Can. J. Math., Vol. XXXIX, No. 4, 1987, pp. 880-892

THREE TEST PROBLEMS FOR QUASISIMILARITY

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1. Kaplansky proposed in [7] three problems with which to test the adequacy of a proposed structure theory of infinite abelian groups. These problems can be rephrased as test problems for a structure theory of operators on Hilbert space. Thus, R. Kadison and I. Singer answered in [6] these test problems for the unitary equivalence of operators. We propose here a study of these problems for quasisimilarity of operators on Hilbert space. We recall first that two (bounded, linear) operators T and T', acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, are said to be quasisimilar if there exist bounded operators $X:\mathcal{H} \to \mathcal{H}'$ and $Y:\mathcal{H}' \to \mathcal{H}$, with densely defined inverses, satisfying the relations $T'X = XT$ and $TY = YT'$. The fact that T and T' are quasisimilar is indicated by $T \sim T'$. The problems mentioned above can now be formulated as follows.

Problem 1. If T and T' are operators acting on Hilbert spaces, and $T \oplus T \sim T' \oplus T'$, is it true that T and T' are quasisimilar?

Problem 2. Assume that T, T', and T'' satisfy the relation $T \oplus T' \sim T \oplus T''$. Does it follow that T' and T'' are quasisimilar?

Problem 3. Assume that T, T_1, T', and T_1' are such that $T \sim T' \oplus T_1'$ and $T' \sim T \oplus T_1$. Does it follow that $T \sim T'$?

As in the case of unitary equivalence, it is clear that Problem 2 has a negative answer, unless some finiteness assumption is made about T. Simple counterexamples can be produced by taking T, T', T'' to be the zero operators on Hilbert spaces of various dimensions. In the case of unitary equivalence the answer to Problem 3 is always yes, and this can be seen by applying a version of the Cantor-Bernstein argument. The reader will easily convince himself that such an argument is bound to fail for quasisimilarity, or even for similarity.

In what follows we will give complete answers to the three problems stated above in the particular case in which all the operators involved are of class C_0. For the reader's convenience we recall some basic definitions (cf. also Chapter III of [8]). An operator T, acting on a Hilbert space, is said to be of class C_0 if it is a completely nonunitary contraction (i.e., $\|T\| \leqq 1$ and T has no unitary direct summands) and $u(T) = 0$ for some u in the algebra H^∞ of all bounded analytic functions on the unit disc

Received January 30, 1985. The author gratefully acknowledges the support of the National Science Foundation and of the Mathematical Sciences Research Institute.

https://doi.org/10.4153/CJM-1987-043-x Published online by Cambridge University Press

$$D = \{\lambda : |\lambda| < 1\}.$$

The latter condition means that

$$\lim_{r \nearrow 1} u(rT) = 0$$

in the strong operator topology, where $u(rT)$ is given, for example, by the Riesz-Dunford functional calculus. If T is an operator of class C_0, the ideal $\{u \in H^\infty : u(T) = 0\}$ is principal, and it is generated by an (essentially unique) inner function denoted m_T.

The simplest operators of class C_0 are the Jordan blocks which we presently define. Denote by H^2 the usual Hardy space for D, that is,

$$H^2 = \left\{ f : f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, |\lambda| < 1, \sum_{n=0}^{\infty} |a_n|^2 < +\infty \right\},$$

and denote by S the shift operator on H^2 defined by

$$(Sf)(\lambda) = \lambda f(\lambda), \quad f \in H^2, \lambda \in D.$$

For every inner function $\theta \in H^\infty$ we set

$$\mathscr{H}(\theta) = H^2 \ominus \theta H^2,$$

and denote by $S(\theta)$ the compression of S to $\mathscr{H}(\theta)$:

$$S(\theta) = P_{\mathscr{H}(\theta)} S | \mathscr{H}(\theta).$$

The operator $S(\theta)$ is called a Jordan block; it is an operator of class C_0 and the ideal

$$\{u \in H^\infty : u(S(\theta)) = 0\}$$

is generated by θ. Note that $S(\theta) = S(\theta')$ if and only if $\theta' = \gamma\theta$ for some $\gamma \in \mathbf{C}$, $|\gamma| = 1$. We write $\theta \equiv \theta'$ if $S(\theta) = S(\theta')$.

