA}^\$ + \Sigma_k\text{-LEM} \vdash \varphi^\$ \leftrightarrow \varphi \vee \$$;
2. *If* $\varphi \in \Sigma_k$, $\mathsf{HA}^\$ + \Sigma_k\text{-LEM} \vdash \varphi^\$ \leftrightarrow \varphi \vee \$$.

Note that Σ_k-LEM *is an axiom scheme in the language of* HA (*which does not contain* \$).

PROOF. By simultaneous induction on k. The base case is by Lemma 3.4. Assume items 1 and 2 for k to show those for $k+1$. First, for the first item, let $\varphi :\equiv \forall x \varphi_1$ where $\varphi_1 \in \Sigma_k$. By the induction hypothesis, we have $\mathsf{HA}^\$ + \Sigma_k\text{-LEM} \vdash \varphi_1^\$ \leftrightarrow \varphi_1 \vee \$$. Note that $\mathsf{HA}^\$$ proves $\forall x \varphi_1 \vee \$ \rightarrow \forall x(\varphi_1 \vee \$)$. In the following, we show the converse $\forall x(\varphi_1 \vee \$) \rightarrow \forall x \varphi_1 \vee \$$ inside $\mathsf{HA}^\$ + \Sigma_{k+1}\text{-LEM}$. Since $\neg\varphi_1$ has some equivalent formula in Π_k in the presence of Σ_{k-1}-DNE (cf. Remark 5.3), by Σ_{k+1}-LEM, we have now $\exists x \neg \varphi_1 \vee \neg\exists x \neg\varphi_1$. In the former case, we have \$ by using our assumption $\forall x(\varphi_1 \vee \$)$. In the latter case, we have $\forall x \varphi_1$ since $\neg \exists x \neg \varphi_1 \leftrightarrow \forall x \neg\neg\varphi_1$ and Σ_{k+1}-LEM implies Σ_{k+1}-DNE. Thus $\mathsf{HA}^\$ + \Sigma_{k+1}\text{-LEM}$ proves $\forall x(\varphi_1 \vee \$) \rightarrow \forall x \varphi_1 \vee \$$. Then we have that $\mathsf{HA}^\$ + \Sigma_{k+1}\text{-LEM}$ proves

$$\varphi^\$ \equiv \forall x \varphi_1^\$ \underset{\text{[I.H.] } \Sigma_k\text{-LEM}}{\longleftrightarrow} \forall x(\varphi_1 \vee \$) \underset{\Sigma_{k+1}\text{-LEM}}{\longleftrightarrow} \forall x \varphi_1 \vee \$.$$

Next, for the second item, let $\varphi :\equiv \exists x \varphi_1$ where $\varphi_1 \in \Pi_k$. Note that $\varphi^\$$ is $\neg_\$\neg_\$\exists x \varphi_1^\$$. By the induction hypothesis, we have $\mathsf{HA}^\$ + \Sigma_k\text{-LEM}$ proves $\varphi_1^\$ \leftrightarrow \varphi_1 \vee \$$, and hence, $\varphi^\$ \leftrightarrow \neg_\$\neg_\$\exists x\varphi_1$. Then it is trivial that $\mathsf{HA}^\$ + \Sigma_k\text{-LEM}$ proves $\exists x \varphi_1 \vee \$ \rightarrow \varphi^\$$. In the following, we show the converse direction inside $\mathsf{HA}^\$ + \Sigma_{k+1}\text{-LEM}$. By Σ_{k+1}-LEM, we have now $\exists x\varphi_1 \vee \neg\exists x \varphi_1$. Then it suffices to show $\neg\exists x\varphi_1 \wedge \neg_\$\neg_\$\exists x\varphi_1 \rightarrow \$$, which is trivial since $\neg\exists x \varphi_1 \rightarrow \neg_\$\exists x\varphi_1$. ⊣

COROLLARY 3.6. *For a formula* φ *of* HA, *if* $\varphi \equiv \exists x \varphi_1$ *with* $\varphi_1 \in \Pi_k$, *then* $\mathsf{HA}^\$ + \Sigma_k\text{-LEM} \vdash \exists x\left({\varphi_1}^\$\right) \leftrightarrow \varphi \vee \$$.

PROOF. Since $\exists x \varphi_1 \vee \$ \leftrightarrow \exists x(\varphi_1 \vee \$)$, this is trivial by Lemma 3.5.(1). ⊣

In the context of intuitionistic/semi-classical arithmetic, a formula does not have an equivalent formula of the prenex normal form (namely, formula in Σ_k or Π_k) while it does in classical arithmetic. Because of this fact, the conservation theorem only for prenex formulas is not applicable in many practical cases. On the other hand, Akama et al. [1] introduced the classes U_k and E_k of formulas which correspond to classical Π_k and Σ_k respectively in the sense that every formula in U_k (resp. E_k) is equivalent over PA to some formula in Π_k (resp. Σ_k) and vice versa. In addition, the authors introduced in [7] the classes U_k^+ and E_k^+, which are cumulative versions of U_k and E_k. For obtaining the conservation results for the classes as large as possible, we introduce classes $\mathcal{R}_k$ and $\mathcal{J}_k$ (see Definition 3.11), which relativize $\mathcal{R}$ and $\mathcal{J}$ in [9] respectively with regard to the formulas of degree $\leq k$ in the sense of [1, 7].

To make the definitions absolutely clear, we recall some notions in [1, 7]: An *alternation path* is a finite sequence of + and – in which + and – appear alternatively. For an alternation path s, let $i(s)$ denote the first symbol of s if $s \not\equiv \langle\,\rangle$ (empty sequence); $\times$ if $s \equiv \langle\,\rangle$. Let $s^\perp$ denote the alternation path which is obtained by switching + and – in s, and let $l(s)$ denote the length of s. For a formula φ, the set of alternation paths $Alt(\varphi)$ of φ is defined as follows:

- If φ is prime, then $Alt(\varphi) := \{\langle\rangle\}$;
- Otherwise, $Alt(\varphi)$ is defined inductively by the following clauses:
  - If $\varphi \equiv \varphi_1 \wedge \varphi_2$ or $\varphi \equiv \varphi_1 \vee \varphi_2$, then $Alt(\varphi) := Alt(\varphi_1) \cup Alt(\varphi_2)$;
  - If $\varphi \equiv \varphi_1 \to \varphi_2$, then $Alt(\varphi) := \{s^\perp \mid s \in Alt(\varphi_1)\} \cup Alt(\varphi_2)$;
  - If $\varphi \equiv \forall x \varphi_1$, then $Alt(\varphi) := \{s \mid s \in Alt(\varphi_1) \text{ and } i(s) \equiv -\} \cup \{-s \mid s \in Alt(\varphi_1) \text{ and } i(s) \not\equiv -\}$;
  - If $\varphi \equiv \exists x \varphi_1$, then $Alt(\varphi) := \{s \mid s \in Alt(\varphi_1) \text{ and } i(s) \equiv +\} \cup \{+s \mid s \in Alt(\varphi_1) \text{ and } i(s) \not\equiv +\}$.

In addition, for a formula φ, the degree $deg(\varphi)$ of φ is defined as

$$deg(\varphi) := \max\{l(s) \mid s \in Alt(\varphi)\}.$$

DEFINITION 3.7 (cf. [1, Definition 2.4] and [7, Definition 2.11]). The classes $\mathrm{F}_k, \mathrm{U}_k, \mathrm{E}_k, \mathrm{F}_k^+, \mathrm{U}_k^+$ and E_k^+ of HA-formulas are defined as follows:

- $\mathrm{F}_k := \{\varphi \mid deg(\varphi) = k\}$; $\mathrm{F}_k^+ := \{\varphi \mid deg(\varphi) \leq k\}$;
- $\mathrm{U}_0 := \mathrm{E}_0 := \mathrm{F}_0\ (= \Sigma_0 = \Pi_0)$;
- $\mathrm{U}_{k+1} := \{\varphi \in \mathrm{F}_{k+1} \mid i(s) \equiv - \text{ for all } s \in Alt(\varphi) \text{ such that } l(s) = k+1\}$;
- $\mathrm{E}_{k+1} := \{\varphi \in \mathrm{F}_{k+1} \mid i(s) \equiv + \text{ for all } s \in Alt(\varphi) \text{ such that } l(s) = k+1\}$;
- $\mathrm{U}_k^+ := \mathrm{U}_k \cup \bigcup_{i<k} \mathrm{F}_i$; $\mathrm{E}_k^+ := \mathrm{E}_k \cup \bigcup_{i<k} \mathrm{F}_i$.

REMARK 3.8. As shown in [7, Proposition 4.6], for any $\varphi \in \mathrm{U}_k^+$ and $\psi \in \mathrm{E}_k^+$, there exist $\varphi' \in \mathrm{U}_k$ and $\psi' \in \mathrm{E}_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathrm{FV}(\psi) = \mathrm{FV}(\psi')$, $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi'$ and $\mathsf{HA} \vdash \psi \leftrightarrow \psi'$. Then it also follows that for any $\varphi \in \mathrm{F}_k^+$, there exists $\varphi' \in \mathrm{F}_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi'$. Thus one may identify $\mathrm{E}_k^+, \mathrm{U}_k^+$ and F_k^+ with $\mathrm{E}_k, \mathrm{U}_k$ and F_k respectively over HA without loss of generality.

Then the authors showed the following prenex normal form theorem:

THEOREM 3.9 (cf. [7, Theorem 5.3] which corrects [1, Theorem 2.7]). *For a* HA-*formula* φ, *the following hold*:

1. *If* $\varphi \in \mathrm{E}_k^+$, *then there exists* $\varphi' \in \Sigma_k$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and*
$$\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k\text{-DNS} \vdash \varphi \leftrightarrow \varphi';$$
2. *If* $\varphi \in \mathrm{U}_k^+$, *then there exists* $\varphi' \in \Pi_k$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and*
$$\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \varphi \leftrightarrow \varphi';$$

where U_k-DNS *is the axiom scheme of the double-negation-shift restricted to formulas in* U_k *and* $(\Pi_k \vee \Pi_k)$-DNE *is DNE restricted to formulas of the form* $\varphi \vee \psi$ *with* $\varphi, \psi \in \Pi_k$.

REMARK 3.10. $\mathsf{HA} + \Sigma_k$-LEM proves Σ_k-DNE, U_k-DNS and $(\Pi_k \vee \Pi_k)$-DNE. Then the prenex normal form theorems for E_k^+ and U_k^+ are available in $\mathsf{HA} + \Sigma_k$-LEM.

DEFINITION 3.11 (cf. [9, Definition 6]). Define $\mathcal{R}_0 := \mathcal{J}_0 := \Sigma_0\ (= \Pi_0)$. In addition, we define simultaneously classes $\mathcal{R}_{k+1}$ and $\mathcal{J}_{k+1}$ as follows: Let F range over formulas in F_k^+, R and R' over those in $\mathcal{R}_{k+1}$, and J and J' over those in $\mathcal{J}_{k+1}$ respectively. Then $\mathcal{R}_{k+1}$ and $\mathcal{J}_{k+1}$ are inductively generated by the clauses

1. $F, R \wedge R', R \vee R', \forall x R, J \to R \in \mathcal{R}_{k+1}$;
2. $F, J \wedge J', J \vee J', \exists x J, R \to J \in \mathcal{J}_{k+1}$.

LEMMA 3.12 (A relativized version of [9, Proposition 7(2, 3)]). *For a* HA*-formula* φ, *the following hold*:

1. *If* $\varphi \in \mathcal{R}_{k+1}$, *then* $\mathsf{HA}^{\$} + \Sigma_k$-LEM *proves* $\neg_{\$}\neg\varphi \to \varphi^{\$}$;
2. *If* $\varphi \in \mathcal{J}_{k+1}$, *then* $\mathsf{HA}^{\$} + \Sigma_k$-LEM *proves* $\varphi^{\$} \to \neg_{\$}\neg_{\$}\varphi$.

PROOF. We show items 1 and 2 simultaneously by induction on the structure of formulas.

Let φ be prime. Since φ is in F_0, we have $\varphi \in \mathcal{R}_{k+1} \cap \mathcal{J}_{k+1}$. Since $\mathsf{HA} \vdash \varphi \vee \neg\varphi$, we have $\mathsf{HA}^{\$} \vdash \neg_{\$}\neg\varphi \to \varphi \vee \$$. Then we have item 1 by Lemma 3.4. Item 2 is trivial.

The induction step is the same as that for [9, Proposition 7] in addition with the cases of $\varphi :\equiv \forall x \varphi_1 \in \mathcal{J}_{k+1}$ and $\varphi :\equiv \exists x \varphi_1 \in \mathcal{R}_{k+1}$:

If $\varphi :\equiv \forall x \varphi_1 \in \mathcal{J}_{k+1}$, then we have $\varphi \in \mathrm{F}_k^+$, and hence, $\varphi \in \mathrm{U}_k^+$. By Remark 3.10, one may assume $\varphi \in \Pi_k$. By Lemma 3.5.(1), we have $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi^{\$} \leftrightarrow \varphi \vee \$$. Since $\varphi \vee \$$ implies $\neg_{\$}\neg_{\$}\varphi$, we have $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi^{\$} \to \neg_{\$}\neg_{\$}\varphi$.

If $\varphi :\equiv \exists x \varphi_1 \in \mathcal{R}_{k+1}$, then we have $\varphi \in \mathrm{F}_k^+$, and hence, $\varphi \in \mathrm{E}_k^+$ (and $k > 0$). By Remark 3.10, one may assume $\varphi_1 \in \Pi_{k-1}$. Reason in $\mathsf{HA}^{\$} + \Sigma_k$-LEM. Now we have $\exists x \varphi_1 \vee \neg\exists x \varphi_1$. In the latter case, we have \$ in the presence of $\neg_{\$}\neg\exists x \varphi_1$. Thus we have $\neg_{\$}\neg\exists x \varphi_1 \to \exists x \varphi_1 \vee \$$. By Corollary 3.6, we have that $\neg_{\$}\neg\exists x \varphi_1$ implies $\exists x \left(\varphi_1{}^{\$}\right)$, and hence, $(\exists x \varphi_1)^{\$}$. ⊣

DEFINITION 3.13 (cf. [9, Definition 6]). Define $\mathcal{Q}_0 := \Sigma_0\ (= \Pi_0)$. In addition, we define a class $\mathcal{Q}_{k+1}$ as follows. Let P range over prime formulas, Q and Q' over formulas in $\mathcal{Q}_{k+1}$, and J over those in $\mathcal{J}_{k+1}$. Then $\mathcal{Q}_{k+1}$ is inductively generated by the clause

$$P, Q \wedge Q', Q \vee Q', \forall x Q, \exists x Q, J \to Q \in \mathcal{Q}_{k+1}.$$

LEMMA 3.14 (A relativized version of [9, Proposition 7(1)]). *For a* HA*-formula* φ, *if* $\varphi \in \mathcal{Q}_{k+1}$, *then* $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi \to \varphi^{\$}$.

PROOF. By induction on the structure of formulas, we show that for any HA-formula φ, if $\varphi \in \mathcal{Q}_{k+1}$, then $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi \to \varphi^{\$}$.

If φ is prime, then we have $\mathsf{HA}^{\$} \vdash \varphi \to \varphi^{\$}$ trivially by the definition of $\varphi^{\$}$. If $\varphi :\equiv \varphi_1 \wedge \varphi_2, \varphi :\equiv \varphi_1 \vee \varphi_2, \varphi :\equiv \forall x \varphi_1$ or $\varphi :\equiv \exists x \varphi_1$, we have $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi \to \varphi^{\$}$ in a straightforward way by using the induction hypothesis (as for [9, Proposition 7(1)]).

Assume $\varphi :\equiv \varphi_1 \to \varphi_2 \in \mathcal{Q}_{k+1}$. Then we have $\varphi_1 \in \mathcal{J}_{k+1}$ and $\varphi_2 \in \mathcal{Q}_{k+1}$. By the induction hypothesis, we have $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi_2 \to \varphi_2^{\$}$. On the other hand, by Lemma 3.12.(2), we have $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi_1^{\$} \to \neg_{\$}\neg_{\$}\varphi_1$. Since $\mathsf{HA}^{\$} \vdash \neg_{\$}\neg_{\$}\varphi_2^{\$} \leftrightarrow \varphi_2^{\$}$ by Proposition 3.2.(1), we have that $\mathsf{HA}^{\$} + \Sigma_k$-LEM proves

$$\begin{aligned}
(\varphi_1 \to \varphi_2) &\underset{[\text{I.H.}]\ \Sigma_k\text{-LEM}}{\longrightarrow} (\varphi_1 \to \varphi_2^{\$}) \\
&\longrightarrow (\neg_{\$}\neg_{\$}\varphi_1 \to \neg_{\$}\neg_{\$}\varphi_2^{\$}) \\
&\underset{\Sigma_k\text{-LEM}}{\longrightarrow} (\varphi_1^{\$} \to \neg_{\$}\neg_{\$}\varphi_2^{\$}) \\
&\longleftrightarrow (\varphi_1^{\$} \to \varphi_2^{\$}).
\end{aligned}$$

⊣

Now we define a class $\mathcal{V}_k$ of HA-formulas by using the class $\mathcal{J}_k$ in Definitions 3.11.

