Linear Substitutions and their Invariants.

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Introduction.

The main points of this paper are :—

(i) The construction of a linear substitution from its poles or linear invariants and its multipliers (§§ 3, 7) ;

(ii) a formula for r repetitions of a substitution (§ 4) ;

(iii) a specification of the types of linear substitutions of order r, with examples of the simplest of those types (§§ 9 ff.) ;

(iv) The geometrical illustration of the case of three variables (§§ 15 ff.)

§ 1. Take a triangle, sides a, b, c. Form a second triangle with sides a', b', c' equal to the medians of the first, and a third with sides a'', b'', c'' equal to the medians of the second. From the relations between sides and medians

$$a'^2 = -\tfrac{1}{4}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2, \text{ etc.,}$$

it follows that

$$\text{(i)} \qquad \frac{a''^2}{a^2} = \frac{b''^2}{b^2} = \frac{c''^2}{c^2} = \tfrac{9}{16},$$

so that the third triangle is similar to the first;

$$\text{(ii)} \qquad \frac{a'^2 + b'^2 + c'^2}{a^2 + b^2 + c^2} = \tfrac{3}{4},$$

$$\frac{b'^2 - c'^2}{b^2 - c^2} = \frac{c'^2 - a'^2}{c^2 - a^2} = \frac{a'^2 - b'^2}{a^2 - b^2} = -\tfrac{3}{4};$$

or the four functions of the sides which form the denominators are *invariants* for the transformation, except for the numerical factors

$+\frac{3}{4}$ in the first case, $-\frac{3}{4}$ in the others. We are evidently dealing with a linear substitution, expressing the *squares* of the medians in terms of those of the sides; it is a substitution of order 2, and possesses independent linear invariants equal to the number (three) of the variables. Increasing the coefficients in the ratio 4 : 3, we reduce the determinant of the substitution to unity, and it then takes the form, with x_1, x_2, x_3, in place of a^2, b^2, c^2,

$$\left.\begin{aligned} x_1' &= -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{2}{3}x_3, \\ x_2' &= \tfrac{2}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3, \\ x_3' &= \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3, \end{aligned}\right\}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

giving

$$\frac{x_1''}{x_1} = \frac{x_2''}{x_2} = \frac{x_3''}{x_3} = 1,$$

$$\frac{x_1' + x_2' + x_3'}{x_1 + x_2 + x_3} = 1,$$

$$\frac{x_2' - x_3'}{x_2 - x_3} = \frac{x_3' - x_1'}{x_3 - x_1} = \frac{x_1' - x_2'}{x_1 - x_2} = -1.$$

Since

$$16\Delta^2 = 2\Sigma b^2c^2 - \Sigma a^4$$

$$= \tfrac{1}{3}(a^2 + b^2 + c^2)^2 - \tfrac{2}{3}\Sigma(b^2 - c^2)^2,$$

the expression for the area is also invariant, and the medians will always form a triangle.

§ 2. *The General Linear Substitution.*

Let $\alpha, \beta, \ldots$ be symbols taking the values $1, 2, \ldots n$. Then we denote by (l) the linear substitution in n variables

$$x'_\alpha = \sum_\beta l_{\alpha\beta}\, x_\beta. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(l) followed by (m) leads to the substitution $(m)(l)$ given by

$$x_\gamma'' = \sum_\alpha m_{\gamma\alpha} x'_\alpha$$

$$= \sum_\beta \Big(\sum_\alpha m_{\gamma\alpha} l_{\alpha\beta}\Big) x_\beta.$$

For the result of r repetitions of (l) we shall use the notations $(l)^r$, and

$$x_\alpha^{(r)} = \sum_\beta l_{\alpha\beta}^{(r)}\, x_\beta.$$

The variables may be regarded as homogeneous "point" coordinates in space of $(n-1)$ dimensions. A set of values

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$x_{a,\epsilon}$ $(a=1, 2, \ldots n)$, will then define a "point" P_ϵ, and a linear equation

$$L \equiv \sum_a p_a x_a = 0$$

may be said to specify an "$(n-2)$-plane," *i.e.* a linear $(n-2)$-dimensional locus. Two sets of values of the x's determine the same point when the ratios of corresponding values are equal; and likewise for the p's. The point P′ into which P is changed by the substitution will be called the *transformed* of P.