We can now define a more general class of operators, called the Jordan operators. Assume that for each ordinal number α we are given an inner function $\theta_\alpha \in H^\infty$ such that

(i) θ_α divides θ_β whenever $\alpha \geqq \beta$;

(ii) $\theta_\alpha \equiv \theta_\beta$ whenever $\mathrm{card}(\alpha) = \mathrm{card}(\beta)$; and

(iii) $\theta_\alpha \equiv 1$ for some α (and hence $\theta_\beta \equiv 1$ for $\beta \geqq \alpha$).

In this case the operator

$$T = \bigoplus_{\theta_\alpha \not\equiv 1} S(\theta_\alpha)$$

is called a Jordan operator; T is of class C_0 and $m_T \equiv \theta_0$. The following result, proved in [**3**] and [**1**], shows why Jordan operators are important in the study of the class C_0.

THEOREM 4. *Every operator T of class C_0 is quasisimilar to a unique Jordan operator, called the Jordan model of T.*

https://doi.org/10.4153/CJM-1987-043-x Published online by Cambridge University Press

We are now able to answer Problem 1 for the class C_0.

PROPOSITION 5. *Assume that T and T' are operators of class C_0. If $T \oplus T \sim T' \oplus T'$, then $T \sim T'$.*

Proof. The idea is that the Jordan model of T can be determined if we know the Jordan model $T \oplus T$. Indeed, assume that

$$\bigoplus_\alpha S(\varphi_\alpha)$$

is the Jordan model of T, and define inner functions ψ_α as follows:

$$\psi_\alpha = \varphi_k \quad \text{if } \alpha = 2k \text{ or } \alpha = 2k + 1, \, k < \omega,$$
$$\psi_\alpha = \varphi_\alpha \quad \text{if } \alpha \geqq \omega.$$

(Here ω denotes, as usual, the first transfinite ordinal.) It is easy to check that

$$\bigoplus_\alpha S(\psi_\alpha)$$

is a Jordan operator, and every inner function θ appears twice as often among the ψ_α than among the φ_α; this follows from conditions (ii) above, and the equality $\aleph = 2\aleph$ for infinite cardinals $\aleph$. Thus $T \oplus T$ is quasisimilar to

$$\bigoplus_\alpha S(\psi_\alpha),$$

and hence $\oplus_\alpha S(\psi_\alpha)$ must be the Jordan model of $T \oplus T$. Now, it is clear that

$$\varphi_\alpha = \psi_{2\alpha}, \quad \alpha < \omega, \quad \text{and}$$
$$\varphi_\alpha = \psi_\alpha, \quad \alpha \geqq \omega,$$

so that the Jordan model of T can be obtained from the Jordan model of $T \oplus T$, as claimed. Now, if $T \oplus T \sim T' \oplus T'$, it follows that $T \oplus T$ and $T' \oplus T'$ have the same Jordan model. Consequently T and T' have the same Jordan model, and hence $T \sim T'$ by Theorem 4, as desired.

Likewise, Problem 3 has a positive answer whose proof is based on Jordan operators. In fact. B. Sz.-Nagy and C. Foias proved in **[9]** a much stronger result which we state below. We recall that an operator T can be injected into an operator T' if there exists a continuous one-to-one operator X satisfying the equation $T'X = XT$. We indicate by $T \overset{i}{\prec} T'$ the fact that T can be injected into T'. Then the relevant result in **[9]** is as follows.

THEOREM 6. *Assume that T and T' are operators of class C_0. If $T \overset{i}{\prec} T'$ and $T' \overset{i}{\prec} T$, then $T \sim T'$.*

This clearly answers Problem 3, since $T \sim T' \oplus T'_1$ implies easily that $T' \overset{i}{\prec} T$ and, likewise, $T' \sim T \oplus T_1$ implies that $T \overset{i}{\prec} T'$.

Problem 2 is more difficult to answer, and its solution will occupy the rest of this paper. As mentioned above, a positive answer to Problem 2 can only be obtained under some additional finiteness assumption (we note here that the zero operator on any Hilbert space is an operator of class C_0, so that the counterexample mentioned above does apply to the class C_0). In order to arrive at the right finiteness assumption we need some preliminaries, and we begin with some simple combinatorics.