DEFINITION 3.15. Let J range over formulas in $\mathcal{J}_k$, V and V' over those in $\mathcal{V}_k$. Then $\mathcal{V}_k$ is inductively generated by the clause

$$J, V \wedge V', \forall x V \in \mathcal{V}_k.$$

For our conservation result, we use the following fact on substitution.

LEMMA 3.16 (cf. [3, Theorem 6.2.4] and [7, Lemma 6.10]). *Let X be a set of* HA-*sentences and φ be a* $\mathsf{HA}^{\$}$-*formula. If* $\mathsf{HA}^{\$} + X \vdash \varphi$, *then* $\mathsf{HA} + X \vdash \varphi[\psi/\$]$ *for any* HA-*formula ψ such that the free variables of ψ are not bounded in φ, where $\varphi[\psi/\$]$ is the* HA-*formula obtained from φ by replacing all the occurrences of* \$ *in φ with ψ.*

THEOREM 3.17. *For any* HA-*formulas $\varphi \in \mathcal{V}_{k+1}$ and $\psi \in \mathcal{Q}_{k+1}$, if* $\mathsf{PA} \vdash \psi \to \varphi$, *then* $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi$.

PROOF. Since one can freely replace the bounded variables, it suffices to show that for any HA-formulas $\varphi \in \mathcal{V}_{k+1}$ and $\psi \in \mathcal{Q}_{k+1}$ such that the free variables of φ are not bounded in ψ, if $\mathsf{PA} \vdash \psi \to \varphi$, then $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi$. We show this assertion by induction on the structure of formulas in $\mathcal{V}_{k+1}$.

Case of $\varphi \in \mathcal{J}_{k+1}$: Fix $\psi \in \mathcal{Q}_{k+1}$ such that the free variables of φ are not bounded in ψ. Suppose $\mathsf{PA} \vdash \psi \to \varphi$. Then, by Proposition 3.2.(2), we have $\mathsf{HA}^{\$} \vdash \psi^{\$} \to \varphi^{\$}$. By Lemma 3.14 and Lemma 3.12.(2), we have $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \psi \to \neg_{\$}\neg_{\$}\varphi$. By Lemma 3.16, we have that $\mathsf{HA} + \Sigma_k\text{-LEM}$ proves $\psi \to ((\varphi \to \varphi) \to \varphi)$, equivalently, $\psi \to \varphi$.

Case of $\varphi :\equiv \varphi_1 \wedge \varphi_2 \in \mathcal{V}_{k+1}$: Then $\varphi_1, \varphi_2 \in \mathcal{V}_{k+1}$. Fix $\psi \in \mathcal{Q}_{k+1}$ such that the free variables of $\varphi_1 \wedge \varphi_2$ are not bounded in ψ. Suppose $\mathsf{PA} \vdash \psi \to \varphi_1 \wedge \varphi_2$. Then $\mathsf{PA} \vdash \psi \to \varphi_1$ and $\mathsf{PA} \vdash \psi \to \varphi_2$. By the induction hypothesis, we have $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi_1$ and $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi_2$, and hence, $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi_1 \wedge \varphi_2$.

Case of $\varphi :\equiv \forall x \varphi_1 \in \mathcal{V}_{k+1}$: Then $\varphi_1 \in \mathcal{V}_{k+1}$. Fix $\psi \in \mathcal{Q}_{k+1}$ such that the free variables of $\forall x \varphi_1$ are not bounded in ψ. In addition, assume that x does not appear in ψ without loss of generality. Suppose $\mathsf{PA} \vdash \psi \to \forall x \varphi_1$. Then $\mathsf{PA} \vdash \psi \to \varphi_1$. By the induction hypothesis, we have that $\mathsf{HA} + \Sigma_k\text{-LEM}$ proves $\psi \to \varphi_1$. Since $x \notin \mathrm{FV}(\psi)$, we have $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \forall x \varphi_1$. ⊣

REMARK 3.18. Since Π_{k+2} is a sub-class of $\mathcal{V}_{k+1}$ and $\mathcal{Q}_{k+1}$ contains all prenex formulas, we have [7, Theorem 6.14] (and a-fortiori Proposition 1.1) as a corollary of Theorem 3.17.

COROLLARY 3.19. *Let X be a set of* HA-*sentences in $\mathcal{Q}_{k+1}$. For any* HA-*formulas $\varphi \in \mathcal{V}_{k+1}$ and $\psi \in \mathcal{Q}_{k+1}$, if* $\mathsf{PA} + X \vdash \psi \to \varphi$, *then* $\mathsf{HA} + X + \Sigma_k\text{-LEM} \vdash \psi \to \varphi$.

PROOF. Assume $\mathsf{PA} + X \vdash \psi \to \varphi$. Then there exists a finite number of sentences $\psi_0, \dots, \psi_m \in X$ such that $\mathsf{PA} + \psi_0 + \cdots + \psi_m \vdash \psi \to \varphi$. Since PA satisfies the deduction theorem, we have $\mathsf{PA} \vdash \psi_0 \wedge \cdots \wedge \psi_m \wedge \psi \to \varphi$. Since $\psi_0 \wedge \cdots \wedge \psi_m \wedge \psi \in \mathcal{Q}_{k+1}$, by Theorem 3.17, we have $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi_0 \wedge \cdots \wedge \psi_m \wedge \psi \to \varphi$, and hence, $\mathsf{HA} + X + \Sigma_k\text{-LEM} \vdash \psi \to \varphi$. ⊣

§4. The relation of the classes $\mathcal{R}_k$ and $\mathcal{J}_k$ with the existing classes U_k and E_k. In the following, we show that our classes $\mathcal{R}_k$ and $\mathcal{J}_k$ in Definition 3.11 are in fact equivalent over HA to U_k and E_k (see Definition 3.7) respectively.

Proposition 4.1. $\mathrm{U}_k^+ = \mathcal{R}_k$ *and* $\mathrm{E}_k^+ = \mathcal{J}_k$.

Proof. By induction on k. The base case is trivial. For the induction step, assume $\mathrm{U}_k^+ = \mathcal{R}_k$ and $\mathrm{E}_k^+ = \mathcal{J}_k$. We show

1. $\varphi \in \mathrm{U}_{k+1}^+$ if and only if $\varphi \in \mathcal{R}_{k+1}$,
2. $\varphi \in \mathrm{E}_{k+1}^+$ if and only if $\varphi \in \mathcal{J}_{k+1}$,

simultaneously by induction on the structure of formulas. If φ is prime, since $\varphi \in \mathrm{F}_0$, we are done. Assume that items 1 and 2 hold for φ_1 and φ_2. Using [7, Lemma 4.5(1)], we have

$$\varphi_1 \wedge \varphi_2 \in \mathrm{U}_{k+1}^+ \Leftrightarrow \varphi_1, \varphi_2 \in \mathrm{U}_{k+1}^+ \underset{\text{I.H.}}{\Longleftrightarrow} \varphi_1, \varphi_2 \in \mathcal{R}_{k+1} \Leftrightarrow \varphi_1 \wedge \varphi_2 \in \mathcal{R}_{k+1}.$$

In the same manner, we also have $\varphi_1 \wedge \varphi_2 \in \mathrm{E}_{k+1}^+ \Leftrightarrow \varphi_1 \wedge \varphi_2 \in \mathcal{J}_{k+1}$, $\varphi_1 \vee \varphi_2 \in \mathrm{U}_{k+1}^+ \Leftrightarrow \varphi_1 \vee \varphi_2 \in \mathcal{R}_{k+1}$, $\varphi_1 \vee \varphi_2 \in \mathrm{E}_{k+1}^+ \Leftrightarrow \varphi_1 \vee \varphi_2 \in \mathcal{J}_{k+1}$. For $\varphi_1 \to \varphi_2$, using [7, Lemma 4.5(3)] we have

$$\begin{aligned} & \varphi_1 \to \varphi_2 \in \mathrm{U}_{k+1}^+ \\ \Longleftrightarrow \quad & \varphi_1 \in \mathrm{E}_{k+1}^+ \text{ and } \varphi_2 \in \mathrm{U}_{k+1}^+ \\ \underset{\text{I.H.}}{\Longleftrightarrow} \quad & \varphi_1 \in J_{k+1} \text{ and } \varphi_2 \in \mathcal{R}_{k+1} \\ \Longleftrightarrow \quad & \varphi_1 \to \varphi_2 \in \mathcal{R}_{k+1}. \end{aligned}$$

In the same manner, we also have $\varphi_1 \to \varphi_2 \in \mathrm{E}_{k+1}^+ \Leftrightarrow \varphi_1 \to \varphi_2 \in \mathcal{J}_{k+1}$. For $\forall x \varphi_1$, using [7, Lemma 4.5(4,6)], we have

$$\forall x \varphi_1 \in \mathrm{U}_{k+1}^+ \Leftrightarrow \varphi_1 \in \mathrm{U}_{k+1}^+ \underset{\text{I.H.}}{\Longleftrightarrow} \varphi_1 \in \mathcal{R}_{k+1} \Leftrightarrow \forall x \varphi_1 \in \mathcal{R}_{k+1}$$

and

$$\forall x \varphi_1 \in \mathrm{E}_{k+1}^+ \Leftrightarrow \forall x \varphi_1 \in \mathrm{U}_k^+ \Leftrightarrow \forall x \varphi_1 \in \mathrm{F}_k^+ \Leftrightarrow \forall x \varphi_1 \in \mathcal{J}_{k+1}.$$

In the same manner, we also have $\exists x \varphi_1 \in \mathrm{U}_{k+1}^+ \Leftrightarrow \exists x \varphi_1 \in \mathcal{R}_{k+1}$ and $\exists x \varphi_1 \in \mathrm{E}_{k+1}^+ \Leftrightarrow \exists x \varphi_1 \in \mathcal{J}_{k+1}$. ⊣

Corollary 4.2. $\mathrm{U}_k = \mathcal{R}_k$ *and* $\mathrm{E}_k = \mathcal{J}_k$.

Proof. Immediate by Proposition 4.1 and Remark 3.8. ⊣

Corollary 4.3. *For a set X of* HA-*sentences in* $\mathcal{Q}_{k+1}$, PA + X *is* E_{k+1}-*conservative over* HA + X + Σ_k-LEM.

Proof. Immediate from Corollaries 3.19 and 4.2 since $\mathcal{J}_{k+1} \subseteq \mathcal{V}_{k+1}$. ⊣

Remark 4.4. Corollary 4.3 deals with the conservativity of the class of formulas in E_{k+1}, which seems to be strictly stronger than that for sentences in E_{k+1} (cf. Section 6.1).

Remark 4.5. Similar to Definition 3.11, define the classes $\mathcal{R}_k'$ and $\mathcal{J}_k'$ as follows. Define $\mathcal{R}_0' := \mathcal{J}_0' := \Sigma_0 \, (= \Pi_0)$ and $\mathcal{R}_{k+1}'$ and $\mathcal{J}_{k+1}'$ simultaneously as follows: Let E range over formulas in E_k^+, U over those in U_k^+, R and R' over those in $\mathcal{R}_{k+1}'$, and J and J' over those in $\mathcal{J}_{k+1}'$ respectively. Then $\mathcal{R}_{k+1}'$ and $\mathcal{J}_{k+1}'$ are inductively generated by the clauses

1. $E, R \wedge R', R \vee R', \forall x R, J \to R \in \mathcal{R}'_{k+1}$;
2. $U, J \wedge J', J \vee J', \exists x J, R \to J \in \mathcal{J}'_{k+1}$.

Then the proof of Proposition 4.1 shows that $\mathrm{U}_k^+ = \mathcal{R}'_k$ and $\mathrm{E}_k^+ = \mathcal{J}'_k$. Hence $\mathcal{R}_k = \mathcal{R}'_k$ and $\mathcal{J}_k = \mathcal{J}'_k$.

REMARK 4.6. Define $\mathcal{R}''_{k+1}$ and $\mathcal{J}''_{k+1}$ as for $\mathcal{R}'_{k+1}$ and $\mathcal{J}'_{k+1}$ in Remark 4.5 with replacing E_k^+ and U_k^+ by Σ_k and Π_k. Then, as in the proof of Proposition 4.1 with using the prenex normal form theorems in $\mathsf{HA} + \Sigma_k$-LEM (cf. Remark 3.10), one can show $\mathrm{U}_{k+1}^+ = \mathcal{R}''_{k+1}$ and $\mathrm{E}_{k+1}^+ = \mathcal{J}''_{k+1}$ over $\mathsf{HA} + \Sigma_k$-LEM.

As described in Definition 3.7, the classes E_k and U_k are originally defined by using the notion of alternation path. On the other hand, Remark 4.6 reveals that one can define these classes (via Remark 3.8) inductively without using the notion of alternation path. A technical advantage of this usual way of defining classes is that one can prove properties of these classes by induction on the structure of formulas in those classes.

§5. Conservation theorems for the classes of formulas. In this section, we explore the notion that PA is Γ-conservative over T for semi-classical arithmetic T and a class Γ of formulas (especially, $\Pi_k, \Sigma_k, \mathrm{U}_k, \mathrm{E}_k, \mathrm{F}_k$ etc.).

DEFINITION 5.1. For classes of HA-formulas Γ and Γ', $\Gamma \vee \Gamma'$ is the class of formulas of form $\varphi \vee \psi$ where $\varphi \in \Gamma$ and $\psi \in \Gamma'$.

We recall the notion of duals for prenex formulas from [1, 6].

DEFINITION 5.2 (cf. [6, Definition 3.2]). For any formula φ in prenex normal form, we define the dual $\varphi^\perp$ of φ inductively as follows:

1. $\varphi^\perp :\equiv \neg\varphi$ if φ is quantifier-free;
2. $(\forall x \varphi)^\perp :\equiv \exists x (\varphi)^\perp$;
3. $(\exists x \varphi)^\perp :\equiv \forall x (\varphi)^\perp$.

REMARK 5.3. For φ in Σ_k (resp. Π_k), $\varphi^\perp$ is in Π_k (resp. Σ_k), $\mathrm{FV}\left(\varphi^\perp\right) = \mathrm{FV}(\varphi)$ and $\left(\varphi^\perp\right)^\perp$ is equivalent to φ over HA. For each prenex formula φ, $\varphi^\perp$ implies $\neg\varphi$ intuitionistically. On the other hand, the converse direction for formulas in Σ_k (resp. Π_k) is equivalent to Σ_{k-1}-DNE (resp. Σ_k-DNE). Then it follows that for $\varphi \in \Sigma_k$ there exists $\varphi' \in \Pi_k$ such that $\mathrm{FV}(\varphi') = \mathrm{FV}(\varphi)$ and $\mathsf{HA} + \Sigma_{k-1}\text{-DNE} \vdash \varphi' \leftrightarrow \neg\varphi$ (cf. [7, Lemma 4.8(2)]). In addition, $\neg\varphi^\perp$ implies $\neg\neg\varphi$ in the presence of Σ_{k-1}-DNE for the both cases of $\varphi \in \Sigma_k$ and $\varphi \in \Pi_k$. Note also that PA proves $\varphi \vee \varphi^\perp$ for each prenex formula φ. We refer the reader to [6, Section 3] for more information about the dual principles for prenex formulas in semi-classical arithmetic.

5.1. Conservation theorems for $\Pi_k, \Sigma_k, \mathrm{E}_k$ and F_k.

DEFINITION 5.4. Let T be a theory in the language of HA and Γ be a class of HA-formulas.

- T is closed under Γ-DNE-R if $T \vdash \neg\neg\varphi$ implies $T \vdash \varphi$ for all $\varphi \in \Gamma$.
- T is closed under Γ-CD-R if $T \vdash \forall x(\varphi \vee \psi)$ implies $T \vdash \varphi \vee \forall x \psi$ for all $\varphi, \psi \in \Gamma$ such that $x \notin \mathrm{FV}(\varphi)$.

