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CONNEXIONS AND PROLONGATIONS

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I. Connexions as Maps on Tangent Bundles

1. Introduction. Computation of the velocity of a given motion depends on measurement of nearby position changes only. Computation of acceleration, on the other hand, depends on measurement of nearby changes in velocity. But since velocity vectors are attached to positions so that even nearby ones are not *a priori* comparable, acceleration is not computable until a rule for comparison of vectors along a curve is given. Such a rule – parallel translation or linear connexion – exists automatically in Euclidean spaces. For motions in more general manifolds, for example (semi-) Riemannian ones, parallel translation is a less obvious consequence of the metric properties.

If $\xi(t)$ is a motion in a manifold M then its velocity field $\xi'(t)$ can be viewed as a motion in the tangent manifold TM and the iterated velocity field $\xi''(t)$ as a motion in the iterated tangent manifold $T(TM) = T^2(M)$. We can speak of acceleration in M only when we have a way of converting $\xi''(t)$ to a motion in TM. This leads naturally to the idea that a linear connexion is a way of reducing second order data to first order data, in effect a map from the second order tangent bundle to the first order tangent bundle (P. Dombrowski [**1**]). In preference to equivalent standard definitions of connexions as operators on pairs of smooth vector fields (covariant derivatives), as distributions on the frame bundle, as horizontal maps and splittings of the tangent bundle [**2**]— this seems like the alternative best suited to any categorical theory such as that of prolongations which concerns higher order properties of manifolds and maps. (See Part II.)

The view that a linear connexion (given in terms of a covariant derivative operator ∇) is a map C from T^2M to TM depends very simply on the relationship $C(x'Y) = \nabla_{x'}Y$ where $x'Y$ means the second order tangent vector which describes the change of the vector field Y on M in the direction of the tangent vector x'. In Theorem 1 we characterize such maps and so must recognize explicitly the two natural and isomorphic, yet distinct vector bundle structures on the pair of spaces T^2M, TM. Formally it is easier to keep them apart when we deal with general vector bundles over the manifold M rather than just the tangent bundle, and this is why we treat connexions in the setting of vector bundles. One of the two structures on T^2M, TM is in fact more than just a vector bundle: the fibres are T-modules where T is a two-dimensional real

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algebra. Because this extra bit of structure seems unusual we have spelled it out in local coordinates as well.

The isomorphism between the two vector bundle structures on the space of second order tangents is caused by the well known symmetry of T^2M [**4**]. This symmetry reflects a much more general situation, namely the equivalence of iteration of higher order operations [**8**] which appears explicitly in Part II. We show how this symmetry helps produce a simple, intrinsic formula for the torsion tensor of any linear connexion, and likewise how it exhibits the curvature tensor as a map on the third order tangent bundle (Theorems 3 and 2). Beyond these particular formulas obtained, the methods of computation may have some interest because we refer to the elements of T^2M in a geometric way.

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2. Notation. We will stay entirely within the category of C^∞ maps and manifolds. (B, M, p, F, G) will mean that the map $p: B \to M$ gives B the structure of a fiber bundle space over the manifold M, with standard fiber F and group G. Usually we suppress the mentioning of F and G, and just write (B, M, p) for brevity. If $x_i (i = 1, 2, \ldots d)$ are local coordinate functions in M and if $m \in M$ then the notation $m \leftrightarrow (m_i)$ is just shorthand for: the d-tuple (m_i) is a list of the local coordinates $x_i(m)$. If $f: M \to N$ is a map then $Tf: TM \to TN$ will be its differential (rather than df or f_*).

In keeping with the functorial view of the second part of this paper we will regard tangent vectors as prolongations. Let T be the two-dimensional real algebra spanned by 1 and τ where 1 is unity and $\tau^2 = 0$. Thus T is a local algebra in the sense that it is the direct sum $R + I$ where R is the real numbers and I is a maximal ideal. (Henceforth T will therefore have two meanings: as this algebra or as the tangent functor, with context deciding between them.) A *tangent vector* x' at m in the manifold M is an algebra homomorphism from the algebra DM of C^∞ functions on M to the algebra T subject to the additional condition that for any $f \in DM$ the real component of $x'(f)$ is $f(m)$. Thus $x'(f) = f(m) + Lx'(f)\tau$ where $Lx'(f)$ is the component of $x'(f)$ along τ. The linearity of x' on DM implies that Lx' is a linear functional and the multiplicative property $x'(fg) = x'(f)x'(g)$ that Lx' is a derivation on DM. In this way we recapture from x' a tangent vector Lx' in the usual sense.