We denote by $\mathcal{S}$ the set of all bounded sequences $\{x_n:n \geqq 0\}$ of nonnegative real numbers, and by $\mathcal{S}_0$ the collection of all nonincreasing sequences in $\mathcal{S}$. We can define a sorting operation sort: $\mathcal{S} \to \mathcal{S}_0$ as follows. Let $x = \{x_n:n \geqq 0\}$ be an element of $\mathcal{S}$, and set

$$\sigma_0 = 0, \quad \sigma_n = \sup\{x_{i_1} + x_{i_2} < \ldots + x_{i_n}: 0 \leqq i_1 < i_2 < \ldots < i_n\} \quad \text{for } n \geqq 1.$$

It is clear that the sequence $\{\sigma_n:n \geqq 0\}$ is nondecreasing, and we set $\text{sort}(x) = y$, where $y = \{y_n:n \geqq 0\}$ is given by

$$y_n = \sigma_{n+1} - \sigma_n, \quad n \geqq 0.$$

The following result shows that sort(x) belongs to $\mathcal{S}_0$, and that sort is indeed a sorting in many cases (it is instructive to calculate sort(x) in case x is an increasing sequence).

LEMMA 7. *Let* $x = \{x_n:n \geqq 0\}$ *and* $y = \{y_n:n \geqq 0\}$ *be such that* $x \in \mathcal{S}$ *and* $\text{sort}(x) = y$. *Then for every integer* $n \geqq 0$, *and every positive real number* t *the following assertions are equivalent*:

(i) $t \geqq y_n$;

(ii) $\text{card}\{i:x_i > t\} \leqq n$.

Proof. Assume first that (ii) holds, and let $0 \leqq i_1 < i_2 < \ldots < i_{n+1}$ be integers. Then it follows that there exists some k, $1 \leqq k \leqq n + 1$, such that $x_{i_k} \leqq t$. Thus

$$\begin{aligned} x_{i_1} + x_{i_2} + \ldots + x_{i_{n+1}} &= \sum_{j \neq k} x_{i_j} + x_{i_k} \\ &\leqq \sum_{j \neq k} x_{i_j} + t \\ &\leqq \sigma_n + t, \end{aligned}$$

and, since $i_1, i_2, \ldots, i_{n+1}$ were arbitrarily chosen, we deduce that

$$\sigma_{n+1} \leqq \sigma_n + t.$$

Thus

$$t \geqq \sigma_{n+1} - \sigma_n = y_n,$$

that is (i).

Conversely, assume that (ii) fails, so that

$$\operatorname{card}\{i:x_i > t\} \geqq n + 1,$$

and choose ϵ such that actually

$$\operatorname{card}\{i:x_i \geqq t + \epsilon\} \geqq n + 1.$$

Let $0 \leqq i_1 < i_2 < \ldots < i_n$ be a sequence of integers. Then there must exist $i \notin \{i_1, i_2, \ldots, i_n\}$ such that $s_i \geqq t + \epsilon$; we infer

$$\sum_{j=1}^{n} s_{i_j} + t + \epsilon \leqq \sum_{j=1}^{n} x_{i_j} + x_i \leqq \sigma_{n+1}$$

and, as before, this implies that

$$\sigma_n + t + \epsilon \leqq \sigma_{n+1}.$$

Equivalently, we proved that $t + \epsilon \leqq y_n$, and hence (i) fails. The lemma is proved.

If $x = \{x_n:n \geqq 0\}$ and $y = \{y_n:n \geqq 0\}$ are elements of $\mathscr{S}$, we denote by $x \cup y$ the sequence $x_0, y_0, x_1, y_1, \ldots$ i.e.,

$$x \cup y = \{z_n:n \geqq 0\},$$

where $z_{2k} = x_k$ and $z_{2k+1} = y_k$, $k \geqq 0$.

LEMMA 8. *Assume that x is a sequence in $\mathscr{S}_0$. The map*

$$y \rightarrow \operatorname{sort}(x \cup y)$$

is one-to-one on $\mathscr{S}_0$ if and only if x converges to zero.

Proof. Assume first that $x = \{x_n:n \geqq 0\}$ does not converge to zero, and let $y = \{y_n:n \geqq 0\}$ be defined by $y_n = t$, $n \geqq 0$, where t is any number satisfying

$$0 < t < \lim_{n\rightarrow\infty} x_n.$$

It is easy to verify that $x = \operatorname{sort}(x \cup y)$, and hence the map

$$y \rightarrow \operatorname{sort}(x \cup y)$$

is not one-to-one.