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

- T is closed under Γ-DML-R (resp. Γ-DML$^\perp$-R) if $T \vdash \neg(\varphi \wedge \psi)$ implies $T \vdash \neg\varphi \vee \neg\psi$ (resp. $T \vdash \varphi^\perp \vee \psi^\perp$) for all $\varphi, \psi \in \Gamma$.

Note that φ and ψ in the above may contain free variables.

As mentioned in [14, Section 3.5.1], Σ_1-DNE-R is known as Markov's rule (for primitive recursive predicates). The fact that PA is Σ_1-conservative (equivalently, Π_2-conservative) over HA implies that HA is closed under Markov's rule (Σ_1-DNE-R), and vice versa. The generalization Σ_k-DNE-R of Markov's rule is already mentioned in [8, Section 4.4]. It is easy to see that for semi-classical arithmetic T, if PA is Σ_k-conservative over T, then T is closed under Σ_k-DNE-R. Then it is natural to ask about the converse. As we show in Theorem 5.9, this is also the case (note that the case for $k = 2$ is essentially shown in the proof of [12, Proposition 3.3]).

The following are our "reversal" results.

LEMMA 5.5. *Let T be a theory containing* HA. *If* PA *is* $(\Sigma_k \vee \Pi_k)$*-conservative over T, then* $T \vdash \Sigma_k$-LEM.

PROOF. Fix $\xi \in \Sigma_k$. Let $\xi^\perp \in \Pi_k$ be the dual of ξ. Since PA $\vdash \xi \vee \xi^\perp$, by our assumption, we have $T \vdash \xi \vee \xi^\perp$, and hence, $T \vdash \xi \vee \neg\xi$. ⊣

LEMMA 5.6. *Let T be a theory containing* HA. *If T is closed under* Σ_{k+1}-DNE-R, *then T proves* Σ_k-LEM.

PROOF. We show that for all $m \le k$, T proves Σ_m-LEM, by induction on m. Since T contains HA, the base case is trivial. Assume $m + 1 \le k$ and $T \vdash \Sigma_m$-LEM. Let $\varphi \in \Sigma_{m+1}$. Since HA $\vdash \neg\neg(\varphi \vee \neg\varphi)$, by Remark 5.3 and the fact that Σ_m-LEM implies Σ_m-DNE, we have $T \vdash \neg\neg(\varphi \vee \varphi^\perp)$ where $\varphi^\perp \in \Pi_{m+1}$. Since $\varphi \vee \varphi^\perp$ is equivalent over HA to some formula in Σ_{m+2} (cf. [7, Lemma 4.4]), by Σ_{k+1}-DNE-R, we have $T \vdash \varphi \vee \varphi^\perp$, and hence, $\varphi \vee \neg\varphi$. Thus we have shown $T \vdash \Sigma_{m+1}$-LEM. ⊣

LEMMA 5.7. *Let T be a theory containing* HA. *If T is closed under* Σ_k-CD-R, *then T proves* Σ_k-LEM.

PROOF. We show that for all $m \le k$, T proves Σ_m-LEM, by induction on m. Since T contains HA, the base case is trivial. Assume $m + 1 \le k$ and $T \vdash \Sigma_m$-LEM. Let $\varphi :\equiv \exists x \varphi_1$ where $\varphi_1 \in \Pi_m$. Since T proves Π_m-LEM and Σ_m-DNE, we have $T \vdash \varphi_1 \vee \neg\varphi_1$, and hence, $T \vdash \varphi_1 \vee \varphi_1^\perp$ (cf. Remark 5.3). Then $T \vdash \forall x(\exists x\varphi_1 \vee \varphi_1^\perp)$ follows. Since $\exists x\varphi_1, \varphi_1^\perp \in \Sigma_{m+1}$, by Σ_k-CD-R, we have $T \vdash \exists x\varphi_1 \vee \forall x\varphi_1^\perp$, and hence, $T \vdash \exists x\varphi_1 \vee \neg\exists x\varphi_1$. Thus we have shown $T \vdash \Sigma_{m+1}$-LEM. ⊣

LEMMA 5.8. *Let T be a theory containing* HA. *Then T is closed under* Π_k-DML$^\perp$-R *if and only if T is closed under* Σ_k-DNE-R.

PROOF. We first show the "only if" direction. Assume that T is closed under Π_k-DML$^\perp$-R and $T \vdash \neg\neg\varphi$ where $\varphi \in \Sigma_k$. Since $\neg\neg\varphi$ is equivalent over HA to $\neg(\neg\varphi \wedge \neg\varphi)$, by Remark 5.3, we have

$$T \vdash \neg(\varphi^\perp \wedge \varphi^\perp).$$

Since $\varphi^\perp \in \Pi_k$, by Π_k-DML$^\perp$-R, we have $T \vdash \left(\varphi^\perp\right)^\perp \vee \left(\varphi^\perp\right)^\perp$, and hence, $T \vdash \varphi$ (cf. Remark 5.3).

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

For the converse direction, assume that T is closed under Σ_k-DNE-R and $T \vdash \neg(\varphi \wedge \psi)$ where $\varphi, \psi \in \Pi_k$. Since $\neg(\varphi \wedge \psi)$ is intuitionistically equivalent to $\neg(\neg\neg\varphi \wedge \neg\neg\psi)$, by Lemma 5.6 and Remark 5.3 (note that Σ_{k-1}-LEM implies Σ_{k-1}-DNE), we have $T \vdash \neg\left(\neg\varphi^{\perp} \wedge \neg\psi^{\perp}\right)$ where $\varphi^{\perp}, \psi^{\perp} \in \Sigma_k$. Then $T \vdash \neg\neg\left(\varphi^{\perp} \vee \psi^{\perp}\right)$ follows. By Σ_k-DNE-R, we have $T \vdash \varphi^{\perp} \vee \psi^{\perp}$. ⊣

THEOREM 5.9. *Let T be semi-classical arithmetic and X be a set of* HA*-sentences in $\mathcal{Q}_{k+1}$. The following are pairwise equivalent*:

1. PA $+ X$ *is $\mathcal{V}_{k+1}$-conservative over $T + X$;*
2. PA $+ X$ *is Π_{k+2}-conservative over $T + X$;*
3. PA $+ X$ *is Σ_{k+1}-conservative over $T + X$;*
4. $T + X$ *is closed under* Σ_{k+1}-DNE-R;
5. $T + X$ *is closed under* Π_{k+1}-DML$^{\perp}$-R;
6. PA $+ X$ *is* E_{k+1}*-conservative over $T + X$;*
7. PA $+ X$ *is* F_k*-conservative over $T + X$;*
8. PA $+ X$ *is* $(\Sigma_k \vee \Pi_k)$*-conservative over $T + X$;*
9. $T + X \vdash \Sigma_k$-LEM;
10. $T + X \vdash \Sigma_k$-CD;
11. $T + X$ *is closed under* Σ_k-CD-R;

where Σ_k-CD *is the scheme* $\forall x(\varphi \vee \psi) \to \varphi \vee \forall x \psi$ *with* $\varphi, \psi \in \Sigma_k$ *such that* $x \notin \mathrm{FV}(\varphi)$ (*cf.* [6, *Section* 7]).

PROOF. The implications (1) → (6) → (7) → (8), (1) → (2) → (3) → (4) and (9) → (10) → (11) are trivial (cf. Corollary 4.3 and Remark 3.18). The implications (8) → (9), (4) → (9), (11) → (9) and (9) → (1) are by Lemmata 5.5, 5.6, 5.7 and Corollary 3.19 respectively. The equivalence (4) ↔ (5) is by Lemma 5.8. ⊣

5.2. Conservation theorem for U_k**.** In contrast to the fact that E_{k+1}-conservativity and F_k-conservativity are characterized by Σ_k-LEM (see Theorem 5.9), U_{k+1}-conservativity requires more than Σ_k-LEM:

PROPOSITION 5.10. PA *is not* $(\Pi_1 \vee \Pi_1)$*-conservative over* HA.

PROOF. We use the same argument as in [7, Section 3]. Suppose that PA is conservative over HA for all formulas $\varphi \vee \psi$ with $\varphi, \psi \in \Pi_1$. Let $\Phi(x)$ be the following formula:

$$\forall u \neg(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) = 0) \vee \forall u \neg(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) \neq 0), \tag{2}$$

where T and U are the standard primitive recursive predicate and function from the Kleene normal form theorem. Since

$$\neg\left(\exists u(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) = 0) \wedge \exists u(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) \neq 0)\right)$$

is provable in HA, we have PA $\vdash \Phi(x)$. Then, by our assumption, we have HA $\vdash \Phi(x)$, and hence, HA $\vdash \forall x \Phi(x)$. On the other hand, as shown in the proof of [7, Proposition 3.1], $\neg\forall x \Phi(x)$ is provable in HA $+ \mathrm{CT}_0$ where CT_0 is the arithmetical form of Church's thesis from [13, Section 3.2.14]. Then we have HA $+ \mathrm{CT}_0 \vdash \perp$, which is a contradiction by [13, Section 3.2.22]. ⊣

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

Let T be semi-classical arithmetic. By Theorems 3.9.(2) and 5.9, if T proves $(\Pi_{k+1} \vee \Pi_{k+1})$-DNE, then PA is U_{k+1}-conservative (and hence, a-fortiori $(\Pi_{k+1} \vee \Pi_{k+1})$-conservative) over T. On the other hand, if PA is $(\Pi_{k+1} \vee \Pi_{k+1})$-conservative over T, then T proves Σ_k-LEM by Lemma 5.5 and the fact that both of Σ_k and Π_k can be seen as sub-classes of Π_{k+1}. Thus $(\Pi_{k+1} \vee \Pi_{k+1})$-DNE implies the U_{k+1}-conservativity, which implies the $(\Pi_{k+1} \vee \Pi_{k+1})$-conservativity, which implies Σ_k-LEM and not vice versa. For further studying the relation of the $\mathrm{U}_{k+1}/(\Pi_{k+1} \vee \Pi_{k+1})$-conservativity and semi-classical arithmetic, we introduce some extended classes of Π_k and Σ_k.

DEFINITION 5.11.

- $\bigvee \Pi_k$ denotes the class consisting of disjunctions of formulas in Π_k.
- A class $\mathrm{E}\Pi_k$ is defined by the following clauses:
  - $\varphi \in \Pi_k$;
  - If $\varphi, \psi \in \mathrm{E}\Pi_k$, then $\varphi \vee \psi \in \mathrm{E}\Pi_k$;
  - If $\varphi \in \mathrm{E}\Pi_k$, then $\forall x \varphi \in \mathrm{E}\Pi_k$.
- $\mathrm{E}\Sigma_{k+1}$ denotes the class consisting of formulas of the form $\exists x_1, \dots, x_n \varphi$ where $\varphi \in \mathrm{E}\Pi_k$.

REMARK 5.12. $\Pi_k \subseteq \Pi_k \vee \Pi_k \subseteq \bigvee \Pi_k \subseteq \mathrm{E}\Pi_k \subseteq \mathrm{E}\Sigma_{k+1}$.

LEMMA 5.13. *For any* HA*-formulas* $\varphi, \psi \in \mathrm{E}\Pi_k$, *there exists* $\xi \in \mathrm{E}\Pi_k$ *such that* $\mathrm{FV}(\xi) = \mathrm{FV}(\varphi \wedge \psi)$ *and* $\mathsf{HA} \vdash \xi \leftrightarrow \varphi \wedge \psi$.

PROOF. By induction on the sum of the complexity of φ and ψ.

If both of φ and ψ are in Π_k, then we are done by [7, Lemma 4.3(2)].

Suppose $\psi :\equiv \psi_1 \vee \psi_2$ where $\psi_1, \psi_2 \in \mathrm{E}\Pi_k$. By the induction hypothesis, there exist $\xi_1, \xi_2 \in \mathrm{E}\Pi_k$ such that $\mathrm{FV}(\xi_1) = \mathrm{FV}(\varphi \wedge \psi_1)$, $\mathrm{FV}(\xi_2) = \mathrm{FV}(\varphi \wedge \psi_2)$, $\mathsf{HA} \vdash \xi_1 \leftrightarrow \varphi \wedge \psi_1$ and $\mathsf{HA} \vdash \xi_2 \leftrightarrow \varphi \wedge \psi_2$. Then we have that

$$\mathrm{FV}(\xi_1 \vee \xi_2) = \mathrm{FV}(\xi_1) \cup \mathrm{FV}(\xi_2) = \mathrm{FV}(\varphi \wedge \psi_1) \cup \mathrm{FV}(\varphi \wedge \psi_2) = \mathrm{FV}(\varphi \wedge \psi)$$

and that HA proves

$$\xi_1 \vee \xi_2 \leftrightarrow (\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2) \leftrightarrow \varphi \wedge (\psi_1 \vee \psi_2) \equiv \varphi \wedge \psi.$$

Thus one can take $\xi_1 \vee \xi_2 \in \mathrm{E}\Pi_k$ as a witness.

Suppose $\psi :\equiv \forall x \psi_1$ where $\psi_1 \in \mathrm{E}\Pi_k$. Without loss of generality, assume $x \notin \mathrm{FV}(\varphi)$. By the induction hypothesis, there exists $\xi_1 \in \mathrm{E}\Pi_k$ such that $\mathrm{FV}(\xi_1) = \mathrm{FV}(\varphi \wedge \psi_1)$ and $\mathsf{HA} \vdash \xi_1 \leftrightarrow \varphi \wedge \psi_1$. Then we have

$$\mathrm{FV}(\forall x \xi_1) = \mathrm{FV}(\varphi \wedge \psi_1) \setminus \{x\} = \mathrm{FV}(\varphi \wedge \forall x \psi_1)$$

and that HA proves

$$\forall x \xi_1 \leftrightarrow \forall x(\varphi \wedge \psi_1) \leftrightarrow \varphi \wedge \forall x \psi_1.$$

Thus one can take $\forall x \xi_1 \in \mathrm{E}\Pi_k$ as a witness. ⊣

In what follows, we use [7, Lemma 4.5] many times implicitly.

LEMMA 5.14. *For a* HA*-formula* φ, *the following hold*:

1. *If* $\varphi \in \mathrm{U}^+_{k+1}$, *then there exists* $\varphi' \in \mathrm{E}\Pi_{k+1}$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \varphi' \rightarrow \varphi$ *and* $\mathsf{PA} \vdash \varphi \rightarrow \varphi'$;
2. *If* $\varphi \in \mathrm{E}^+_{k+1}$, *then there exists* $\varphi' \in \mathrm{E}\Pi_{k+1}$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \varphi' \rightarrow \neg\varphi$ *and* $\mathsf{PA} \vdash \neg\varphi \rightarrow \varphi'$.

PROOF. We show items 1 and 2 by simultaneous induction on the structure of formulas. We suppress the arguments on free variables when they are clear from the context.

If φ is prime, then items 1 and 2 are trivial since φ is decidable in HA. For the induction step, assume items 1 and 2 hold for φ_1 and φ_2.

Case of $\varphi :\equiv \varphi_1 \vee \varphi_2$: For item 1, suppose $\varphi_1 \vee \varphi_2 \in \mathrm{U}^+_{k+1}$. Then $\varphi_1, \varphi_2 \in \mathrm{U}^+_{k+1}$. By using the induction hypothesis, there exist $\varphi_1', \varphi_2' \in \mathrm{E}\Pi_{k+1}$ such that $\mathsf{HA} + \Sigma_k$-LEM proves $\varphi_1' \rightarrow \varphi_1$ and $\varphi_2' \rightarrow \varphi_2$ and PA proves $\varphi_1 \rightarrow \varphi_1'$ and $\varphi_2 \rightarrow \varphi_2'$. Now $\varphi_1' \vee \varphi_2' \in \mathrm{E}\Pi_{k+1}$ and $\mathsf{HA} + \Sigma_k$-LEM proves

$$\varphi_1' \vee \varphi_2' \underset{\text{[I.H.] } \Sigma_k\text{-LEM}}{\longrightarrow} \varphi_1 \vee \varphi_2.$$

On the other hand, PA proves the converse. For item 2, suppose $\varphi_1 \vee \varphi_2 \in \mathrm{E}^+_{k+1}$. Then $\varphi_1, \varphi_2 \in \mathrm{E}^+_{k+1}$. By the induction hypothesis, there exist $\varphi_1', \varphi_2' \in \mathrm{E}\Pi_{k+1}$ such that $\mathsf{HA} + \Sigma_k$-LEM proves $\varphi_1' \rightarrow \neg\varphi_1$ and $\varphi_2' \rightarrow \neg\varphi_2$ and PA proves $\neg\varphi_1 \rightarrow \varphi_1'$ and $\neg\varphi_2 \rightarrow \varphi_2'$. By Lemma 5.13, there exists $\varphi' \in \mathrm{E}\Pi_{k+1}$ such that $\mathrm{FV}(\varphi') = \mathrm{FV}(\varphi_1' \wedge \varphi_2')$ and $\mathsf{HA} \vdash \varphi' \leftrightarrow \varphi_1' \wedge \varphi_2'$. Then we have that $\mathsf{HA} + \Sigma_k$-LEM proves

$$\varphi' \leftrightarrow \varphi_1' \wedge \varphi_2' \underset{\text{[I.H.] } \Sigma_k\text{-LEM}}{\longrightarrow} \neg\varphi_1 \wedge \neg\varphi_2 \leftrightarrow \neg(\varphi_1 \vee \varphi_2)$$

and also PA proves the converse.