If a point is unaltered by the substitution, its coordinates must satisfy for some value of k the equations

$$kx_a = \sum_\beta l_{a\beta} x_\beta, \quad (a=1, 2, \ldots n). \qquad (3)$$

Eliminating the x's, we obtain the *characteristic* equation for k:

$$\prod_{\epsilon=1}^{n}(k-k_\epsilon) \equiv \begin{vmatrix} l_{11}-k, & l_{12}, & \ldots l_{1n} \\ l_{21}, & l_{22}-k, & \ldots l_{2n} \\ \ldots & \ldots & \ldots \\ l_{n1}, & l_{n2}, & \ldots l_{nn}-k \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (4)$$

Assuming for the present that no root is repeated, each root k_ϵ determines an *invariant point* or *pole** P_ϵ of the substitution.

§ 3. Given the n poles P_ϵ, assumed not to lie on a plane locus, and the n corresponding roots k_ϵ, assumed all different, we can construct the substitution uniquely. The coordinates of P_ϵ being $x_{a,\epsilon}$, $(a=1, 2, \ldots n)$, put

$$D \equiv \begin{vmatrix} x_{11}, & x_{21}, & \ldots & x_{n1} \\ x_{12}, & x_{22}, & \ldots & x_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ x_{1n}, & x_{2n}, & \ldots & x_{nn} \end{vmatrix},$$

which does not vanish. Substituting the coordinates of the P_ϵ in (3), and the appropriate values of k, we have n^2 equations of the type

$$l_{a1}x_{1\epsilon} + l_{a2}x_{2\epsilon} + \ldots + l_{an}x_{n\epsilon} = k_\epsilon x_{a\epsilon}.$$

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* Hilton, *Finite Groups*, III. 6.

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Solving for the l's the n equations obtained by keeping α constant and making ϵ vary, we find

$$D \, . \, l_{\alpha\beta} = \sum_{\epsilon} k_{\epsilon} \, x_{\alpha\epsilon} X_{\beta\epsilon}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where $X_{\alpha\beta}$ is the co-factor of $x_{\alpha\beta}$ in D.

These expressions for the l's, substituted in (2), give the substitution in terms of the poles and roots alone.

The substitution (2) contains n^2 coefficients, and therefore $(n^2 - 1)$ independent constants. The knowledge of P′, the *transformed* of a given point P, involves $(n-1)$ relations between these constants. Thus $(n+1)$ such pairs will in general determine the substitution. The poles constitute n pairs, being points which coincide with their transformeds; and the knowledge of the roots is equivalent to that of an $(n+1)^{\text{th}}$ pair, as is clear from (2) and (5).

§ 4. Consider the effect, on the coordinates of the pole P_{ϵ}, of repeated application of (l). One application changes $x_{\alpha\epsilon}$ into $kx_{\alpha\epsilon}$; hence r applications will change it into $k^r x_{\alpha\epsilon}$. The substitution $(l)^r$ has the same poles as (l), but the root associated with each pole, or what we may call the *multiplier* of the pole, is the r^{th} power of the old multiplier. Hence the coefficients of $(l)^r$ are given by

$$D \, . \, l^{(r)}_{\alpha\beta} = \sum_{\epsilon} k^r_{\epsilon} \, x_{\alpha\epsilon} X_{\beta\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

In words, *the coefficients of the r-times repeated substitution are linear functions of the r^{th} powers of the roots of the characteristic equation of the original substitution, with coefficients independent of r.*

As an example, (6) may be used to obtain the result of two or more repetitions of the substitution of § 3.

§ 5. In order that a given substitution may be of order r, we must have

$$l^{(r)}_{\alpha\beta} = 0, \quad (\alpha \neq \beta),$$

$$l^{(r)}_{\alpha\alpha} = \text{a constant, say } \lambda^r.$$

Since $\sum_{\epsilon} x_{\alpha\epsilon} X_{\beta\epsilon}$ has the value D or zero according as α, β are equal or not, we must therefore have

$$k_{\epsilon}^{r} = \lambda^{r}, \ (\epsilon = 1, 2, \dots n) ;$$

whence $\qquad k_{\epsilon} = \lambda \rho_{\epsilon}$, ρ_{ϵ} being an r^{th} root of unity.