Conversely, assume that

$$\lim_{n\rightarrow\infty} x_n = 0,$$

and let $y' = \{y'_n:n \geqq 0\}$ and $y'' = \{y''_q:n \geqq 0\}$ be two distinct elements of $\mathscr{S}_0$. Denote by q the first integer such that $y'_q \neq y''_q$ and assume for definiteness that $y'_q < y''_q$. Denote next by p the first integer satisfying the inequality $x_p < y''_q$; such an integer exists because

$$\lim_{n\to\infty} x_n = 0.$$

We will show that

$$\text{sort}(x \cup y') = \{z'_n : n \geqq 0\}$$

is different from

$$\text{sort}(x \cup y'') = \{z''_n : n \geqq 0\}$$

by proving that

$$\sigma'_{p+q+1} = z'_0 + z'_1 + \ldots + z'_{p+q+1} \neq \sigma''_{p+q+1}$$
$$= z''_0 + z''_1 + \ldots + z''_{p+q+1}.$$

The definition of sort, and that fact that $x, y' \in \mathcal{S}_0$, shows that

$$\sigma'_{p+q+1} = \max\{x_0 + x_1 + \ldots + x_j + y'_0 + y'_1 + \ldots + y'_{p+q-j-1} : -1 \leqq j \leqq p + q\},$$

where the x terms [respectively the y terms] are absent if $j = -1$ [respectively $j = p + q$]. We claim that the maximum is attained either for $j = p - 1$, or for $j = p$. Indeed, if $j \leqq p - 2$, we have

$$j + 1 \leqq p - 1 \quad \text{and} \quad p + q - j - 1 \geqq q + 1,$$

so that

$$x_{j+1} \geqq x_{p-1} > y''_q > y'_q \geqq y'_{q+1} \geqq y'_{p+q-j-1}.$$

It is then easy to see that

$$x_0 + x_1 + \ldots + x_j + y'_0 + y'_1 + \ldots + y'_{p+q-j-1}$$
$$< x_0 + x_1 + \ldots + x_{j+1} + y'_0 + y'_1 + \ldots + y'_{p+q-j-2},$$

and hence the maximum is not attained for $j \leqq p - 2$. Analogously, if $j \geqq p + 1$, then $p + q - j \leqq q - 1$, from which we deduce

$$x_j \leqq x_p < y''_q \leqq y''_{q-1} = y'_{q-1} \leqq y'_{p+q-j},$$

and

$$x_0 + x_1 + \ldots + x_j + y'_0 + y'_1 + \ldots + y'_{p+q-j-1}$$
$$< x_0 + x_1 + \ldots + x_{j-1} + y'_0 + y'_1 + \ldots + y'_{p+q-j}.$$

We conclude that

$$\begin{aligned}\sigma'_{p+q+1} &= \max\{x_0 + x_1 + \ldots + x_{p-1} + y'_0 + y'_1 + \ldots + y'_q,\\ &\qquad x_0 + x_1 + \ldots + x_p + y'_0 + y'_1 + \ldots + y'_{q-1}\}\\ &= x_0 + x_1 + \ldots + x_{p-1} + y'_0 + y'_1 + \ldots + y'_{q-1}\\ &\quad + \max\{x_p, y'_q\}.\end{aligned}$$

Analogously,

$$\begin{aligned}\sigma''_{p+q+1} &= x_0 + x_1 + \ldots + x_{p-1} + y''_0 + y''_1 + \ldots + y''_{q-1}\\ &\qquad\qquad + \max\{x_p, y''_q\}\\ &= x_0 + x_1 + \ldots + x_{p-1} + y'_0 + y'_1 + \ldots + y'_{q-1} + y''_q,\end{aligned}$$

and it is readily seen that

$$\sigma'_{p+q+1} < \sigma''_{p+q+1}.$$

Therefore the map $y \rightarrow \operatorname{sort}(x \cup y)$ is one-to-one, as desired. The lemma follows.