Case of $\varphi :\equiv \varphi_1 \wedge \varphi_2$: For item 1, suppose $\varphi_1 \wedge \varphi_2 \in \mathrm{U}^+_{k+1}$. Then $\varphi_1, \varphi_2 \in \mathrm{U}^+_{k+1}$. By using the induction hypothesis and Lemma 5.13, one can take a witness for $\varphi_1 \wedge \varphi_2$ in a straightforward way. Item 2 follows from the induction hypothesis as in the case of $\varphi :\equiv \varphi_1 \vee \varphi_2$: $\varphi_1' \vee \varphi_2' \in \mathrm{E}\Pi_{k+1}$ is the witness since $\mathsf{HA} + \Sigma_k$-LEM proves

$$\varphi_1' \vee \varphi_2' \underset{\text{[I.H.] } \Sigma_k\text{-LEM}}{\longrightarrow} \neg\varphi_1 \vee \neg\varphi_2 \rightarrow \neg(\varphi_1 \wedge \varphi_2)$$

and PA proves the converse.

Case of $\varphi :\equiv \varphi_1 \rightarrow \varphi_2$: For item 1, suppose $\varphi_1 \rightarrow \varphi_2 \in \mathrm{U}^+_{k+1}$. Then $\varphi_1 \in \mathrm{E}^+_{k+1}$ and $\varphi_2 \in \mathrm{U}^+_{k+1}$. By the induction hypothesis, there exist $\varphi_1', \varphi_2' \in \mathrm{E}\Pi_{k+1}$ such that $\mathsf{HA} + \Sigma_k$-LEM proves $\varphi_1' \rightarrow \neg\varphi_1$ and $\varphi_2' \rightarrow \varphi_2$ and PA proves $\neg\varphi_1 \rightarrow \varphi_1'$ and $\varphi_2 \rightarrow \varphi_2'$. Now $\varphi_1' \vee \varphi_2' \in \mathrm{E}\Pi_{k+1}$ and $\mathsf{HA} + \Sigma_k$-LEM proves

$$\varphi_1' \vee \varphi_2' \underset{\text{[I.H.] } \Sigma_k\text{-LEM}}{\longrightarrow} \neg\varphi_1 \vee \varphi_2 \rightarrow (\varphi_1 \rightarrow \varphi_2).$$

On the other hand, PA proves the converse. For item 2, suppose $\varphi_1 \rightarrow \varphi_2 \in \mathrm{E}^+_{k+1}$. Then $\varphi_1 \in \mathrm{U}^+_{k+1}$ and $\varphi_2 \in \mathrm{E}^+_{k+1}$. By the induction hypothesis, there exist $\varphi_1', \varphi_2' \in \mathrm{E}\Pi_{k+1}$ such that $\mathsf{HA} + \Sigma_k$-LEM proves $\varphi_1' \rightarrow \varphi_1$ and $\varphi_2' \rightarrow \neg\varphi_2$ and PA proves $\varphi_1 \rightarrow \varphi_1'$ and $\neg\varphi_2 \rightarrow \varphi_2'$. By Lemma 5.13, there exists $\varphi' \in \mathrm{E}\Pi_{k+1}$ such that $\mathrm{FV}(\varphi') =$

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

$\mathrm{FV}(\varphi_1' \wedge \varphi_2')$ and $\mathsf{HA} \vdash \varphi' \leftrightarrow \varphi_1' \wedge \varphi_2'$. Then we have that $\mathsf{HA} + \Sigma_k$-LEM proves

$$\varphi' \leftrightarrow \varphi_1' \wedge \varphi_2' \underset{[\text{I.H.}]\,\Sigma_k\text{-LEM}}{\longrightarrow} \varphi_1 \wedge \neg\varphi_2 \rightarrow \neg(\varphi_1 \rightarrow \varphi_2)$$

and also that PA proves the converse.

Case of $\varphi :\equiv \exists x\varphi_1$: For item 1, suppose $\exists x\varphi_1 \in \mathrm{U}^+_{k+1}$. Then $\exists x\varphi_1 \in \mathrm{E}^+_k$. By Remark 3.10, there exists $\varphi' \in \Sigma_k$ such that $\mathrm{FV}(\varphi') = \mathrm{FV}(\varphi)$ and $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \varphi' \leftrightarrow \varphi$. Since Σ_k can be seen as a subclass of Π_{k+1}, we are done. For item 2, suppose $\exists x\varphi_1 \in \mathrm{E}^+_{k+1}$. Then $\varphi_1 \in \mathrm{E}^+_{k+1}$. By the induction hypothesis, there exists $\varphi_1' \in \mathrm{E}\Pi_{k+1}$ such that $\mathrm{FV}(\varphi_1') = \mathrm{FV}(\varphi_1)$, $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \varphi_1' \rightarrow \neg\varphi_1$ and $\mathsf{PA} \vdash \neg\varphi_1 \rightarrow \varphi_1'$. Now $\forall x\varphi_1' \in \mathrm{E}\Pi_{k+1}$ and $\mathrm{FV}(\forall x\varphi_1') = \mathrm{FV}(\exists x\varphi_1)$. Then we have that $\mathsf{HA} + \Sigma_k$-LEM proves

$$\forall x\varphi_1' \underset{[\text{I.H.}]\,\Sigma_k\text{-LEM}}{\longrightarrow} \forall x\neg\varphi_1 \leftrightarrow \neg\exists x\varphi_1$$

and also that PA proves the converse.

Case of $\varphi :\equiv \forall x\varphi_1$: For item 1, suppose $\forall x\varphi_1 \in \mathrm{U}^+_{k+1}$. Then $\varphi_1 \in \mathrm{U}^+_{k+1}$. By the induction hypothesis, there exists $\varphi_1' \in \mathrm{E}\Pi_{k+1}$ such that $\mathrm{FV}(\varphi_1') = \mathrm{FV}(\varphi_1)$, $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \varphi_1' \rightarrow \varphi_1$ and $\mathsf{PA} \vdash \varphi_1 \rightarrow \varphi_1'$. It is straightforward to see that $\forall x\varphi_1' \in \mathrm{E}\Pi_{k+1}$ is a witness for $\forall x\varphi_1 \in \mathrm{U}^+_{k+1}$. For item 2, suppose $\forall x\varphi_1 \in \mathrm{E}^+_{k+1}$. Then $\forall x\varphi_1 \in \mathrm{U}^+_k$. By Remark 3.10, there exists $\varphi' \in \Pi_k$ such that $\mathrm{FV}(\varphi') = \mathrm{FV}(\varphi)$ and $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \varphi' \leftrightarrow \varphi$. Since $\neg\varphi'$ is equivalent to some $\varphi'' \in \Sigma_k$ in the presence of Σ_k-DNE (cf. Remark 5.3), we are done. ⊣

LEMMA 5.15. *Let T be a theory containing* HA *and X be a set of* HA*-sentences. If* $\mathsf{PA} + X$ *is* $\mathrm{E}\Pi_{k+1}$*-conservative over* $T + X$*, then so is* U_{k+1}*-conservative.*

PROOF. Let $\varphi \in \mathrm{U}_{k+1}$. Suppose $\mathsf{PA} + X \vdash \varphi$. By Lemma 5.14, there exists $\varphi' \in \mathrm{E}\Pi_{k+1}$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \varphi' \rightarrow \varphi$ and $\mathsf{PA} \vdash \varphi \rightarrow \varphi'$. Then $\mathsf{PA} + X \vdash \varphi'$. By our assumption, we have $T + X \vdash \varphi'$. As in the proof of Lemma 5.5, one can show $T + X \vdash \Sigma_k$-LEM by using the $\mathrm{E}\Pi_{k+1}$-conservativity. Then $T + X \vdash \varphi$ follows. ⊣

THEOREM 5.16. *Let T be semi-classical arithmetic and X be a set of* HA*-sentences in $\mathcal{Q}_{k+1}$. Then the following are pairwise equivalent*:

1. $\mathsf{PA} + X$ *is* U_{k+1}*-conservative over* $T + X$;
2. $\mathsf{PA} + X$ *is* $\mathrm{E}\Pi_{k+1}$*-conservative over* $T + X$;
3. $T + X$ *is closed under* $\mathrm{E}\Pi_{k+1}$-DNE-R;
4. $T + X$ *is closed under* $\mathrm{E}\Pi_{k+1}$-CD-R;
5. $T + X$ *is closed under* U_{k+1}-DNE-R;
6. $T + X$ *is closed under* U_{k+1}-CD-R.

PROOF. Since $\mathrm{E}\Pi_{k+1} \subseteq \mathrm{U}_{k+1}$, the equivalence between (1) and (2) follows immediately from Lemma 5.15.
($2 \rightarrow 3$): Let $\varphi \in \mathrm{E}\Pi_{k+1}$ and assume $T + X \vdash \neg\neg\varphi$. Since $T + X \subseteq \mathsf{PA} + X$, we have $\mathsf{PA} + X \vdash \varphi$. By (2), we have $T + X \vdash \varphi$.
($3 \rightarrow 4$): Let $\varphi, \psi(x) \in \mathrm{E}\Pi_{k+1}$ and $x \notin \mathrm{FV}(\varphi)$. Assume $T + X \vdash \forall x(\varphi \vee \psi(x))$. Since HA proves $\neg\neg(\varphi \vee \neg\varphi)$ and $(\varphi \vee \neg\varphi) \wedge \forall x(\varphi \vee \psi(x)) \rightarrow \varphi \vee \forall x\psi(x)$, we have $T + X \vdash \neg\neg(\varphi \vee \forall x\psi(x))$. Since $\varphi \vee \forall x\psi(x) \in \mathrm{E}\Pi_{k+1}$, by $\mathrm{E}\Pi_{k+1}$-DNE-R, we have $T + X \vdash \varphi \vee \forall x\psi(x)$.

(4 → 2): Assume that $T + X$ is closed under $\mathrm{E}\Pi_{k+1}$-CD-R. By Lemma 5.7, we have $T + X \vdash \Sigma_k$-LEM. We show that $\mathsf{PA} + X \vdash \varphi_1 \vee \cdots \vee \varphi_n$ implies $T + X \vdash \varphi_1 \vee \cdots \vee \varphi_n$ for any $\varphi_1, \dots, \varphi_n \in \mathrm{E}\Pi_{k+1}$ by induction on the sum of the complexity of $\varphi_1, \dots, \varphi_n \in \mathrm{E}\Pi_{k+1}$.

First, suppose that all of $\varphi_1, \dots, \varphi_n$ are in Π_{k+1}. Let $\varphi_i :\equiv \forall x_i \varphi_i'$ with $\varphi_i' \in \Sigma_k$ for each $i \in \{1, \dots, n\}$. Assume $\mathsf{PA} + X \vdash \varphi_1 \vee \cdots \vee \varphi_n$. Then $\mathsf{PA} + X \vdash \varphi_1' \vee \cdots \vee \varphi_n'$. Since $T + X \vdash \Sigma_k$-LEM and $X \subseteq \mathcal{Q}_{k+1}$, by Corollary 4.3, we have $T + X \vdash \varphi_1' \vee \cdots \vee \varphi_n'$. Then $T + X \vdash \forall x_1(\varphi_1' \vee \cdots \vee \varphi_n')$ follows. By $\mathrm{E}\Pi_{k+1}$-CD-R, we have $T + X \vdash \forall x_1 \varphi_1' \vee \varphi_2' \vee \cdots \vee \varphi_n'$. Iterating this procedure for more $n-1$ times, we have $T + X \vdash \forall x_1 \varphi_1' \vee \cdots \vee \forall x_n \varphi_n'$.

Secondly, suppose $\varphi_1, \dots, \varphi_n \in \mathrm{E}\Pi_{k+1}$ and $\varphi_n :\equiv \varphi_n' \vee \varphi_n''$ with $\varphi_n', \varphi_n'' \in \mathrm{E}\Pi_{k+1}$. Without loss of generality, let $n > 1$. Assume $\mathsf{PA} + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n$, equivalently, $\mathsf{PA} + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n' \vee \varphi_n''$. By the induction hypothesis, we have $T + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n' \vee \varphi_n''$, equivalently, $T + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n$.

Finally, suppose $\varphi_1, \dots, \varphi_n \in \mathrm{E}\Pi_{k+1}$ and $\varphi_n :\equiv \forall x_n \varphi_n'$ with $\varphi_n' \in \mathrm{E}\Pi_{k+1}$. Without loss of generality, let $n > 1$. Assume $\mathsf{PA} + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n$. Then $\mathsf{PA} + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n'$ follows. By the induction hypothesis, we have $T + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n'$, and hence, $T + X \vdash \forall x_n(\varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n')$. By $\mathrm{E}\Pi_{k+1}$-CD-R, we have $T + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n$.

The implications (1 → 5) and (5 → 6) are shown as for (2 → 3) and (3 → 4) respectively. In addition, (6 → 4) is trivial. ⊣

Next, we characterize the $(\Pi_{k+1} \vee \Pi_{k+1})$-conservativity by several rules.

LEMMA 5.17. *Let T be a theory containing* HA. *If T is closed under* $(\Pi_k \vee \Pi_k)$-DNE-R, *then so is* Π_k-CD-R.

PROOF. The proof of (3 → 4) of Theorem 5.16 works. ⊣

LEMMA 5.18. *Let T be a theory containing* HA. *Then T is closed under* Σ_k-DML$^\perp$-R *if and only if T is closed under* $(\Pi_k \vee \Pi_k)$-DNE-R.

PROOF. One can show the "only if" direction as in the proof of that in Lemma 5.8. For the converse direction, again by the corresponding proof in Lemma 5.8, it suffices to show that if T is closed under $(\Pi_k \vee \Pi_k)$-DNE-R, then T proves Σ_{k-1}-LEM. The latter is the case by Lemmata 5.17 and 5.7. ⊣

THEOREM 5.19. *Let T be semi-classical arithmetic and X be a set of* HA-*sentences in $\mathcal{Q}_{k+1}$. Then the following are pairwise equivalent*:

1. $\mathsf{PA} + X$ *is* $(\Pi_{k+1} \vee \Pi_{k+1})$-*conservative over* $T + X$;
2. $T + X$ *is closed under* $(\Pi_{k+1} \vee \Pi_{k+1})$-DNE-R;
3. $T + X$ *is closed under* Π_{k+1}-CD-R;
4. $T + X$ *is closed under* Σ_{k+1}-DML$^\perp$-R;
5. $T + X$ *is closed under* Σ_{k+1}-DML-R *and* $T + X$ *proves* Σ_k-DNE.

PROOF. One can show (1 → 2) as in the proof of (2 → 3) of Theorem 5.16. The implication (2 → 3) is by Lemma 5.17.