Hence the substitution (l) is of order r, provided

$$D \cdot l_{\alpha\beta} = \lambda \sum_{\epsilon} \rho_{\epsilon} x_{\alpha\epsilon} X_{\beta\epsilon}, \qquad \dots\dots(7)$$

where λ is a constant, and ρ_{ϵ} an r^{th} root of unity.

We may without loss assume $\lambda = 1$; and we can now construct substitutions of order r with a given set of poles.

§ 6. Consider the assumptions made up to this point.

(i) We have assumed that the characteristic equation (4) has no repeated root. But (7) defines a substitution of order r, whether there are equalities among the ρ_{ϵ} or not. When the ρ_{ϵ} are all equal, the substitution is identical; but short of this, equalities among them will determine distinct valid types of substitution.

Consider now the derivation of the poles from the substitution (§ 2). If *e.g.* $k_1 = k_2$, the others being distinct, the poles $P_3, P_4, \dots P_n$ are uniquely determined as before; but only $(n-2)$ of the n equations obtained from (3) by writing k_1 instead of k will be independent. Hence to the repeated root k_1 there correspond, not two unique poles, but a *line* (linear one-fold) of such; each point on the locus satisfies the conditions for a pole, and any two of them will serve for P_1, P_2. Similarly in other cases: equalities among the roots do not prevent us from obtaining n poles, but simply impair the uniqueness of that determination. Thus, in constructing a substitution from a given set of poles, the association of equal roots with two poles P_1, P_2, confers the polar property on every point of the line P_1P_2; and the association of equal roots with $(s+1)$ poles confers the polar property on every point of the s-plane which they determine.

(ii) We have assumed that D the determinant of the poles (§3) does not vanish. If it does, there are n poles on an $(n-2)$-plane; which is impossible, unless every point of the $(n-2)$-plane is a pole. But then all the n poles may be taken arbitrarily on this $(n-2)$-plane, which involves all the roots k_ϵ being equal, and the substitution reducing to identity.

§ 7. The substitution (2) changes the linear function $\sum_\alpha p_\alpha x_\alpha$ into

$\sum_\alpha\left(p_\alpha \sum_\beta l_{\alpha\beta} x_\beta\right), \;= \sum_\beta\left(\sum_\alpha l_{\alpha\beta} p_\alpha\right) x_\beta.$

If this function is invariant we must have for some value of k

$$\sum_\alpha l_{\alpha\beta} p_\alpha = k p_\beta \qquad (8)$$

Elimination of the p's leads to the same equation for k as in § 2, rows and columns of the determinant, however, being interchanged. Thus, associated with each root k_ϵ, there is not only a pole or invariant point P_ϵ, but also an invariant linear function or $(n-2)$-plane L_ϵ; and the substitution can be constructed from the L's and k's as readily as from the P's and k's. It is clear that L_ϵ must be the $(n-2)$-plane determined by the $(n-1)$ poles other than P_ϵ; and this will now be formally proved.

Denoting the coefficients or coordinates of L_ϵ by $p_{1\epsilon}, p_{2\epsilon}, \ldots p_{n\epsilon}$, put

$$E \equiv \begin{vmatrix} p_{11}, & p_{21}, & \cdots p_{n1} \\ p_{21}, & p_{22}, & \cdots p_{n2} \\ \cdots & \cdots & \cdots \\ p_{n1}, & p_{n2}, & \cdots p_{nn} \end{vmatrix},$$

and let the co-factor of $p_{\alpha\beta}$ be $P_{\alpha\beta}$. Let (l') be the substitution constructed from the L's and k's.

Then by (8),

$$l'_{1\beta} p_{1\epsilon} + l'_{2\beta} p_{2\epsilon} + \ldots + l'_{\alpha\beta} p_{\alpha\epsilon} + \ldots + l'_{n\beta} p_{n\epsilon} = k_\epsilon p_{\alpha\epsilon}, \; (\epsilon = 1, 2, \ldots n).$$

Solving for the l''s,

$$E \,.\, l'_{\alpha\beta} = \sum_\epsilon k_\epsilon p_{\beta\epsilon} P_{\alpha\epsilon} \qquad (9)$$

But if L_ϵ is the $(n-2)$-plane determined by the $(n-1)$ poles other than P_ϵ, clearly

$$p_{\beta\epsilon} = X_{\beta\epsilon}, \; P_{\alpha\epsilon} = D^{n-2} x_{\alpha\epsilon}, \; E = D^{n-1};$$

thus (9) becomes

$$D \,.\, l'_{\alpha\beta} = \sum_\epsilon k_\epsilon x_{\alpha\epsilon} X_{\beta\epsilon};$$

whence, by (5), $(l') \equiv (l)$.