Remarks. (1) The isomorphism between the set of tangent vectors as prolongations and the set of real derivations is not natural because it depends on the choice of a basis for the maximal ideal of T. We fix one such choice, τ, once and for all.

(2) During most of the first part of this paper the reader may interpret tangent vectors as usual, but in the discussion of the structure of the iterated tangent bundle and in Lemmas 5 and 6 our interpretation is significant.

3. Vector bundles. There are two distinct though isomorphic ways of viewing the iterated tangent vectors in a manifold as a bundle over the tangent vectors. Firstly, if $\pi: TM \to M$ takes any tangent vector to its point of attachment in the manifold (basepoint projection), then the differential $T\pi$ makes the iterated tangents $T(TM) = T^2M$ into a vector bundle $(T^2M, TM, T\pi)$ where addition $+_{T\pi}$ and scalar multiplication $L_{T\pi}$ in the fibers are essentially the differentials $T(+)$ and $T(L)$ of addition and scalar multiplication in (TM, M, π). Secondly, the basepoint projection from $T(TM)$ to TM, also denoted π, yields the usual tangent structure (T^2M, TM, π). To reduce the effort of distinguishing two different bundle structures on the same pair of spaces (T^2M, TM) we generalize slightly. Thus (E, M, p) will be a real vector bundle over the manifold M. Then (TE, E, π) is a vector bundle, but the "differentiated" bundle (TE, TM, Tp) has fibers which are in fact modules with respect to the two-dimensional algebra T. How this T-module structure comes about is implied in Lemma 2, Part II, and below we describe it in local coordinates. The two bundles are connected via the commutative diagram

$$\begin{tikzcd}
 & TE \arrow[dl, "\pi"'] \arrow[dr, "Tp"] & \\
E \arrow[dr, "p"'] & & TM \arrow[dl, "\pi"] \\
 & M &
\end{tikzcd}$$

Let $x_i (i = 1, 2, \ldots d)$ be local coordinates in M. Then $(x_{i0}, x_{i1}) = (x_{10}, \ldots, x_{d0}, x_{11}, \ldots, x_{d1})$ are local coordinates in TM defined by $x'(x_i) = x_{i0}(x') + x_{i1}(x')\tau$ for any $x' \in TM$. If $(x_i \circ p, y_j)$ are local coordinates in $E (j = 1, 2, \ldots, e)$, induced local coordinates in TE will be $((x_i \circ p)_0, y_{j0}, (x_i \circ p)_1, y_{j1})$. With these conventions we list explicitly:

(a) $v' \leftrightarrow (m_i, v_j, a_i, b_j)$ in TE implies

$$\pi(v') \leftrightarrow (m_i, v_j) \quad \text{in } E$$
$$Tp(v') \leftrightarrow (m_i, a_i) \quad \text{in } TM.$$

(b) If $\pi(v') = \pi(w')$ and hence

$$v' \leftrightarrow (m_i, v_j, a_i, b_j), \qquad w' \leftrightarrow (m_i, v_j, c_i, d_j)$$

then in (TE, E, π)

$$v' +_\pi w' \leftrightarrow (m_i, v_j, a_i + c_i, b_j + d_j)$$
$$a_\pi v' \leftrightarrow (m_i, v_j, aa_i, ab_j).$$

(c) If $TP(v') = Tp(w')$ and hence

$$v' \leftrightarrow (m_i, v_j, a_i, b_j), \qquad w' \leftrightarrow (m_i, w_j, a_i, d_j)$$

then in (TE, TM, Tp)

$$v' +_{Tp} w' \leftrightarrow (m_i, v_j + w_j, a_i, b_j + d_j)$$

and for any $t = a + b\tau \in T$

$$tv'_{Tp} \leftrightarrow (m_i, av_j, a_i, ab_j + bv_j).$$

In particular if $t = a$ is real,

$$a_{Tp}v' \leftrightarrow (m_i, av_j, a_i, ab_j),$$

and if $t = \tau$

$$\tau_{Tp}v' \leftrightarrow (m_i, 0, a_i, v_j).$$

Next we restate in bundle language how any vector space is naturally isomorphic to its tangent space at 0 (when given the differentiable structure which includes its linear coordinate systems).