In order to see what is the relevance of sorting to the theory of the class C_0, we study the Jordan models of certain operators acting on separable spaces. Let $\{\varphi_j : j < \omega\}$ be a sequence of inner functions in H^∞. We remark that the operator

$$T = \bigoplus_{j<\omega} S(\varphi_j)$$

(not assumed to be a Jordan operator) is of class C_0 if and only if the family $\{\varphi_j : j < \omega\}$ admits a least common inner multiple, denoted $\bigvee\{\varphi_j : j < \omega\}$. Moreover, if T is of class C_0, then

$$m_T \equiv \bigvee\{\varphi_j : j < \omega\}.$$

Assume now that

$$T = \bigoplus_{j<\omega} S(\varphi_j)$$

is an operator of class C_0, and consider the natural question of determining the Jordan model of T. Let

$$\bigoplus_{j<\omega} S(\theta_j)$$

denote the Jordan model of T (we have $\theta_\omega \equiv 1$ because T acts on a separable space). By the results of [**4**], we have

$$\theta_j \equiv d_{j+1}/d_j, \quad j < \omega,$$

where $d_0 = 1$, and, for $j \geqq 1$, d_j is the least scalar multiple of the jth exterior power of the characteristic function of T. Now, the characteristic function of T clearly coincides (in the sense of [**8**], Chapter VI) with the diagonal matrix whose diagonal entries are the functions φ_j, $j < \omega$. Thus we see that the jth exterior power of this characteristic function coincides with a diagonal matrix whose diagonal entries are all the possible products

$$\varphi_{i_1}\varphi_{i_2} \ldots \varphi_{i_j} \quad \text{with } i_1 < i_2 < \ldots < i_j.$$

We deduce at once the formula

$$d_j \equiv \vee\{\varphi_{i_1}\varphi_{i_2} \ldots \varphi_{i_j}: i_1 < i_2 < \ldots i_j\}, \quad j \geqq 1.$$

The formula for θ_j can thus be viewed as a multiplicative analogue of the sorting operation in H^∞. The analogy can be made more precise by the use of the factorization theory for inner functions.

We recall (cf., e.g., [5]) the definitions of Blaschke products and of singular inner functions. For every point $a \in D$ we set

$$B_a(\lambda) = \lambda \quad \text{if } a = 0, \quad \text{and}$$

$$B_a(\lambda) = \frac{|a|}{a}\frac{a - \lambda}{1 - \bar{a}\lambda}, \quad a \neq 0, \lambda \in D.$$

Assume now that $\mu:D \to \{0, 1, 2, \ldots\}$ is a function satisfying the condition

$$\sum_{a \in D} \mu(a)(1 - |a|) < +\infty.$$

Then the Blaschke product b_μ determined by μ is defined as

$$b_\mu(\lambda) = \prod_{a \in D} B_a(\lambda)^{\mu(a)}, \quad \lambda \in D,$$

and it is an inner function. Let now ν be a finite positive Borel measure on

$$\mathbf{T} = \{\zeta:|\zeta| = 1\},$$

singular to Lebesgue (arclength) measure, and define the singular inner function s_ν by

$$s_\nu(\lambda) = \exp\left(-\int_{\mathbf{T}} \frac{\zeta + \lambda}{\zeta - \lambda} d\mu(\zeta)\right).$$

It is known that every inner function $\theta \in H^\infty$ can be written uniquely as

$$\theta = \gamma b_\mu s_\nu,$$

where $\gamma \in \mathbf{T}$, b_μ is a Blaschke product, and s_ν is a singular inner function, as defined above. Moreover, if $\theta' = \gamma' b_{\mu'} s_{\nu'}$ is another inner function, then

$$\theta\theta' = \gamma\gamma' b_{\mu+\mu'} s_{\nu+\nu'}.$$

In particular, it follows that θ divides θ' if and only if $\mu \leqq \mu'$ and $\nu \leqq \nu'$ (these inequalities are defined in the obvious way).