§ 8. The simplest expression of a substitution will be that in which the invariant system is made the system of reference. This is always possible, as we have seen, though, in the case of equal roots k_ϵ, not unique. If then P_ϵ is defined by

$$\begin{cases} x_{\alpha\epsilon} = 0, \ (\alpha \neq \epsilon), \\ x_{\epsilon\epsilon} = 1, \end{cases}$$

the substitution takes the form

$$x'_\alpha = k_\alpha x_\alpha, \ (\alpha = 1, 2, \dots n), \qquad (10)$$

and the x_α are themselves the invariant linear functions.

§ 9. Substitutions fall into types according to the equalities among the roots k_ϵ. If s_1 of them are equal to ρ_1, s_2 to ρ_2, etc., we have the type $(s_1, s_2, \dots)$, the order of the numbers within the brackets being immaterial. Since the number of roots is n, and among them there are not more than r different values, the number of types for given integers n, r is the number of partitions of n into r or fewer parts. Thus, the types for $r = 2$ fall under the symbol $(n - s, s)$, and for $r = 3$ take one of the forms $(n - s, s)$, $(n - s - t, s, t)$. We proceed to consider the type $(n - 1, 1)$, which admits of very simple expression.

§ 10. *Type* $(n - 1, 1)$.

Put

$$k_\epsilon = 1, \ (1 \leq \epsilon \leq s),$$
$$k_\epsilon = \rho, \ (\epsilon > s),$$

where ρ is an r^{th} root of unity, other than unity itself. From (5),

$$\left.\begin{aligned} l_{\alpha\beta} &= (1 - \rho) \sum_{\epsilon=1}^{s} x_{\alpha\epsilon} X_{\beta\epsilon}/D, \quad (\beta \neq \alpha), \\ l_{\alpha\alpha} &= (1 - \rho) \sum_{\epsilon=1}^{s} x_{\alpha\epsilon} X_{\alpha\epsilon}/D + \rho. \end{aligned}\right\} \dots\dots\dots\dots (11)$$

In particular, if $s = 1$, and λ_α, μ_α are written respectively for $x_{\alpha 1}$, $X_{\alpha 1}/D$,

$$\left.\begin{aligned} l_{\alpha\beta} &= (1 - \rho)\lambda_\alpha \mu_\beta, \\ l_{\alpha\alpha} &= (1 - \rho)\lambda_\alpha \mu_\alpha + \rho \end{aligned}\right\}, \dots\dots\dots\dots (12)$$

where $\sum_\alpha \lambda_\alpha \mu_\alpha = 1$.

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The reduced number of arbitrary constants in this formula is due to the fact that the coordinates of the poles $P_2, P_3, \ldots P_n$ only enter in the expressions for the co-factors of the coordinates of P_1 in D. This was to be expected, for the equality of the $(n-1)$ roots confers polarity on every point of the $(n-2)$-plane defined by these $(n-1)$ poles; so that these poles are not unique.

Representing the substitution pictorially by the determinant of the coefficients, and removing the factor $(1-\rho)$ from each row to the outside, we have for (12) the form

$$(1-\rho)^n \begin{vmatrix} \lambda_1\mu_1+\sigma, & \lambda_1\mu_2, & \ldots \lambda_1\mu_n \\ \lambda_2\mu_1, & \lambda_2\mu_2+\sigma, & \ldots \lambda_2\mu_n \\ \cdots & \cdots & \cdots \\ \lambda_n\mu_1, & \lambda_n\mu_2, & \ldots \lambda_n\mu_n+\sigma \end{vmatrix}, \qquad \ldots\ldots\ldots\ldots(13)$$

where $\sigma \equiv \rho(1-\rho)^{-1}$. Since this determinant has evidently the value

$$\sigma^{n-1}(\sigma+1) = \rho^{n-1}(1-\rho)^{-n},$$

the determinant of the substitution (12) itself has the value ρ^{n-1}.