We show (3 → 1). Assume that $T + X$ is closed under Π_{k+1}-CD-R. By Lemma 5.7, we have $T + X \vdash \Sigma_k$-LEM. Let $\varphi_1 :\equiv \forall x_1 \psi_1$ and $\varphi_2 :\equiv \forall x_2 \psi_2$ with $\psi_1, \psi_2 \in \Sigma_k$. Suppose $\mathsf{PA} + X \vdash \forall x_1 \psi_1 \vee \forall x_2 \psi_2$. Then $\mathsf{PA} + X \vdash \neg\left(\exists x_1 \psi_1^\perp \wedge \exists x_2 \psi_2^\perp\right)$. Since

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

$\exists x_1\psi_1^{\perp} \wedge \exists_2\psi_2^{\perp}$ is equivalent to a formula in Σ_{k+1} (cf. [7, Lemma 4.3(2)]), by Remark 5.3, there exists $\xi \in \Pi_{k+1}$ such that $\mathrm{FV}(\xi) = \mathrm{FV}(\forall x_1\psi_1 \vee \forall x\psi_2)$ and $\mathsf{HA} + \Sigma_k$-DNE $\vdash \xi \leftrightarrow \neg\left(\exists x_1\psi_1^{\perp} \wedge \exists x_2\psi_2^{\perp}\right)$. Then we have $\mathsf{PA} + X \vdash \xi$. Since $X \subseteq \mathcal{Q}_{k+1}$, by Corollary 3.19, we have $\mathsf{HA} + X + \Sigma_k$-LEM $\vdash \xi$. Since Σ_k-DNE is derivable from Σ_k-LEM, we have that $T + X$ proves $\neg\left(\exists x_1\psi_1^{\perp} \wedge \exists x_2\psi_2^{\perp}\right)$, equivalently, $\forall x_1, x_2\neg\left(\psi_1^{\perp} \wedge \psi_2^{\perp}\right)$. Since $T + X \vdash \Sigma_k$-DNE, again by Remark 5.3, we have that $T + X$ proves $\forall x_1, x_2\neg(\neg\psi_1 \wedge \neg\psi_2)$, equivalently, $\forall x_1, x_2\neg\neg(\psi_1 \vee \psi_2)$. Since $\psi_1 \vee \psi_2$ is equivalent to a formula in Σ_k (cf. [7, Lemma 4.4]), $T + X \vdash \forall x_1, x_2(\psi_1 \vee \psi_2)$ follows. By using Π_{k+1}-CD-R twice, we have $T + X \vdash \forall x_1\psi_1 \vee \forall x_2\psi_2$.

The equivalence ($2 \leftrightarrow 4$) is by Lemma 5.18. The implication ($5 \rightarrow 4$) is by the fact that for $\varphi \in \Sigma_{k+1}$, $\varphi^{\perp}$ is derived from $\neg\varphi$ in the presence of Σ_k-DNE (cf. Remark 5.3). The implication (3 & $4 \rightarrow 5$) is by Lemma 5.7 (note that Σ_k-LEM implies Σ_k-DNE). ⊣

Remark 5.20. From the perspective of Remark 5.12, it is natural to ask the status of the $\bigvee\Pi_{k+1}$-conservativity. As in the proof of Theorem 5.19, one can show the following equivalence:

1. $\mathsf{PA} + X$ is $\bigvee\Pi_{k+1}$-conservative over $T + X$;
2. For any $\varphi_1, \dots, \varphi_n \in \Pi_{k+1}$, if $T + X \vdash \neg\neg(\varphi_1 \vee \cdots \vee \varphi_n)$, then $T + X \vdash \varphi_1 \vee \cdots \vee \varphi_n$;
3. For any $\varphi_1, \dots, \varphi_n \in \Pi_{k+1}$ such that $x \notin \mathrm{FV}(\varphi_1 \vee \cdots \vee \varphi_{n-1})$, if $T + X \vdash \forall x(\varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \varphi_n)$, then $T + X \vdash \varphi_1 \vee \cdots \vee \varphi_{n-1} \vee \forall x\varphi_n$;
4. For any $\varphi_1, \dots, \varphi_n \in \Sigma_{k+1}$, if $T + X \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$, then $T + X \vdash \varphi_1^{\perp} \vee \cdots \vee \varphi_n^{\perp}$;
5. $T + X$ proves Σ_k-DNE and for any $\varphi_1, \dots, \varphi_n \in \Sigma_{k+1}$, if $T + X \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$, then $T + X \vdash \neg\varphi_1 \vee \cdots \vee \neg\varphi_n$;

where $X \subseteq \mathcal{Q}_{k+1}$. This characterization suggests that the $\bigvee\Pi_{k+1}$-conservativity lies strictly between the U_{k+1}-conservativity and the $(\Pi_{k+1} \vee \Pi_{k+1})$-conservativity, but we do not have the proof of the strictness.

Remark 5.21. From the comparison between [6, Corollary 7.6] and the equivalences in Theorem 5.19, it is natural to ask whether the (contrapositive) collection rule restricted to formulas in Π_{k+1} is also equivalent to the items in Theorem 5.19. This question is still open.

§6. Conservation theorems for the classes of sentences. In the study of fragments of PA, the conservativity for classes of sentences has been studied extensively e.g., in [11, Section 2]. The following proposition states that the conservativity for a class of formulas is equivalent to that restricted only to sentences if the class is closed under taking a universal closure:

Proposition 6.1. *Let Γ be a class of* HA*-formulas such that Γ is closed under taking a universal closure. For any theories T and T' containing* HA *in the language of* HA, *if T' is conservative over T for any sentences in Γ, then T' is Γ-conservative over T.*

Proof. Let $\varphi \in \Gamma$. Assume $T' \vdash \varphi$. Then we have $T' \vdash \widetilde{\varphi}$ where $\widetilde{\varphi}$ is the universal closure of φ. Since $\widetilde{\varphi}$ is a sentence in Γ, by our assumption, we have $T \vdash \widetilde{\varphi}$, and hence, $T \vdash \varphi$. ⊣

Therefore, for classes as $\Pi_k, \mathrm{U}_k, \mathrm{E}\Pi_k$ etc., the strength of the conservativity does not vary even if we restrict them only to sentences. On the other hand, since $\Sigma_k, \mathrm{E}_k, \mathrm{F}_k$ etc. are not closed under taking a universal closure, this is not the case for such classes. In what follows, we explore the relation on the notion that PA is Γ-conservative over T for semi-classical arithmetic T and the class Γ of sentences.

DEFINITION 6.2. For a class Γ of HA-formulas, $\underset{\sim}{\Gamma}$ denotes the class of HA-sentences in Γ.

6.1. Conservation theorems for Σ_k sentences and E_k sentences. For the $\underset{\sim}{\Sigma_k}$-conservativity, we have the following:

PROPOSITION 6.3. *Let T be semi-classical arithmetic containing Σ_{k-1}-LEM, and X be a set of HA-sentences in $\mathcal{Q}_k$. Then $\mathsf{PA} + X$ is $\underset{\sim}{\Sigma_{k+1}}$-conservative over $T + X$ if and only if $T + X$ is closed under $\underset{\sim}{\Sigma_{k+1}}$-DNE-R.*

PROOF. We first show the "only if" direction. Let $\varphi \in \underset{\sim}{\Sigma_{k+1}}$. Assume $T + X \vdash \neg\neg\varphi$. Then $\mathsf{PA} + X \vdash \varphi$. Since $\mathsf{PA} + X$ is now $\underset{\sim}{\Sigma_{k+1}}$-conservative over $T + X$, we have $T + X \vdash \varphi$.

In the following, we show the converse direction. Without loss of generality, assume $k > 0$. Let $\exists x \forall y\, \psi \in \underset{\sim}{\Sigma_{k+1}}$ with ψ in Σ_{k-1}. Assume $\mathsf{PA} + X \vdash \exists x \forall y \psi$. By Proposition 3.2.(2), we have $\mathsf{HA}^{\$} + X^{\$} \vdash \neg_{\$}\neg_{\$}\exists x \forall y \psi^{\$}$, and hence, $\mathsf{HA}^{\$} + \Sigma_{k-1}\text{-LEM} + X \vdash \neg_{\$}\neg_{\$}\exists x \forall y \psi^{\$}$ by Lemma 3.14. Using Lemma 3.5.(2), we have $\mathsf{HA}^{\$} + \Sigma_{k-1}\text{-LEM} + X \vdash \neg_{\$}\neg_{\$}\exists x \forall y\, (\psi \vee \$)$. By substituting \$ with $\perp$ (cf. Lemma 3.16), we have $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} + X \vdash \neg\neg\exists x \forall y \psi$. Since T is semi-classical arithmetic containing Σ_{k-1}-LEM, we have $T + X \vdash \neg\neg\exists x \forall y \psi$. By $\underset{\sim}{\Sigma_{k+1}}$-DNE-R, $T + X \vdash \exists x \forall y \psi$ follows. ⊣

Proposition 6.3 is a counterpart of the equivalence between (3) and (4) in Theorem 5.9 for the case of sentences. In what follows, we deal with the $\underset{\sim}{\mathrm{E}_{k+1}}$-conservativity. In particular, we show that the $\underset{\sim}{\mathrm{E}_{k+1}}$-conservativity can be reduced to $\underset{\sim}{\mathrm{E}\Sigma_{k+1}}$-conservativity.

LEMMA 6.4. *For HA-formulas $\varphi_1, \varphi_2 \in \mathrm{E}\Sigma_{k+1}$, there exist $\psi, \xi \in \mathrm{E}\Sigma_{k+1}$ such that $\mathrm{FV}(\psi) = \mathrm{FV}(\varphi_1 \wedge \varphi_2) = \mathrm{FV}(\varphi_1 \vee \varphi_2) = \mathrm{FV}(\xi)$ and HA proves $\psi \leftrightarrow \varphi_1 \wedge \varphi_2$ and $\xi \leftrightarrow \varphi_1 \vee \varphi_2$.*

PROOF. Let $\varphi_1 :\equiv \exists x_1, \dots, x_n \varphi_1'$ and $\varphi_2 :\equiv \exists y_1, \dots, y_m \varphi_2'$ with $\varphi_1', \varphi_2' \in \mathrm{E}\Pi_k$. Without loss of generality, assume $x_1, \dots, x_n \notin \mathrm{FV}(\varphi_2')$ and $y_1, \dots, y_m \notin \mathrm{FV}(\varphi_1')$.

By Lemma 5.13, there exists $\psi' \in \mathrm{E}\Pi_k$ such that $\mathrm{FV}(\psi') = \mathrm{FV}(\varphi_1' \wedge \varphi_2')$ and $\mathsf{HA} \vdash \psi' \leftrightarrow \varphi_1' \wedge \varphi_2'$. Put $\psi :\equiv \exists x_1, \dots, x_n, y_1, \dots, y_m\, \psi'$, which is in $\mathrm{E}\Sigma_{k+1}$. Then it is trivial that $\mathrm{FV}(\psi) = \mathrm{FV}(\varphi_1 \wedge \varphi_2)$ and $\mathsf{HA} \vdash \psi \leftrightarrow \varphi_1 \wedge \varphi_2$.

Put $\xi :\equiv \exists x_1, \dots, x_n, y_1, \dots, y_m\, (\varphi_1' \vee \varphi_2')$, which is in $\mathrm{E}\Sigma_{k+1}$. Since ξ is equivalent to $\exists x_1, \dots, x_n \varphi_1' \vee \exists y_1, \dots, y_m \varphi_2'$ over HA, we have that $\mathrm{FV}(\xi) = \mathrm{FV}(\varphi_1 \vee \varphi_2)$ and $\mathsf{HA} \vdash \xi \leftrightarrow \varphi_1 \vee \varphi_2$. ⊣

LEMMA 6.5. *For a HA-formula φ, the following hold:*

1. *If $\varphi \in \mathrm{U}^+_{k+1}$, then there exists $\varphi' \in \mathrm{E}\Sigma_{k+1}$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \to \neg\varphi$ and $\mathsf{PA} \vdash \neg\varphi \to \varphi'$.*

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

2. *If* $\varphi \in \mathrm{E}^+_{k+1}$, *then there exists* $\varphi' \in \mathrm{E}\Sigma_{k+1}$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \to \varphi$ *and* $\mathsf{PA} \vdash \varphi \to \varphi'$.

Proof. Note that $\mathrm{U}^+_{k+1} = \mathcal{R}'_{k+1}$ and $\mathrm{E}^+_{k+1} = \mathcal{J}'_{k+1}$ where $\mathcal{R}'_{k+1}$ and $\mathcal{J}'_{k+1}$ are the classes defined in Remark 4.5. Then it suffices to show items 1 and 2 where U^+_{k+1} and E^+_{k+1} are replaced by $\mathcal{R}'_{k+1}$ and $\mathcal{J}'_{k+1}$ respectively. In the following, we show the assertions by induction on the constructions of $\mathcal{R}'_{k+1}$ and $\mathcal{J}'_{k+1}$.

For $\varphi \in \mathrm{E}^+_k \subseteq \mathcal{R}'_{k+1}$, by Lemma 5.14, there exists $\varphi' \in \mathrm{E}\Pi_k (\subseteq \mathrm{E}\Sigma_{k+1})$ such that $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \to \neg\varphi$ and $\mathsf{PA} \vdash \neg\varphi \to \varphi'$. For $\varphi \in \mathrm{U}^+_k \subseteq \mathcal{J}'_{k+1}$, by Lemma 5.14, there exists $\varphi' \in \mathrm{E}\Pi_k (\subseteq \mathrm{E}\Sigma_{k+1})$ such that $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \to \varphi$ and $\mathsf{PA} \vdash \varphi \to \varphi'$. For the induction step, let $\varphi_1, \varphi_2 \in \mathcal{R}'_{k+1}$ and $\psi_1, \psi_2 \in \mathcal{J}'_{k+1}$ and $\varphi'_1, \varphi'_2, \psi'_1, \psi'_2 \in \mathrm{E}\Sigma_{k+1}$ satisfy $\mathrm{FV}(\varphi_1) = \mathrm{FV}(\varphi'_1)$, $\mathrm{FV}(\varphi_2) = \mathrm{FV}(\varphi'_2)$, $\mathrm{FV}(\psi_1) = \mathrm{FV}(\psi'_1)$, $\mathrm{FV}(\psi_2) = \mathrm{FV}(\psi'_2)$ and that $\mathsf{HA} + \Sigma_{k-1}\text{-LEM}$ proves $\varphi'_1 \to \neg\varphi_1$, $\varphi'_2 \to \neg\varphi_2$, $\psi'_1 \to \psi_1$, $\psi'_2 \to \psi_2$ and PA proves $\neg\varphi_1 \to \varphi'_1$, $\neg\varphi_2 \to \varphi'_2$, $\psi_1 \to \psi'_1$, $\psi_2 \to \psi'_2$. By Lemma 6.4, for any conjunction and disjunction of $\varphi'_1, \varphi'_2, \psi'_1, \psi'_2 \in \mathrm{E}\Sigma_{k+1}$, there exists an equivalent (over HA) $\xi \in \mathrm{E}\Sigma_{k+1}$ which preserves the free variables. For $\varphi :\equiv \varphi_1 \lor \varphi_2 \in \mathcal{R}'_{k+1}$, take $\varphi' \in \mathrm{E}\Sigma_{k+1}$ as an equivalent of $\varphi'_1 \land \varphi'_2$. For $\varphi :\equiv \psi_1 \lor \psi_2 \in \mathcal{J}'_{k+1}$, take $\varphi' \in \mathrm{E}\Sigma_{k+1}$ as an equivalent of $\psi'_1 \lor \psi'_2$. For $\varphi :\equiv \varphi_1 \land \varphi_2 \in \mathcal{R}'_{k+1}$, take $\varphi' \in \mathrm{E}\Sigma_{k+1}$ as an equivalent of $\varphi'_1 \lor \varphi'_2$. For $\varphi :\equiv \psi_1 \land \psi_2 \in \mathcal{J}'_{k+1}$, take $\varphi' \in \mathrm{E}\Sigma_{k+1}$ as an equivalent of $\psi'_1 \land \psi'_2$. For $\varphi :\equiv \psi_1 \to \varphi_2 \in \mathcal{R}'_{k+1}$, take $\varphi' \in \mathrm{E}\Sigma_{k+1}$ as an equivalent of $\psi'_1 \land \varphi'_2$. For $\varphi :\equiv \varphi_1 \to \psi_2 \in \mathcal{J}'_{k+1}$, take $\varphi' \in \mathrm{E}\Sigma_{k+1}$ as an equivalent of $\varphi'_1 \lor \psi'_2$. For $\varphi :\equiv \forall x \varphi_1 \in \mathcal{R}'_{k+1}$, take $\varphi' :\equiv \exists x \varphi'_1 \in \mathrm{E}\Sigma_{k+1}$. For $\varphi :\equiv \exists x \psi_1 \in \mathcal{J}'_{k+1}$, take $\varphi' :\equiv \exists x \psi'_1 \in \mathrm{E}\Sigma_{k+1}$. We leave the routine verification for the reader. ⊣

Corollary 6.6. *For a* HA*-formula* φ, *the following hold*:

1. *If* $\varphi \in \mathrm{U}^+_{k+1}$, *then there exists* $\varphi' \in \Sigma_{k+1}$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + (\Pi_k \lor \Pi_k)\text{-DNE} \vdash \varphi' \to \neg\varphi$ *and* $\mathsf{PA} \vdash \neg\varphi \to \varphi'$.
2. *If* $\varphi \in \mathrm{E}^+_{k+1}$, *then there exists* $\varphi' \in \Sigma_{k+1}$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + (\Pi_k \lor \Pi_k)\text{-DNE} \vdash \varphi' \to \varphi$ *and* $\mathsf{PA} \vdash \varphi \to \varphi'$.