Let us return now to our operator

$$T = \bigoplus_{j<\omega} S(\varphi_j),$$

which we assume to be of class C_0. Consider an arbitrary inner function

$$\theta = \gamma b_\mu s_\nu$$

such that $\theta(T) = 0$. As noted above, θ must be a common inner multiple of $\{\varphi_j : j < \omega\}$, and hence we can write

$$\varphi_j = \gamma_j b_{\mu_j} s_{\nu_j}, \quad j < \omega,$$

with $\mu_j \leqq \mu$ and $\nu_j \leqq \nu$. Furthermore, we may write $d\nu_j = f_j d\nu$, where f_j is a Borel function defined on $\mathbf{T}$ such that $0 \leqq f_j \leqq 1$, $j < \omega$. We compute next the functions d_j, $1 \leqq j < \omega$, where

$$d_j = \bigvee\{\varphi_{i_1}\varphi_{i_2} \ldots \varphi_{i_j} : i_1 < i_2 < \ldots < i_j\}.$$

Since

$$\varphi_{i_1}\varphi_{i_2} \ldots \varphi_{i_j} = \gamma_{i_1}\gamma_{i_2} \ldots \gamma_{i_j} b_{\mu_{i_1}+\mu_{i_2}+\ldots+\mu_{i_j}} s_{\nu_{i_1}+\nu_{i_2}+\ldots+\nu_{i_j}},$$

it is easily seen that

$$d_j \equiv b_{\mu_j'} s_{\nu_j'}$$

where

$$\mu_j'(a) = \sup\{\mu_{i_1}(a) + \mu_{i_2}(a) + \ldots + \mu_{i_j}(a) : i_1 < i_2 < \ldots < i_j\}, \ a \in D,$$

$$d\nu_j' = f_j' d\nu, \quad \text{and}$$

$$f_j'(\zeta) = \sup\{f_{i_1}(\zeta) + f_{i_2}(\zeta) + \ldots + f_{i_j}(\zeta) : i_1 < i_2 < \ldots < i_j\}.$$

Finally, we see from the formulas $\theta_j = d_{j+1}/d_j$ that the Jordan model

$$\bigoplus_{j<\omega} S(\theta_j)$$

is given by

$$\theta_j \equiv b_{\mu_j''} s_{\nu_j''},$$

where

$$\{\mu_0''(a), \mu_1''(a), \ldots\} = \text{sort}\{\mu_0(a), \mu_1(a), \ldots\}, \quad a \in D,$$

$$d\nu_j'' = f_j'' d\nu, \quad \text{and}$$

$$\{f_0''(\zeta), f_1''(\zeta), \ldots\} = \text{sort}\{f_0(\zeta), f_1(\zeta), \ldots\}, \quad \zeta \in \mathbf{T}.$$

The result just described is the basic ingredient in our solution of Problem 2; we don't state it as a separate proposition because such a statement would be rather lengthy and cumbersome.

Definition 9. An operator T of class C_0 is said to have property (P) if every operator $X \in \{T\}'$, satisfying the condition $\ker X = \{0\}$, has dense range.

Operators with property (P) are characterized by the following result from [**2**].

THEOREM 10. *Let T be an operator of class C_0 with Jordan model*

$$\bigoplus_{\alpha} S(\theta_\alpha).$$

Then T has property (P) if and only if the greatest common inner divisor of $\{\theta_j : j < \omega\}$ is 1, *i.e.*, $\wedge\{\theta_j : j < \omega\} \equiv 1$.

Note that one consequence of the relation $\wedge\{\theta_j : j < \omega\} \equiv 1$ is that $\theta_\omega \equiv 1$, and hence

$$\bigoplus_{\alpha} S(\theta_\alpha) = \bigoplus_{j<\omega} S(\theta_j)$$

acts on a separable space. We can now state the solution of Problem 2 for the class C_0.

THEOREM 11. *Assume that T is an operator of class C_0.*

(i) *If T has property (P), T' and T'' are operators of class C_0, and $T \oplus T' \sim T \oplus T''$, then $T' \sim T''$.*

(ii) *If T does not have property (P), there exists an operator T' of class C_0, acting on a nontrivial Hilbert space, such that $T \oplus T' \sim T$.*

Proof. Since quasisimilarity is an equivalence relation, we may assume that all the operators T, T', T'' involved are Jordan operators. Assume first that

$$T = \bigoplus_{\alpha} S(\theta_\alpha)$$

does not have property (P), so that $\varphi \equiv \wedge\{\theta_j : j < \omega\}$ is not a constant function. We will prove that $T \oplus S(\varphi) \sim T$, and to do this it clearly suffices to show that

$$\left(\bigoplus_{j<\omega} S(\theta_j)\right) \oplus S(\varphi) \sim \bigoplus_{j<\omega} S(\theta_j).$$