Thus the general substitution of order r and type $(n-1, 1)$ is given by

$$x'_a = \rho x_a + (1-\rho)\lambda_a \sum_\beta \mu_\beta x_\beta, \; (a, \beta = 1, 2, \ldots n), \; \ldots\ldots (14)$$

where $\sum_a \lambda_a \mu_a = 1$, and ρ is an r^{th} root of unity other than unity itself. For order 2, $\rho = -1$; for order 3, $\rho = \omega$ or ω^2; for order 4, $\rho = \pm i$; and so on. We do not regard as distinct two substitutions in which the ratio of corresponding coefficients is constant. Thus one different in form, but essentially identical with that just written, would be obtained on multiplying each term upon its right by ρ' any r^{th} root of unity.

Similarly, for the type $(n-2, 2)$ the formula (14) is replaced by

$$x'_a = \rho x_a + (1-\rho)\{\lambda_a \sum_\beta \mu_\beta x_\beta + \lambda'_a \sum_\beta \mu'_\beta x_\beta\},$$

where $\sum_a \lambda_a \mu_a = \sum_a \lambda'_a \mu'_a = 1$; and so in other cases.

§ 11. A simple case of (13) arises when the diagonal elements are all equal. Putting $\lambda_1\mu_1 = \lambda_2\mu_2 = \ldots = \lambda_n\mu_n = \dfrac{1}{n}$, we can write (13) in the form

$$\left(\frac{1-\rho}{n}\right)^n \begin{vmatrix} 1+n\sigma, & \lambda_1\lambda_2^{-1}, & \dots & \lambda_1\lambda_n^{-1} \\ \lambda_2\lambda_1^{-1}, & 1+n\sigma, & \dots & \lambda_2\lambda_n^{-1} \\ \dots & \dots & \dots & \dots \\ \lambda_n\lambda_1^{-1}, & \lambda_n\lambda_2^{-1}, & \dots & 1+n\sigma \end{vmatrix}, \qquad \text{.............(15)}$$

where the λ's are arbitrary. As a still more special case, we can make the determinant symmetrical by writing

$$\pm\lambda_1 = \pm\lambda_2 = \dots = 1.$$

From the table which follows:

r	ρ	σ	$1+n\sigma$
2	-1	$-\frac{1}{2}$	$1-\frac{n}{2}$
3	ω	$\frac{1}{3}(\omega-1)$	$1+\frac{n}{3}(\omega-1)$
4	i	$\frac{1}{2}(i-1)$	$1+\frac{n}{2}(i-1)$

we can deduce the following simple forms:

(i) $n=3,\ r=2.$

$$\left(\frac{2}{3}\right)^3 \begin{vmatrix} -\frac{1}{2}, & 1, & 1 \\ 1, & -\frac{1}{2}, & 1 \\ 1, & 1, & -\frac{1}{2} \end{vmatrix}$$

(ii) $n=3,\ r=3.$

$$\left(\frac{1-\omega}{3}\right)^3 \begin{vmatrix} \omega, & 1, & 1 \\ 1, & \omega, & 1 \\ 1, & 1, & \omega \end{vmatrix}$$

(iii) $n=4,\ r=2.$

$$\left(\frac{1}{2}\right)^4 \begin{vmatrix} -1, & 1, & 1, & 1 \\ 1, & -1, & 1, & 1 \\ 1, & 1, & -1, & 1 \\ 1, & 1, & 1, & -1 \end{vmatrix}$$

(iv) $n=2,\ r=4.$

$$\left(\frac{1-i}{2}\right)^2 \begin{vmatrix} i, & 1 \\ 1, & i \end{vmatrix}$$

In (ii), ω is either imaginary cube root of unity, as in (iv), i is either square root of -1; and the elements on either side of the leading diagonal may be affected if desired with any symmetrical alteration of signs.

(i) is the substitution arrived at in § 1.

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Invariants of Linear Substitutions.

§12. We have seen (§7) that, in the case of any linear substitution, there is associated with each root k_ϵ of the characteristic equation a linear invariant L_ϵ, defining an invariant $(n-2)$-plane in the $(n-1)$-dimensional space. Consider now the invariants of higher degree than the first.