Proof. Since $\varphi' \in \mathrm{E}\Sigma_{k+1}$ is of the form $\exists x \varphi'_1$ where $\varphi'_1 \in \mathrm{E}\Pi_k \subseteq \mathrm{U}^+_k$, by Theorem 3.9.(2), there exists $\psi \in \Sigma_{k+1}$ such that $\mathrm{FV}(\varphi') = \mathrm{FV}(\psi)$ and $\mathsf{HA} + (\Pi_k \lor \Pi_k)\text{-DNE} \vdash \varphi' \leftrightarrow \psi$. Since $\mathsf{HA} + (\Pi_k \lor \Pi_k)\text{-DNE}$ proves $\Sigma_{k-1}\text{-LEM}$, our corollary follows from Lemma 6.5. ⊣

Theorem 6.7. *Let* T *be semi-classical arithmetic and* X *be a set of* HA*-sentences. Then* $\mathsf{PA} + X$ *is* $\widetilde{\mathrm{E}_{k+1}}$*-conservative over* $T + X$ *if and only if* $\mathsf{PA} + X$ *is* $\widetilde{\mathrm{E}\Sigma_{k+1}}$*-conservative over* $T + X$.

Proof. The "only if" direction is trivial since $\widetilde{\mathrm{E}\Sigma_{k+1}} \subseteq \widetilde{\mathrm{E}_{k+1}}$. We show the converse direction. Let $\varphi \in \widetilde{\mathrm{E}_{k+1}}$. Assume $\mathsf{PA} + X \vdash \varphi$. By Lemma 6.5, there exists $\varphi' \in \widetilde{\mathrm{E}\Sigma_{k+1}}$ such that $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \to \varphi$ and $\mathsf{PA} \vdash \varphi \to \varphi'$. Then $\mathsf{PA} + X \vdash \varphi'$. By our assumption, we have $T + X \vdash \varphi'$. On the other hand, as in the proof of Lemma 5.5, one can show $T + X \vdash \Sigma_{k-1}\text{-LEM}$ (note that $\widetilde{\mathrm{E}\Pi_k}$ can be seen as a sub-class of $\widetilde{\mathrm{E}\Sigma_{k+1}}$ and the $\widetilde{\mathrm{E}\Pi_k}$-conservativity implies the $\mathrm{E}\Pi_k$-conservativity by Proposition 6.1). Then we have $T + X \vdash \varphi$. ⊣

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

6.2. Conservation theorem for F_k sentences. Next, we characterize the $\underset{\sim}{\mathrm{F}_k}$-conservativity. To investigate the class F_k, it is convenient to consider the following class:

DEFINITION 6.8. Let B_k^+ be the class of formulas which are constructed from formulas in $\mathrm{E}_k^+ \cup \mathrm{U}_k^+$ by using logical connectives $\wedge, \vee$ and $\rightarrow$. Let B_k^+-LEM be LEM restricted to formulas in B_k^+.

PROPOSITION 6.9. $\mathsf{HA} \vdash \Sigma_k\text{-LEM} \leftrightarrow \mathrm{B}_k^+\text{-LEM}$.

PROOF. First, $\mathsf{HA} + \mathrm{B}_k^+\text{-LEM} \vdash \Sigma_k\text{-LEM}$ is trivial since $\Sigma_k \subseteq \mathrm{E}_k^+$. We show the converse direction. By Remark 3.10, inside $\mathsf{HA} + \Sigma_k$-LEM, one may assume that $\varphi \in \mathrm{B}_k^+$ is constructed from formulas in $\Sigma_k \cup \Pi_k$ by using logical connectives $\wedge, \vee$ and $\rightarrow$. Then we have $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \mathrm{B}_k^+\text{-LEM}$ in a straightforward way. ⊣

PROPOSITION 6.10. $\mathrm{B}_k^+ = \mathrm{F}_k$.

PROOF. Since $\mathrm{F}_k^+ = \mathrm{F}_k$ (cf. Remark 3.8), it suffices to show $\mathrm{B}_k^+ = \mathrm{F}_k^+$.

First, $\mathrm{B}_k^+ \subseteq \mathrm{F}_k^+$ is trivial since $\mathrm{E}_k^+ \subseteq \mathrm{F}_k^+$, $\mathrm{U}_k^+ \subseteq \mathrm{F}_k^+$ and the fact that F_k^+ is closed under $\wedge, \vee$ and $\rightarrow$.

We show that $\varphi \in \mathrm{F}_k^+$ implies $\varphi \in \mathrm{B}_k^+$ for all HA-formulas φ by induction on the structure of formulas. If φ is prime, since $\varphi \in \mathrm{B}_k^+$, then we are done. For the induction step, assume that it holds for φ_1 and φ_2. If $\varphi_1 \wedge \varphi_2 \in \mathrm{F}_k^+$, then $\varphi_1, \varphi_2 \in \mathrm{F}_k^+$ follows. By the induction hypothesis, we have $\varphi_1, \varphi_2 \in \mathrm{B}_k^+$, and hence, $\varphi_1 \wedge \varphi_2 \in \mathrm{B}_k^+$. The cases of $\varphi_1 \vee \varphi_2$ and $\varphi_1 \rightarrow \varphi_2$ are similar. If $\forall x \varphi_1 \in \mathrm{F}_k^+$, by the definition, we have $\forall x \varphi_1 \in \mathrm{U}_k^+$, and hence, $\forall x \varphi_1 \in \mathrm{B}_k^+$. The case of $\exists x \varphi_1 \in \mathrm{F}_k^+$ is similar. ⊣

COROLLARY 6.11 (cf. [1, Corollary 2.8(i)]). $\mathsf{HA} \vdash \Sigma_k\text{-LEM} \leftrightarrow \mathrm{F}_k\text{-LEM}$.

PROOF. Immediate from Propositions 6.9 and 6.10. ⊣

REMARK 6.12. By using Proposition 6.10 and Theorem 3.9, one can show the following: If $\varphi \in \mathrm{F}_k$, then $\mathsf{HA}^\$ + \Sigma_k\text{-LEM} \vdash \varphi^\$ \leftrightarrow \varphi \vee \$$. This is an extension of Lemma 3.5.

LEMMA 6.13. *For all $\varphi \in \mathrm{B}_k^+$, there exist φ' and φ'' which are constructed from formulas in $\mathrm{E}\Pi_k \bigcup \Sigma_k$ by using $\wedge$ and $\vee$ only, and satisfy* $\mathrm{FV}(\varphi') = \mathrm{FV}(\varphi'') = \mathrm{FV}(\varphi)$, $\mathsf{HA} + \Sigma_{k-1}$-LEM *proves $\varphi' \rightarrow \varphi$ and $\varphi'' \rightarrow \neg\varphi$, and* PA *proves $\varphi \rightarrow \varphi'$ and $\neg\varphi \rightarrow \varphi''$.*

PROOF. By induction on the construction of B_k^+.

For the base case, first assume $\varphi \in \mathrm{U}_k^+$. By Lemma 5.14, there exists $\varphi' \in \mathrm{E}\Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \rightarrow \varphi$ and $\mathsf{PA} \vdash \varphi \rightarrow \varphi'$. By Corollary 6.6, there exists $\varphi'' \in \Sigma_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi'')$, $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi'' \rightarrow \neg\varphi$ (cf. Remark 3.10) and $\mathsf{PA} \vdash \neg\varphi \rightarrow \varphi''$. Next assume $\varphi \in \mathrm{E}_k^+$. By Corollary 6.6, there exists $\varphi' \in \Sigma_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \rightarrow \varphi$ and $\mathsf{PA} \vdash \varphi \rightarrow \varphi'$. By Lemma 5.14, there exists $\varphi'' \in \mathrm{E}\Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi'')$, $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi'' \rightarrow \neg\varphi$ and $\mathsf{PA} \vdash \neg\varphi \rightarrow \varphi''$.

For the induction step, let $\varphi_1, \varphi_2 \in \mathrm{B}_k^+$ and $\varphi_1', \varphi_1'', \varphi_2', \varphi_2''$ constructed from formulas in $\mathrm{E}\Pi_k \bigcup \Sigma_k$ by using $\wedge$ and $\vee$ only satisfy the following: $\mathrm{FV}(\varphi_1') = \mathrm{FV}(\varphi_1'') = \mathrm{FV}(\varphi_1)$, $\mathrm{FV}(\varphi_2') = \mathrm{FV}(\varphi_2'') = \mathrm{FV}(\varphi_2)$, $\mathsf{HA} + \Sigma_{k-1}$-LEM proves $\varphi_1' \rightarrow \varphi_1$, $\varphi_2' \rightarrow \varphi_2$, $\varphi_1'' \rightarrow \neg\varphi_1$, $\varphi_2'' \rightarrow \neg\varphi_2$ and PA proves $\varphi_1 \rightarrow \varphi_1'$, $\varphi_2 \rightarrow \varphi_2'$, $\neg\varphi_1 \rightarrow \varphi_1''$,

$\neg\varphi_2 \to \varphi_2''$. For $\varphi :\equiv \varphi_1 \wedge \varphi_2$, take $\varphi' :\equiv \varphi_1' \wedge \varphi_2'$ and $\varphi'' :\equiv \varphi_1'' \vee \varphi_2''$. For $\varphi :\equiv \varphi_1 \vee \varphi_2$, take $\varphi' :\equiv \varphi_1' \vee \varphi_2'$ and $\varphi'' :\equiv \varphi_1'' \wedge \varphi_2''$. For $\varphi :\equiv \varphi_1 \to \varphi_2$, take $\varphi' :\equiv \varphi_1'' \vee \varphi_2'$ and $\varphi'' :\equiv \varphi_1' \wedge \varphi_2''$. We leave the routine verification for the reader. ⊣

Theorem 6.14. *Let T be semi-classical arithmetic and X be a set of* HA*-sentences. Then* $\mathsf{PA} + X$ *is* $\underset{\sim}{\mathrm{F}}_k$*-conservative over* $T + X$ *if and only if* $\mathsf{PA} + X$ *is* $\left(\underset{\sim}{\Sigma}_k \vee \mathrm{E}\underset{\sim}{\Pi}_k\right)$*-conservative over* $T + X$.

Proof. The "only if" direction is trivial since $\underset{\sim}{\Sigma}_k \vee \mathrm{E}\underset{\sim}{\Pi}_k \subseteq \underset{\sim}{\mathrm{F}}_k$. We show the converse direction. Let $\varphi \in \underset{\sim}{\mathrm{F}}_k$. Assume $\mathsf{PA} + X \vdash \varphi$. By Lemma 6.13 and Proposition 6.10, there exist φ' which is constructed from formulas in $\underset{\sim}{\Sigma}_k \bigcup \mathrm{E}\underset{\sim}{\Pi}_k$ by using $\wedge$ and $\vee$ only, and satisfy $\mathsf{HA} + \Sigma_{k-1}\text{-LEM} \vdash \varphi' \to \varphi$ and $\mathsf{PA} \vdash \varphi \to \varphi'$. Without loss of generality, one may assume that φ' is of conjunctive normal form such that each conjunct is a disjunction of sentences in $\underset{\sim}{\Sigma}_k \bigcup \mathrm{E}\underset{\sim}{\Pi}_k$. Since disjunction of sentences in $\underset{\sim}{\Sigma}_k$ is equivalent to a sentence in $\underset{\sim}{\Sigma}_k$ over HA and $\mathrm{E}\underset{\sim}{\Pi}_k$ is closed under $\vee$, each conjunct can be assumed to be of the form $\psi \vee \xi$ where $\psi \in \underset{\sim}{\Sigma}_k$ and $\xi \in \mathrm{E}\underset{\sim}{\Pi}_k$. Let $\varphi' :\equiv \bigwedge_{1 \le i \le n} (\psi_i \vee \xi_i)$ where $\psi_i \in \underset{\sim}{\Sigma}_k$ and $\xi_i \in \mathrm{E}\underset{\sim}{\Pi}_k$. Since $\mathsf{PA} + X \vdash \varphi'$, by the $\left(\underset{\sim}{\Sigma}_k \vee \mathrm{E}\underset{\sim}{\Pi}_k\right)$-conservativity, we have that $T + X$ proves $\psi_i \vee \xi_i$ for each i. Then we have $T + X \vdash \varphi'$. Since $\mathsf{PA} + X$ is now $\mathrm{E}\Pi_k$-conservative over $T + X$ (cf. Proposition 6.1), as in the proof of Lemma 5.5, we have $T + X \vdash \Sigma_{k-1}\text{-LEM}$. Then $T + X \vdash \varphi$ follows. ⊣

In what follows, by further investigating the $\left(\underset{\sim}{\Sigma}_k \vee \mathrm{E}\underset{\sim}{\Pi}_k\right)$-conservativity in Theorem 6.14, we give a characterization of the $\underset{\sim}{\mathrm{F}}_k$-conservativity by axiom schemata.

Definition 6.15. Let Γ be a class of HA-formulas. We introduce the following axiom schemata:

- Γ-$\underset{\sim}{\mathrm{DNE}}$: $\widetilde{\neg\neg\varphi} \to \widetilde{\varphi}$;
- Γ-$\underset{\sim}{\mathrm{DNS}}$: $\widetilde{\neg\neg\varphi} \to \neg\neg\widetilde{\varphi}$;

where $\varphi \in \Gamma$ and $\widetilde{\neg\neg\varphi}$ and $\widetilde{\varphi}$ are universal closures of $\neg\neg\varphi$ and φ respectively.

Proposition 6.16. *Let Γ be a class of* HA*-formulas such that Γ is closed under taking a universal closure. Then* Γ-$\underset{\sim}{\mathrm{DNE}}$ *is equivalent to* Γ-$\underset{\sim}{\mathrm{DNS}} + \underset{\sim}{\Gamma}$-DNE *over* HA.

Proof. It is trivial that Γ-$\underset{\sim}{\mathrm{DNE}}$ implies Γ-$\underset{\sim}{\mathrm{DNS}}$ and also $\underset{\sim}{\Gamma}$-DNE. We show $\mathsf{HA} + \Gamma$-$\underset{\sim}{\mathrm{DNS}} + \underset{\sim}{\Gamma}$-DNE $\vdash \Gamma$-$\underset{\sim}{\mathrm{DNE}}$. Let $\varphi \in \Gamma$. By Γ-$\underset{\sim}{\mathrm{DNS}}$, $\widetilde{\neg\neg\varphi}$ implies $\neg\neg\widetilde{\varphi}$. Since $\widetilde{\varphi}$ is now in $\underset{\sim}{\Gamma}$, by $\underset{\sim}{\Gamma}$-DNE, $\neg\neg\widetilde{\varphi}$ implies $\widetilde{\varphi}$. Thus we have $\mathsf{HA} + \Gamma$-$\underset{\sim}{\mathrm{DNS}} + \underset{\sim}{\Gamma}$-DNE $\vdash \widetilde{\neg\neg\varphi} \to \widetilde{\varphi}$. ⊣

Lemma 6.17. *Let T be a theory containing* HA *and satisfying the deduction theorem, and X be a set of* HA*-sentences in $\mathcal{Q}_k$. If $T + X$ proves* $\underset{\sim}{\Sigma}_k$-LEM *and $T + X$ is closed under* $\mathrm{E}\Pi_k$-DNE-R *with assumptions of sentences in* $\underset{\sim}{\Pi}_k$: $T + X \vdash \psi \to \neg\neg\varphi$ *implies $T + X \vdash \psi \to \varphi$ for all $\psi \in \underset{\sim}{\Pi}_k$ and $\varphi \in \mathrm{E}\Pi_k$, then* $\mathsf{PA} + X$ *is* $\left(\underset{\sim}{\Sigma}_k \vee \mathrm{E}\underset{\sim}{\Pi}_k\right)$*-conservative over* $T + X$.