Choose an inner multiple $\theta = \gamma b_\mu s_\nu$ of θ_0, and write

$$\theta_j = \gamma_j b_{\mu_j} s_{\nu_j}, \quad \mu_j \leqq \mu, \quad d\nu_j = f_j d\nu, \quad 0 \leqq f_j \leqq 1 \text{ for } j < \omega,$$

and

$$\varphi = \gamma' b_{\mu'} s_{\nu'}, \quad \mu' \leqq \mu, \quad d\nu' = f' d\nu, \quad 0 \leqq f' \leqq 1.$$

Since $\oplus_{j<\omega} S(\theta_j)$ is a Jordan model and $\boldsymbol{\varphi}$ divides θ_j for all $j < \omega$, we have

$$\mu'(a) \leqq \mu_{j+1}(a) \leqq \mu_j(a), \quad a \in D, j < \omega,$$

and, upon a ν-negligible modification of the functions f_j, we may also assume that

$$f'(\zeta) \leqq f_{j+1}(\zeta) \leqq f_j(\zeta), \quad \zeta \in \mathbf{T}, j < \omega.$$

We prove now that the Jordan model of

$$\left(\bigoplus_{j<\omega} S(\theta_j)\right) \oplus S(\boldsymbol{\varphi})$$

is precisely

$$\bigoplus_{j<\omega} S(\theta_j).$$

By the remarks preceding Definition 9, it suffices to show that

$$\text{sort}\{\mu'(a), \mu_0(a), \mu_1(a), \dots\} = \{\mu_0(a), \mu_1(a), \dots\}, \quad a \in D,$$

and

$$\text{sort}\{f'(\zeta), f_0(\zeta), f_1(\zeta), \dots\} = \{f_0(\zeta), f_1(\zeta), \dots\}, \quad \zeta \in \mathbf{T}.$$

These relations are quite obvious from the definition of sort, and thus (ii) is proved (note that $S(\boldsymbol{\varphi})$ acts on a trivial Hilbert space if and only if $\boldsymbol{\varphi} \equiv 1$).

Assume now that

$$T = \bigoplus_{j<\omega} S(\theta_j)$$

has property (P), so that $\wedge\{\theta_j : j < \omega\} \equiv 1$, and let

$$T' = \bigoplus_{\alpha} S(\theta'_\alpha)$$

be an arbitrary Jordan operator. Let us consider the Jordan model

$$\bigoplus_{j<\omega} S(\psi_j)$$

of

$$\left(\bigoplus_{j<\omega} S(\theta_j)\right) \oplus \left(\bigoplus_{j<\omega} S(\theta'_j)\right).$$

According to the general recipe, and taking into account the fact that

$$\bigoplus_{j<\omega} S(\theta_j) \quad \text{and} \quad \bigoplus_{j<\omega} S(\theta'_j)$$

are Jordan operators, we have $\psi_j \equiv d_{j+1}/d_j$, where $d_0 = 1$, and

$$d_j = \vee\{\theta_0\theta_1 \ldots \theta_{p-1}\theta'_0\theta'_1 \ldots \theta'_{j-p-1}:0 \leqq p \leqq j\}, \quad j \geqq 1.$$

Since θ'_j divides θ'_{j-p} for $0 \leqq p \leqq j$, $d'_j\theta'_j$ divides the function

$$\vee\{\theta_0\theta_1 \ldots \theta_{p-1}\theta'_0\theta'_1 \ldots \theta'_{j-p-1}\theta'_{j-p}:0 \leqq p \leqq j\}$$

which in turn divides d_{j+1}. Thus θ'_j divides $d_j/d_{j+1} \equiv \psi_j$ for all $j < \omega$. Now, for $\alpha \geqq \omega$ and $j < \omega$, θ'_α divides θ'_j, and hence θ'_α divides ψ_j. We conclude that the operator

$$\left(\bigoplus_{j<\omega} S(\psi_j)\right) \oplus \left(\bigoplus_{\alpha\geqq\omega} S(\theta'_\alpha)\right)$$

is a Jordan operator, and hence is the Jordan model of $T \oplus T'$. The important conclusion is that the nonseparable part of the Jordan model of $T \oplus T'$ contains precisely the same functions as those in the nonseparable part of T'. Thus, in the proof of (i), we may restrict ourselves to the case in which T' and T'' act on separable spaces.