The general homogeneous function of degree r in n variables $f_r(x_1, x_2, \ldots x_n)$, has terms in number

$$ {}_nH_r \equiv \frac{n(n+1)\ldots(n+r-1)}{r\,!}. $$

If this function remains unaltered under the substitution, except for a numerical factor $k^{(r)}$, we obtain at once on equating coefficients of corresponding terms in the x's, and thereafter eliminating the coefficients of f_r, a determinantal equation, constructed of the r-dimensional products of the $l_{\alpha\beta}$, and of degree ${}_nH_r$ in $k^{(r)}$. Assuming the roots of this equation for the present all different, we have therefore ${}_nH_r$ distinct invariants of degree r.

But this number tallies with that of the r-dimensional products of the linear invariants L_ϵ; which latter therefore comprise the complete system of invariants of degree r. Further, the roots of the equation in $k^{(r)}$ must be no other than the r-dimensional products of the roots k_ϵ of the original substitution.

§13. Now suppose that the roots of the equation in $k^{(r)}$ are not all distinct, *i.e.* suppose one or more relations of the form

$$ \prod_\epsilon k_\epsilon^{s_\epsilon} = \prod_\epsilon k_\epsilon^{s'_\epsilon}, \left(\sum_\epsilon s_\epsilon = \sum_\epsilon s'_\epsilon = r\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16) $$

subsist between the k_ϵ. Then the corresponding invariants will have equal multipliers, and hence any linear function of them will also be invariant. The number of *independent* invariants is unaltered, but the system ceases to be unique; or in other words, invariants appear with one or more arbitrary constants.

Suppose (i) $k_1 = k_2$. Then we shall not have, associated with this repeated root, two uniquely determined *linear* invariants, but a single infinity of such, any two of which may be chosen as L_1, L_2, and the others being of the form $L_1 + \lambda L_2$ (λ arbitrary).

Consequently, among the invariants of degree r we shall have the type

$$\Sigma\lambda_s L_1^s L_2^{r-s},\ (\lambda_1,\ \lambda_2,\ \ldots \text{ arbitrary}),$$

and others, such as

$$L_\gamma^t \,.\, \sum_s \lambda_s L_1^s L_2^{r-s-t}, \text{ etc.}$$

Similarly for other equalities among the roots.

Suppose (ii) that $k_1^r = k_2^s k_3^{r-s}$. We then have the invariant containing an arbitrary constant

$$L_1^r + \lambda L_2^s L_3^{r-s} \qquad \ldots\ldots(17)$$

§ 14. Relations of the type (16) are especially likely to arise when the substitution is itself of order r. The imaginary r^{th} roots of unity consist of conjugate pairs, the real root $+1$ (and -1 when it occurs) being for this purpose self-conjugate. Hence relations may arise of the type

$$k_\gamma k_{\gamma'} = k_\delta k_{\delta'} = \ \ldots = 1,$$

leading to the quadratic invariant, in which the λ's are arbitrary,

$$\lambda_\gamma L_\gamma L_{\gamma'} + \lambda_\delta L_\delta L_{\delta'} + \ldots;$$

and similarly invariants of higher degree may arise.

Further, the relations

$$k_1^r = k_2^r = \ldots = k_n^r = 1$$

confer invariancy on the form

$$\sum_s \lambda_s L_s^r,\ (\lambda_1, \lambda_2 \ldots \text{ arbitrary}) \qquad \ldots\ldots(18)$$

Examples of this will occur in the sequel.

§ 15. We now turn to the case of three variables, with its geometrical interpretation. We saw (§ 3) that a substitution in three variables is determined by four points and their *transformeds.* Let P, Q, R, S be given along with their transformeds P′, Q′, R′, S′. Then since cross-ratio is unaltered by a linear substitution, the transformed T′ of any fifth point T is the intersection of the rays P′T′, Q′T′, which satisfy

$$P'(Q'R'S'T') = P(QRST),$$
$$Q'(P'R'S'T') = Q(PRST).$$

The poles P_1, P_2, P_3 coincide each with its own transformed; hence the positions of one other point Q and its transformed Q′ will specify the substitution. For any fifth point T it is easily shewn that

$$P_1(P_2TP_3T') = P_1(P_2QP_3Q') = k_3/k_2,$$

and so for the pencils at P_2, P_3.