Proof. Let $\varphi \in \underset{\sim}{\Sigma}_k$ and $\psi \in \mathrm{E}\underset{\sim}{\Pi}_k$. Assume $\mathsf{PA} + X \vdash \varphi \vee \psi$. Since T satisfies the deduction theorem and $\varphi^\perp \in \underset{\sim}{\Pi}_k$, by our second assumption, we have that $T + X +$

$\varphi^{\perp}$ is closed under $\mathrm{E}\Pi_k$-DNE-R. Since $\varphi^{\perp} \in \mathcal{Q}_k$, by Theorem 5.16, we have that $\mathsf{PA} + X + \varphi^{\perp}$ is $\mathrm{E}\Pi_k$-conservative over $T + X + \varphi^{\perp}$. Since $\mathsf{PA} + X + \varphi^{\perp} \vdash \psi$, we have $T + X + \varphi^{\perp} \vdash \psi$, and hence,

$$T + X \vdash \varphi^{\perp} \to \psi \tag{3}$$

by the deduction theorem. In addition, by our second assumption and Theorem 5.16, we have that $T + X$ is closed under $\mathrm{E}\Pi_k$-CD-R, and hence, $T + X \vdash \Sigma_{k-1}$-LEM by Lemma 5.7. Then, by Remark 5.3, we have $T + X \vdash \neg\varphi \to \varphi^{\perp}$, and hence, $T + X \vdash \neg\varphi \to \psi$ by (3). On the other hand, by our first assumption, we have $T + X \vdash \varphi \vee \neg\varphi$. Then $T + X \vdash \varphi \vee \psi$ follows. ⊣

Theorem 6.18. *Let T be semi-classical arithmetic satisfying the deduction theorem and X be a set of* HA*-sentences in $\mathcal{Q}_k$. Then the following are pairwise equivalent*:

1. $\mathsf{PA} + X$ *is $\underset{\sim}{\mathrm{F}_k}$-conservative over* $T + X$;
2. $T + X$ *proves* $\underset{\sim}{\mathrm{F}_k}$-LEM *and* U_k-$\underset{\sim}{\mathrm{DNS}}$;
3. $T + X$ *proves* $\underset{\sim}{\Sigma_k}$-LEM *and* U_k-$\underset{\sim}{\mathrm{DNE}}$;
4. $T + X$ *proves* $\underset{\sim}{\Sigma_k}$-LEM *and* $\mathrm{E}\Pi_k$-$\underset{\sim}{\mathrm{DNE}}$.

Proof. (1 → 2): Let $\varphi \in \underset{\sim}{\mathrm{F}_k}$. Then $\varphi \vee \neg\varphi \in \underset{\sim}{\mathrm{F}_k}$. Since $\mathsf{PA} \vdash \varphi \vee \neg\varphi$, we have $T + X \vdash \varphi \vee \neg\varphi$ by (1). Let $\psi \in \mathrm{U}_k$. Then $\widetilde{\neg\neg\psi} \to \neg\neg\widetilde{\psi} \in \underset{\sim}{\mathrm{F}_k}$. Since $\mathsf{PA} \vdash \widetilde{\neg\neg\psi} \to \neg\neg\widetilde{\psi}$, we have $T + X \vdash \widetilde{\neg\neg\psi} \to \neg\neg\widetilde{\psi}$ by (1).

(2 → 3): It suffices to show U_k-$\underset{\sim}{\mathrm{DNE}}$ by using $\underset{\sim}{\mathrm{F}_k}$-LEM and U_k-$\underset{\sim}{\mathrm{DNS}}$. Since $\underset{\sim}{\mathrm{U}_k} \subseteq \underset{\sim}{\mathrm{F}_k}$ and $\underset{\sim}{\mathrm{U}_k}$-LEM implies $\underset{\sim}{\mathrm{U}_k}$-DNE, by Proposition 6.16, we are done.

(3 → 4): Trivial.

(4 → 1): By Theorem 6.14 and Lemma 6.17, it suffices for (1) to show that $T + X$ is closed under $\mathrm{E}\Pi_k$-DNE-R with assumptions of Π_k sentences. Let $\psi \in \underset{\sim}{\Pi_k}$ and $\varphi \in \mathrm{E}\Pi_k$. Assume $T + X \vdash \psi \to \neg\neg\varphi$. Then $T + X + \psi \vdash \widetilde{\neg\neg\varphi}$. Since $T + X$ proves $\mathrm{E}\Pi_k$-$\underset{\sim}{\mathrm{DNE}}$ now, we have $T + X + \psi \vdash \widetilde{\varphi}$, and hence, $T + X + \psi \vdash \varphi$. Since T satisfies the deduction theorem, $T + X \vdash \psi \to \varphi$ follows. ⊣

Remark 6.19. U_k-$\underset{\sim}{\mathrm{DNS}}$ in Theorem 6.18.(2) is equivalent over HA to the closed fragment of U_k-DNS:

$$\neg\neg\forall x\varphi \to \forall x\neg\neg\varphi,$$

where $\varphi \in \mathrm{U}_k$ such that $\mathrm{FV}(\varphi) = \{x\}$.

In the following, we show that $\underset{\sim}{\mathrm{F}_k}$-LEM and U_k-$\underset{\sim}{\mathrm{DNS}}$ in Theorem 6.18.(2) are independent over HA.

Proposition 6.20. $\mathsf{HA} + \underset{\sim}{\Gamma}$-LEM $\nvdash (\Pi_1 \vee \Pi_1)$-$\underset{\sim}{\mathrm{DNS}}$ *for any class* Γ *of* HA-*formulas.*

Proof. Suppose $\mathsf{HA} + \underset{\sim}{\Gamma}$-LEM $\vdash (\Pi_1 \vee \Pi_1)$-$\underset{\sim}{\mathrm{DNS}}$. As in the proof of Proposition 5.10, let $\phi(x) \in \Pi_1 \vee \Pi_1$ be formula (2). Since $\mathsf{HA} \vdash \forall x\neg\neg\phi(x)$, we have $\mathsf{HA} + \underset{\sim}{\Gamma}$-LEM $\vdash \neg\neg\forall x\Psi(x)$. Since the double negation of each instance of $\underset{\sim}{\Gamma}$-LEM is provable in HA, by (the proof of) [7, Lemma 4.1], we have $\mathsf{HA} \vdash \neg\neg\forall x\phi(x)$. This is a contradiction as shown in the proof of Proposition 5.10. ⊣

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

PROPOSITION 6.21. $\mathsf{HA} + \mathrm{DNS} \nvdash \underset{\sim}{\Sigma_1}$-LEM *where* DNS *is the axiom scheme of the double-negation-shift* $\forall x(\forall y \neg\neg\varphi(x,y) \to \neg\neg\forall y \varphi(x,y))$.

PROOF. Let φ be a sentence in Π_1 such that $\mathsf{PA} \nvdash \varphi$ and $\mathsf{PA} \nvdash \neg\varphi$ (e.g., the Gödel sentence for Gödel's first incompleteness theorem). Since each instance of DNS is intuitionistically equivalent to a negated sentence (cf. [7, Remark 2.8]), by [13, Theorem 3.1.4 and Lemma 3.1.6], we have that $\mathsf{HA} + \mathrm{DNS}$ has the disjunction property. Suppose $\mathsf{HA} + \mathrm{DNS} \vdash \varphi^\perp \vee \neg\varphi^\perp$ (where $\varphi^\perp \in \underset{\sim}{\Sigma_1}$). Then, by the disjunction property, we have $\mathsf{HA} + \mathrm{DNS} \vdash \varphi^\perp$ or $\mathsf{HA} + \mathrm{DNS} \vdash \neg\varphi^\perp$, and hence, $\mathsf{PA} \vdash \neg\varphi$ or $\mathsf{PA} \vdash \varphi$. This is a contradiction. ⊣

REMARK 6.22. By using the disjunction property of $\mathsf{HA} + \mathrm{DNS}$ as in the proof of Proposition 6.21, one can extend Proposition 5.10 to that PA is not $(\Pi_1 \vee \Pi_1)$-conservative over $\mathsf{HA} + \mathrm{DNS}$: Suppose that PA is $(\Pi_1 \vee \Pi_1)$-conservative over $\mathsf{HA} + \mathrm{DNS}$. Then, by (the proof of) Theorem 5.19, $\mathsf{HA} + \mathrm{DNS}$ is closed under Σ_1-DML$^\perp$-R. Let φ and ψ be sentences in Σ_1 such that HA proves

$$\varphi \leftrightarrow \exists x \left(\mathrm{Pf}\left(x, \ulcorner \varphi^\perp \urcorner\right) \wedge \forall y \le x \neg \mathrm{Pf}\left(y, \ulcorner \psi^\perp \urcorner\right)\right)$$

and

$$\psi \leftrightarrow \exists y \left(\mathrm{Pf}\left(y, \ulcorner \psi^\perp \urcorner\right) \wedge \forall x < y \neg \mathrm{Pf}\left(x, \ulcorner \varphi^\perp \urcorner\right)\right),$$

where $\mathrm{Pf}\left(z, \ulcorner \xi \urcorner\right)$ denotes a proof predicate asserting that z is a code of the proof ξ in $\mathsf{HA} + \mathrm{DNS}$ (cf. [2, Chapter 2]). Since $\mathsf{HA} \vdash \neg(\varphi \wedge \psi)$, by using Σ_1-DML$^\perp$-R, we have $\mathsf{HA} + \mathrm{DNS} \vdash \varphi^\perp \vee \psi^\perp$. Since $\mathsf{HA} + \mathrm{DNS}$ has the disjunction property, we have that $\mathsf{HA} + \mathrm{DNS} \vdash \varphi^\perp$ or $\mathsf{HA} + \mathrm{DNS} \vdash \psi^\perp$. However, in both cases, we have a contradiction by our choice of φ and ψ.

Next, we show that U_k-$\underset{\sim}{\mathrm{DNE}}$, $\mathrm{E}\Pi_k$-$\underset{\sim}{\mathrm{DNE}}$ in Theorem 6.18 and the rule in Lemma 6.17 are pairwise equivalent.

PROPOSITION 6.23. *Let T be semi-classical arithmetic satisfying the deduction theorem and X be a set of* HA*-sentences in $\mathcal{Q}_k$. Then the following are pairwise equivalent*:

1. $T + X \vdash \mathrm{U}_k$-$\underset{\sim}{\mathrm{DNE}}$;
2. $T + X \vdash \mathrm{E}\Pi_k$-$\underset{\sim}{\mathrm{DNE}}$;
3. *$T + X$ is closed under* $\mathrm{E}\Pi_k$-DNE-R *with assumptions of sentences in* Π_k;
4. *For any $\psi \in \underset{\sim}{\Pi_k}$,* $\mathsf{PA} + X + \psi$ *is U_k-conservative over $T + X + \psi$*;
5. *$T + X$ is closed under* U_k-DNE-R *with assumptions of sentences in* U_k;
6. *$T + X$ is closed under* U_k-DNE-R *with assumptions of any sentences.*

PROOF. The implications $(1 \to 2)$ and $(6 \to 5)$ are trivial.
$(2 \to 3)$: By the proof of $(4 \to 1)$ in Theorem 6.18.
$(3 \to 4)$: Fix $\psi \in \underset{\sim}{\Pi_k}$. Let $\varphi \in \mathrm{U}_k$. Assume $\mathsf{PA} + X + \psi \vdash \varphi$. Since $X \cup \{\psi\} \subseteq \mathcal{Q}_k$, by Theorem 5.16, we have $T + X + \psi \vdash \varphi$.
$(4 \to 5)$: Assume $T + X \vdash \psi \to \neg\neg\varphi$ where $\psi \in \underset{\sim}{\mathrm{U}_k}$ and $\varphi \in \mathrm{U}_k$. By Corollary 6.6.(1), there exists $\psi' \in \underset{\sim}{\Sigma_k}$ such that $\mathsf{HA} + \Sigma_{k-1}$-LEM $\vdash \psi' \to \neg\psi$ (cf. Remark 3.10) and $\mathsf{PA} \vdash \neg\psi \to \psi'$. Let $\psi'' :\equiv (\psi')^\perp$. By Remark 5.3, we have $\psi'' \in \underset{\sim}{\Pi_k}$, $\mathsf{HA} + \Sigma_{k-1}$-LEM $\vdash \neg\neg\psi \to \psi''$ and $\mathsf{PA} \vdash \psi'' \to \psi$. Then we have now $\mathsf{PA} + X + \psi'' \vdash \varphi$.

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

By our assumption, $T + X + \psi'' \vdash \varphi$ follows. Since T satisfies the deduction theorem, we have $T + X \vdash \psi'' \to \varphi$. On the other hand, by (the proof of) Lemma 5.5 and our assumption, we have $T + X \vdash \Sigma_{k-1}$-LEM. Then $T + X \vdash \psi \to \varphi$ follows.
(5 $\to$ 1): Let $\varphi \in \mathrm{U}_k$. Note that $\widetilde{\neg\neg\varphi} \in \underset{\sim}{\mathrm{U}_k}$. Since $T + X + \widetilde{\neg\neg\varphi} \vdash \neg\neg\varphi$, by the deduction theorem, we have $T + X \vdash \widetilde{\neg\neg\varphi} \to \neg\neg\varphi$. By our assumption, we have $T + X \vdash \widetilde{\neg\neg\varphi} \to \varphi$, and hence, $T + X \vdash \widetilde{\neg\neg\varphi} \to \widetilde{\varphi}$.
(1 $\to$ 6): Assume $T + X \vdash \psi \to \neg\neg\varphi$ where ψ is a sentence and $\varphi \in \mathrm{U}_k$. Then we have $T + X + \psi \vdash \widetilde{\neg\neg\varphi}$. By our assumption, we have $T + X + \psi \vdash \widetilde{\varphi}$, and hence, $T + X + \psi \vdash \varphi$. Since T satisfies the deduction theorem, $T + X \vdash \psi \to \varphi$ follows. ⊣

Corollary 6.24. *Let X be a set of* HA*-sentences in $\mathcal{Q}_k$. Then* PA $+ X$ *is* U_k*-conservative over* HA $+ X + \mathrm{U}_k$-$\underset{\sim}{\mathrm{DNE}}$.

§7. Interrelations between conservation theorems and logical principles. The $\underset{\sim}{\mathrm{E}_{k+1}}$-conservativity implies both of $\underset{\sim}{\Sigma_{k+1}}$-conservativity and $\underset{\sim}{\mathrm{F}_k}$-conservativity. In what follows, we investigate the relation among them.

Proposition 7.1. *Let T be semi-classical arithmetic and X be a set of* HA*-sentences. If* PA $+ X$ *is* $\underset{\sim}{\Sigma_{k+1}}$*-conservative over $T + X$ and $T + X$ proves* $(\Pi_k \vee \Pi_k)$-DNE, *then* PA $+ X$ *is* $\underset{\sim}{\mathrm{E}_{k+1}}$*-conservative over $T + X$.*

Proof. By Theorem 6.7, it suffices to show $\widetilde{\mathrm{E}\Sigma_{k+1}}$-conservativity instead of the $\underset{\sim}{\mathrm{E}_{k+1}}$-conservativity. Let $\varphi :\equiv \exists x_1, \dots, x_n\, \psi \in \widetilde{\mathrm{E}\Sigma_{k+1}}$ with $\psi \in \mathrm{E}\Pi_k$. Assume PA $+ X \vdash \varphi$. By Theorem 3.9.(2), there exists $\psi' \in \widetilde{\Pi_k}$ such that $\mathrm{FV}(\psi) = \mathrm{FV}(\psi')$ and HA $+ (\Pi_k \vee \Pi_k)$-DNE $\vdash \psi' \leftrightarrow \psi$. Now we have PA $+ X \vdash \exists x_1, \dots, x_n\, \psi'$. Since $\exists x_1, \dots, x_n\, \psi' \in \underset{\sim}{\Sigma_{k+1}}$, by our first assumption, we have that $T + X \vdash \exists x_1, \dots, x_n\, \psi'$. By our second assumption, $T + X \vdash \varphi$ follows. ⊣

Proposition 7.2. *Let T be a theory containing* HA. *If* PA *is* $\underset{\sim}{\Sigma_{k+1}}$*-conservative over T, then T proves* $\underset{\sim}{\Sigma_k}$-LEM *and also* Σ_{k-2}-LEM.