Assume therefore that

$$T' = \bigoplus_{j<\omega} S(\theta'_j), \quad T'' = \bigoplus_{j<\omega} S(\theta''_j), \quad \text{and} \quad T \oplus T' \sim T \oplus T'.$$

Choose an inner function $\theta = \gamma b_\mu s_\nu$ which is a common multiple of θ_0, θ'_0 and θ''_0, and write

$$\theta_j = \gamma_j b_{\mu_j}\theta_{\nu_j}, \quad \mu_j \leqq \mu, \quad d\nu_j = f_j d\nu, \quad 0 \leqq f_j \leqq 1,$$
$$\theta'_j = \gamma'_j b_{\mu'_j}\theta_{\nu'_j}, \quad \mu'_j \leqq \mu, \quad d\nu'_j = f'_j d\nu, \quad 0 \leqq f'_j \leqq 1,$$
$$\theta''_j = \gamma''_j b_{\mu''_j}\theta_{\nu''_j}, \quad \mu''_j \leqq \mu, \quad d\nu''_j = f''_j d\nu, \quad 0 \leqq f''_j \leqq 1,$$

for $j \geqq 0$. The sequences

$$\{\mu_j(a):j \geqq 0\}, \quad \{\mu'_j(a):j \geqq 0\}, \quad \text{and} \quad \{\mu''_j(a):j \geqq 0\}$$

are nonincreasing for $a \in D$, and, upon a ν-negligible modification, we may assume that the sequences

$$\{f_j(\zeta):j \geqq 0\}, \quad \{f'_j(\zeta):j \geqq 0\}, \quad \text{and} \quad \{f''_j(\zeta):j \geqq 0\}$$

are also nonincreasing for $\zeta \in \mathbf{T}$. Furthermore, the condition $\wedge\{\theta_j:j < \omega\} \equiv 1$ implies that

$$\lim_{j\to\infty} \mu_j(a) = 0, \quad a \in D \quad \text{and}$$

$$\lim_{j\to\infty} f_j(\zeta) = 0 \quad \text{for } \nu\text{-almost every } \zeta \in \mathbf{T}.$$

By the remarks preceding Definition 9, the relation $T \oplus T' \sim T \oplus T''$ is equivalent to the relations

$$\text{sort}(\ \{\mu_j(a):j \geqq 0\} \cup \{\mu'_j(a):j \geqq 0\}\ )$$
$$= \text{sort}(\ \{\mu_j(a):j \geqq 0\} \cup \{\mu''_j(a):j \geqq 0\}\ )$$

for $a \in D$, and

$$\text{sort}(\{f_j(\zeta):j \geqq 0\} \cup \{f'_j(\zeta):j \geqq 0\})$$
$$= \text{sort}(\{f_j(\zeta):j \geqq 0\} \cup \{f''_j(\zeta):j \geqq 0\})$$

for ν-almost every $\zeta \in \mathbf{T}$. Lemma 8 implies now that

$$\{\mu'_j(a):j \geqq 0\} = \{\mu''_j(a):j \geqq 0\}, \quad a \in D, \quad \text{and}$$
$$\{f'_j(\zeta):j \geqq 0\} = \{f''_j(\zeta):j \geqq 0\} \quad \text{for } \nu\text{-almost every } \zeta \in \mathbf{T}.$$

We conclude that $\theta'_j \equiv \theta''_j$, $j < \omega$, and hence $T' = T''$, as desired. (The conclusion $T' = T''$ is stronger that $T' \sim T''$ because T' and T'' were taken to be Jordan operators.) The proof of (i), and of the theorem, is now complete.

We remark that Kadison and Singer [**6**] require, for the solution of Problem 2 in the case of unitary equivalence, that the von Neumann algebra $W^*(T)$ generated by T and the commutant $W^*(T)'$ be finite von Neumann algebras. Our condition in Theorem 11 only involves $\{T\}'$. Of course, $\{T\}' \supset W^*(T)'$, so in a sense we require a stronger finiteness condition (which in our case turns out to be necessary as well as sufficient). It would be interesting to know whether Problem 2 has a positive answer for unitary equivalence under the condition that $\{T\}'$ be finite or, more precisely, that T have property (P) (cf. Definition 9 above).

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