When the roots are distinct, k_1 defines the pole P_1 and the invariant line P_2P_3, and so for the other roots, without ambiguity; the lines being invariant in the sense that in general a point on either of them is transformed into another point on the same line.

Thus a substitution of order r will link up the points on each invariant line in sets of r each; *i.e.*, a point Q_1 will take up, on successive applications of the substitution, positions Q_2, Q_3, ... Q_r, Q_1; and the ranges $(Q_\alpha R_\alpha S_\alpha T_\alpha ...)$, $(Q_\beta R_\beta S_\beta T_\beta ..)$ will be homographic. In particular, a substitution of order 2 will set up an involution of points on each invariant line: and similar theorems hold for pencils through the invariant points.

A point Q_1 not on one of the invariant lines will also take up a cycle of positions Q_1, Q_2,.. Q_r, Q_1, but not in a line; and a line M, not through a pole, will take up a cycle of non-concurrent positions.

§16. When $k_2 = k_3$, the pole and line specified by k_1 remain as before; but, instead of definite poles P_2, P_3, we find the condition of invariancy satisfied by every point on the line L_1; which is thus invariant in the special sense, that every point on it is transformed into itself. It follows that every line through P_1, since it cuts L_1 in a second invariant point Π, is invariant in the less special sense. Any point and its transformed are now collinear with P_1; and any line and its transformed are concurrent with L_1. This is *homology** with P_1 as centre, and L_1 as axis; and the *parameter*, or constant cross-ratio $(P_1Q\Pi Q')$, where Q' is the transformed of Q, has the value k_2/k_1.

Repeated applications in this case will transform *any* point Q_1 into positions Q_2, Q_3, ... all on the same line through P_1; and if the substitution is of order r, Q_r will coincide with Q_1. If $r = 2$, the points on *any* line $P_1\Pi$ through P_1 will with their transformeds determine an involution. For all values of $r > 2$ the transformation is imaginary.

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* Russell, *Pure Geometry*, Ch. XXXI.

§ 17. Let a relation of the form

$$k_\alpha^r = k_\beta^s k_\gamma^{r-s}, \quad (0 \leq s \leq r),$$

hold between the three roots k_α, k_β, k_γ. Then by (17) there is an invariant of degree r with an arbitrary constant, viz.,

$$\lambda . L_\alpha^r + L_\beta^s L_\gamma^{r-s}.$$

If $s=0$, two roots k_α^r, k_γ^r of the characteristic equation of $(l)^r$ are equal, and any line $L_\alpha + \lambda L_\gamma$ through P_β will be invariant for *this* substitution, as also any point on L_β; *i.e.* the result of r repetitions of (l) will be a *homology*.

Again, put $r=2$, $s=1$. Then we obtain that for a substitution in which $k_\alpha^2 = k_\beta k_\gamma$, every conic touching L_β, L_γ at their intersections with L_α is invariant. This holds, *e.g.* for a substitution of order 3, with $k_\alpha, k_\beta, k_\gamma = 1, \omega, \omega^2$ respectively.

Lastly, formula (18) shows that for a substitution of order 3 in three variables, there exists a doubly infinite family of invariant cubics, with nine inflexions lying three by three on the invariant lines.

§ 18. The substitution (1) of § 1 is a case of § 16. The equation for k reduces to

$$(k-1)(k+1)^2 = 0.$$

Associated with the multiplier $k=1$ is the pole $P_1(1, 1, 1)$ and the invariant line

$$L_1 \equiv x_1 + x_2 + x_3 = 0.$$

Associated with the multiplier $k = -1$ there are, as poles, all points on L_1; and, as invariant lines, all lines through P_1; and the latter may be expressed in terms of the three symmetrical, but not independent,

$$L'_1 \equiv x_2 - x_3, \; L'_2 \equiv x_3 - x_1, \; L'_3 = x_1 - x_2.$$

It follows that every quadratic of the form

$$\lambda_1 L_1^2 + \Sigma(\lambda'_\alpha L'^2_\alpha + \mu'_\alpha L'_\beta L'_\gamma)$$

is invariant, with multiplier $+1$.

Among these are the symmetrical forms

$$x_1^2 + x_2^2 + x_3^2, \; x_2 x_3 + x_3 x_1 + x_1 x_2.$$