Proof. Assume that PA is $\underset{\sim}{\Sigma_{k+1}}$-conservative over T. Then PA is Π_k-conservative over T (cf. Proposition 6.1), and hence, T proves Σ_{k-2}-LEM by (the proof of) Theorem 5.9. Let $\varphi \in \underset{\sim}{\Sigma_k}$. Then $\varphi^\perp \in \underset{\sim}{\Pi_k}$. Since $\underset{\sim}{\Sigma_k}$ and $\underset{\sim}{\Pi_k}$ can be seen as sub-classes of $\underset{\sim}{\Sigma_{k+1}}$ and $\underset{\sim}{\Sigma_{k+1}}$ is closed under $\vee$ (in the sense of [7, Lemma 4.4]), one may assume $\varphi \vee \varphi^\perp \in \underset{\sim}{\Sigma_{k+1}}$. Since PA $\vdash \varphi \vee \varphi^\perp$, by our assumption, we have $T \vdash \varphi \vee \varphi^\perp$, and hence, $T \vdash \varphi \vee \neg\varphi$. ⊣

Corollary 7.3. *Let T be semi-classical arithmetic satisfying the deduction theorem and X be a set of* HA*-sentences in $\mathcal{Q}_k$. If* PA $+ X$ *is* $\underset{\sim}{\Sigma_{k+1}}$*-conservative over $T + X$ and T proves* U_k-$\underset{\sim}{\mathrm{DNE}}$, *then* PA $+ X$ *is* $\underset{\sim}{\mathrm{F}_k}$*-conservative over $T + X$.*

Proof. Immediate by Theorem 6.18 and Proposition 7.2. ⊣

Remark 7.4. By using Theorem 3.9.(2), one can show that $(\Pi_k \vee \Pi_k)$-DNE implies U_k-$\underset{\sim}{\mathrm{DNE}}$ in a straightforward way. On the other hand, U_k-$\underset{\sim}{\mathrm{DNE}}$ implies the U_k-conservativity by Corollary 6.24. In contrast, $(\Pi_k \vee \Pi_k)$-DNE does not imply $\underset{\sim}{\mathrm{F}_k}$-conservativity since the latter is characterized by $\underset{\sim}{\Sigma_k}$-LEM + U_k-$\underset{\sim}{\mathrm{DNE}}$ (cf. Theorem 6.18) and $(\Pi_k \vee \Pi_k)$-DNE does not imply $\underset{\sim}{\Sigma_k}$-LEM (see [5]).

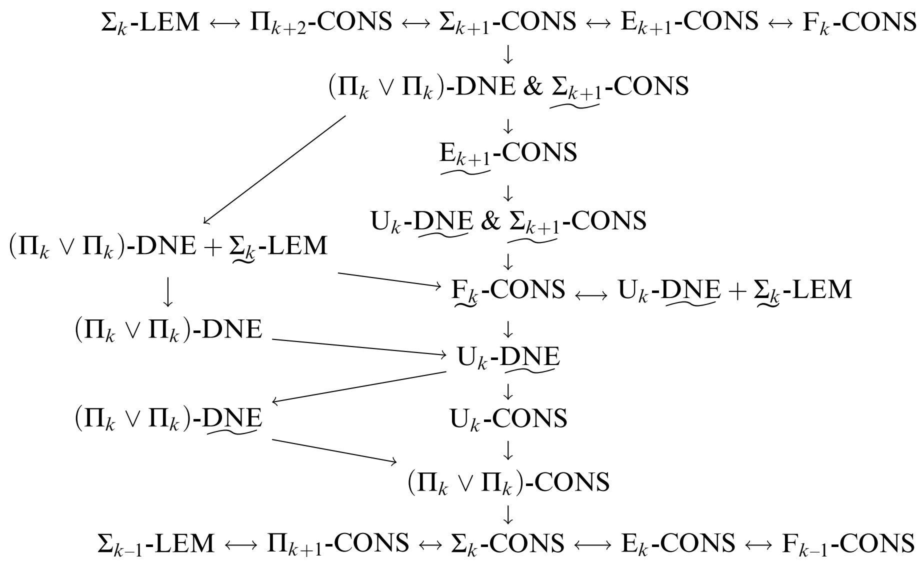

FIGURE 1. Conservation theorems in the arithmetical hierarchy of logical principles.

REMARK 7.5. It is straightforward to see that if a theory T containing HA proves $(\Pi_k \vee \Pi_k)$-$\utilde{\text{DNE}}$, then T is closed under $(\Pi_k \vee \Pi_k)$-DNE-R. Thus $(\Pi_k \vee \Pi_k)$-$\utilde{\text{DNE}}$ implies the $(\Pi_k \vee \Pi_k)$-conservativity (cf. Theorem 5.19). On the other hand, $(\Pi_k \vee \Pi_k)$-$\utilde{\text{DNE}}$ is a fragment of U_k-$\utilde{\text{DNE}}$.

PROPOSITION 7.6. *Let X be a set of* HA*-sentences in $\mathcal{Q}_k$. Then* PA $+ X$ *is* $\utilde{\Sigma_{k+1}}$-*conservative over* HA $+ X + \utilde{\Sigma_{k+1}}$-DNE $+ \Sigma_{k-1}$-LEM.

PROOF. Since HA $+ X + \utilde{\Sigma_{k+1}}$-DNE $+ \Sigma_{k-1}$-LEM contains Σ_{k-1}-LEM and is closed under $\utilde{\Sigma_{k+1}}$-DNE-R, by Proposition 6.3, we are done. ⊣

REMARK 7.7. Propositions 7.6 and 7.2 reveal that the $\utilde{\Sigma_{k+1}}$-conservativity lies between $\utilde{\Sigma_{k+1}}$-LEM $+ \Sigma_{k-1}$-LEM and $\utilde{\Sigma_k}$-LEM $+ \Sigma_{k-2}$-LEM. This seems to be another view of the status of the $\utilde{\Sigma_{k+1}}$-conservativity.

Our results on the relation between conservation theorems and logical principles are summarized in Figure 1 where Γ-CONS denotes the Γ-conservativity for class Γ of HA-formulas. Figure 1 reveals that the logical principle U_k-$\utilde{\text{DNE}}$, which has been first studied in the current paper (cf. Definition 6.15), is closely related to the conservation theorems. For the comprehensive information on the arithmetical hierarchy of logical principles including Σ_k-LEM and $(\Pi_k \vee \Pi_k)$-DNE, we refer the reader to [6]. For the underivability, we know only that Σ_{k-1}-LEM does not imply $(\Pi_k \vee \Pi_k)$-CONS (cf. Proposition 5.10) and that $(\Pi_k \vee \Pi_k)$-DNE does not imply $\utilde{F_k}$-CONS (cf. Remark 7.4). In addition, for $\Gamma \in \{\Sigma_k, \Pi_k, \Pi_k \vee \Pi_k, E_k, F_k, U_k, \utilde{\Sigma_k}\}$, we have characterized Γ-CONS by some fragment of the double-negation-

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

elimination rule DNE-R. On the other hand, we have not achieved that for $\underset{\sim}{\mathrm{E}}_k$ and $\underset{\sim}{\mathrm{F}}_k$.

§8. Appendix: A relativized soundness theorem of the Friedman A-translation for $\mathsf{HA} + \Sigma_k$-LEM. We provide a detailed proof of a relativized soundness theorem of the Friedman A-translation [4] for $\mathsf{HA} + \Sigma_k$-LEM (see Theorem 8.3). In fact, this result was suggested already in [8, Section 4.4] and the detailed proof for $k = 1$ can be found in [12, Lemma 3.1]. The authors, however, couldn't find the proof for arbitrary natural number k anywhere, which is the reason why we present the detailed proof here. For the relativized soundness theorem, we use a variant of Lemma 3.5 with respect to the Friedman A-translation.

We first recall the definition of the Friedman A-translation. In this section, we use symbol $*$ for place holder instead of \$ in the previous sections.

Definition 8.1 (A-translation [4]). For a HA-formula φ, we define φ^* as a formula obtained from φ by replacing all the prime formulas φ_{p} in φ with $\varphi_{\mathrm{p}} \vee *$ (of course, φ^* is officially defined by induction on the logical structure of φ). In particular, $\perp^* :\equiv (\perp \vee *)$, which is equivalent to $*$ over HA^* (HA in the language with a place holder $*$). In what follows, $\neg_* \varphi$ denotes $\varphi \to *$. Note that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi^*)$ for all HA-formulas φ.

The following is a variant of Lemma 3.5 with respect to the Friedman A-translation.

Lemma 8.2. *For a formula φ of HA, the following hold*:

1. *If* $\varphi \in \Pi_k$, $\mathsf{HA}^* + \Sigma_k$-LEM $\vdash \varphi^* \leftrightarrow \varphi \vee *$;
2. *If* $\varphi \in \Sigma_k$, $\mathsf{HA}^* + \Sigma_{k-1}$-LEM $\vdash \varphi^* \leftrightarrow \varphi \vee *$.

Proof. By simultaneous induction on k. The base case is verified by a routine inspection. Assume items 1 and 2 for k to show those for $k+1$. The first item for $k+1$ is shown by using the second item for k as in the proof of Lemma 3.5. For the second item, let $\varphi :\equiv \exists x \varphi_1$ where $\varphi_1 \in \Pi_k$. Then we have that $\mathsf{HA} + \Sigma_k$-LEM proves

$$\varphi^* \equiv \exists x\, (\varphi_1{}^*) \underset{[\text{I.H.}]\ \Sigma_k\text{-LEM}}{\longleftrightarrow} \exists x (\varphi_1 \vee *) \longleftrightarrow \varphi \vee *. \qquad \dashv$$

Theorem 8.3. *If* $\mathsf{HA} + \Sigma_k$-LEM $\vdash \varphi$, *then* $\mathsf{HA}^* + \Sigma_k$-LEM $\vdash \varphi^*$.

Proof. By induction on the length of the proof of φ in $\mathsf{HA} + \Sigma_k$-LEM. By (the proof of) [4, Lemma 2], it suffices to show $\mathsf{HA}^* + \Sigma_k$-LEM $\vdash \varphi^*$ for each instance φ of Σ_k-LEM. Fix $\varphi :\equiv \exists x \varphi_1 \vee \neg \exists x \varphi_1$ with $\varphi_1 \in \Pi_{k-1}$. By Lemma 8.2.(1), $\mathsf{HA}^* + \Sigma_{k-1}$-LEM proves

$$\begin{array}{rcl} \varphi^* & \longleftrightarrow & \exists x\, (\varphi_1{}^*) \vee \neg_* \exists x\, (\varphi_1{}^*) \\ & \underset{\Sigma_{k-1}\text{-LEM}}{\longleftrightarrow} & \exists x\, (\varphi_1 \vee *) \vee \neg_* \exists x\, (\varphi_1 \vee *) \\ & \longleftrightarrow & \exists x\, (\varphi_1 \vee *) \vee \neg_* \exists x \varphi_1, \end{array}$$

which is derived from $\exists x \varphi_1 \vee \neg \exists x \varphi_1$ over HA^*. Thus $\mathsf{HA}^* + \Sigma_k$-LEM proves φ^*. $\dashv$

By the relativized soundness theorem of the Friedman A-translation combined with the usual negative translation, one can show Proposition 1.1 as follows:

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

PROOF SKETCH OF PROPOSITION 1.1. Assume $\mathsf{PA} \vdash \forall x \exists y \varphi$ where $\varphi \in \Pi_k$. By using Kuroda's negative translation (cf. [7, Proposition 6.4]), we have $\mathsf{HA} \vdash \forall x \neg\neg \exists y \varphi_N$ where φ_N is defined as in [7, Definition 6.1]. Since $\mathsf{HA} + \Sigma_{k-1}$-LEM proves Σ_{k-1}-DNE, we have $\mathsf{HA} + \Sigma_{k-1}$-LEM $\vdash \neg\neg\exists y \varphi$ (cf. [7, Lemma 6.5(2)]). By Theorem 8.3, we have $\mathsf{HA}^* + \Sigma_{k-1}$-LEM $\vdash \neg_* \neg_* \exists y \varphi^*$, and hence, $\mathsf{HA}^* + \Sigma_k$-LEM $\vdash \neg_* \neg_* \exists y \varphi$ by Lemma 8.2.(1). By substituting $*$ with $\exists y \varphi$ (cf. Lemma 3.16), we have that $\mathsf{HA} + \Sigma_k$-LEM proves $\exists y \varphi$, and hence, $\forall x \exists y \varphi$. ⊣

The proof of [14, Theorem 3.5.5] (due to Visser) shows that any theory T which contains HA and is sound for the Friedman A-translation is closed under the independence-of-premise rule:

$$T \vdash \neg\varphi \to \exists x \psi \text{ implies } T \vdash \exists x\,(\neg\varphi \to \psi)\,,$$

where $x \notin \mathrm{FV}(\neg\varphi)$. Then, by using Theorem 8.3, we also have the following:

THEOREM 8.4. $\mathsf{HA} + \Sigma_k$-LEM *is closed under the independence-of-premise rule.*

Acknowledgements. The first author was supported by JSPS KAKENHI Grant Numbers JP19J01239 and JP20K14354, and the second author by JP19K14586.

REFERENCES

[1] Y. AKAMA, S. BERARDI, S. HAYASHI, and U. KOHLENBACH, ***An arithmetical hierarchy of the law of excluded middle and related principles. Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS'04)***, 2004, pp. 192–301.

[2] G. BOOLOS, ***The Logic of Provability***, Cambridge University Press, Cambridge, 1993.

[3] D. VAN DALEN, ***Logic and Structure***, fifth ed., Universitext, Springer, London, 2013.

[4] H. FRIEDMAN, *Classically and intuitionistically provably recursive functions*, ***Higher Set Theory*** (G. H. Müller and D. S. Scott, editors), Springer, Berlin; Heidelberg, 1978, pp. 21–27.

[5] M. FUJIWARA, H. ISHIHARA, T. NEMOTO, N.-Y. SUZUKI, and K. YOKOYAMA, *Extended frames and separations of logical principles*, submitted, 2021. Available at https://researchmap.jp/makotofujiwara/misc/30348506.

[6] M. FUJIWARA and T. KURAHASHI, *Refining the arithmetical hierarchy of classical principles*, submitted, 2020. Available at https://arxiv.org/abs/2010.11527.

[7] ———, *Prenex normal form theorems in semi-classical arithmetic*, ***Journal of Symbolic Logic***, vol. 86 (2021), no. 3, pp. 1124–1153.

[8] S. HAYASHI and M. NAKATA, *Towards limit computable mathematics*, ***Types for Proofs and Programs*** (P. Callaghan, Z. Luo, J. McKinna, R. Pollack, and R. Pollack, editors), Springer, Berlin; Heidelberg, 2002, pp. 125–144.

[9] H. ISHIHARA, *A note on the Gödel–Gentzen translation*. ***Mathematical Logic Quarterly***, vol. 46 (2000), no. 1, pp. 135–137.

[10] ———, *Some conservative extension results on classical and intuitionistic sequent calculi*, ***Logic, Construction, Computation*** (U. Berger, H. Diener, P. Schuster, and S. Monika, editors), Ontos Mathematical Logic, De Gruyter, Berlin; Boston, 2012, pp. 289–304.

[11] R. KAYE, J. PARIS, and C. DIMITRACOPOULOS, *On parameter free induction schemas*, ***Journal of Symbolic Logic***, vol. 53 (1988), no. 4, pp. 1082–1097.

[12] U. KOHLENBACH and P. SAFARIK, *Fluctuations, effective learnability and metastability in analysis*. ***Annals of Pure and Applied Logic***, vol. 165 (2014), no. 1, pp. 266–304.

[13] A. S. TROELSTRA (editor), ***Metamathematical Investigation of Intuitionistic Arithmetic and Analysis***, Lecture Notes in Mathematics, 344, Springer, Berlin; New York, 1973.

[14] A. S. TROELSTRA and D. VAN DALEN, ***Constructivism in Mathematics, An Introduction***, ***vol.*** I, Studies in Logic and the Foundations of Mathematics, 121, North-Holland, Amsterdam, 1988.

https://doi.org/10.1017/jsl.2022.25 Published online by Cambridge University Press

SCHOOL OF SCIENCE AND TECHNOLOGY
MEIJI UNIVERSITY
1-1-1 HIGASHI-MITA, TAMA-KU, KAWASAKI-SHI
KANAGAWA 214-8571, JAPAN
E-mail: makotofujiwara@meiji.ac.jp

GRADUATE SCHOOL OF SYSTEM INFORMATICS
KOBE UNIVERSITY
1-1 ROKKODAI, NADA, KOBE 657-8501, JAPAN
E-mail: kurahashi@people.kobe-u.ac.jp