LEMMA 1. *If (E, M, p) is a vector bundle there is a natural identification map ν from E into TE which for each $m \in M$ is a linear isomorphism from the fiber E_m to the tangent space to this fiber at* 0.

Explicitly if for any $v \in E_m$, ρ is the ray $t \rightsquigarrow tv$ in E_m, then the tangent $\rho'(0)$ to ρ at 0 is $\nu(v)$.

If the local coordinates of v are (m_i, v_j) then the corresponding local coordinates of $\nu(v)$ are $(m_i, 0, 0, v_j)$.

Because there are two fiber structures on the space TE there are also two kinds of zero vectors. We clarify the relationship between them and at the same time obtain a different view of $\nu(E)$ as precisely those tangent vectors to E which project to 0 by both π and Tp. First let 0_T: $M \to TM$ and 0_E: $M \to E$ be the zero cross-sections in (TM, M, π) and (E, M, p) respectively. Then the differential $T0_E$: $TM \to TE$ is the zero cross-section in (TE, TM, Tp). The various zero cross-sections are related by the commutative diagram

$$\begin{tikzcd}
& TE & \\
E \arrow[ur, "0_T"] & & TM \arrow[ul, "T0_E"'] \\
& M \arrow[ul, "0_E"] \arrow[ur, "0_T"'] &
\end{tikzcd}$$

LEMMA 2. (1) $\pi^{-1}(0_E M) = \nu(E) +_\pi T0_E(TM)$.
(2) $Tp^{-1}(0_T M) = \nu(E) +_{Tp} 0_T(E)$.
(3) $\nu(E) = \pi^{-1}(0_E M) \cap Tp^{-1}(0_T M)$.

Proof. (1) The meaning of this formula is that every $v' \in TE$ such that $\pi(v') = 0$ is, in the standard tangent structure (TE, E, π), the unique sum $u' +_\pi w'$ where $u' \in \nu(E)$ and $w' \in T0_E(TM)$. $T0_E$ is as usual the differential of the zero cross-section 0_E: $M \to E$. Note first that $w' = T0_E(x')$ if and only if in the usual local coordinates with $\alpha = 0, 1$,

$$(x_i \circ p)_\alpha(w') = (T0_E(x')(x_i \circ p))_\alpha = (x'(x_i \circ p \circ 0_E))_\alpha = (x'(x_i))_\alpha$$
$$y_{j\alpha}(w') = (T0_E(x')(y_j))_\alpha = (x'(y_j \circ 0_E))_\alpha = 0.$$

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Hence $w' \in T0_E(TM)$ if and only if $w' \leftrightarrow (m_i, 0, a_i, 0)$. (1) now follows by the observation that $\pi(v') = 0$ if and only if $v' \leftrightarrow (m_i, 0, a_i, b_j)$, so that if $u' \leftrightarrow (m_i, 0, 0, b_j)$ and $w' \leftrightarrow (m_i, 0, a_i, 0)$ then $u' \in \nu(E)$ and $w' \in T0_E(TM)$ and $v' = u' +_\pi w'$ uniquely.

The proofs of (2) and (3) are analogous.

4. Connexions. Recall that a connexion in the vector bundle (E, M, p) is a map which associates with every vector field X on M and every cross-section $Y: M \to E$ another cross-section $\nabla_X Y$ in E subject to these conditions:

(1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

(2) $\nabla_{X+U}(Y) = \nabla_X Y + \nabla_U Y$

(3) $\nabla_X(fY) = LX(f)Y + f\nabla_X Y$

(4) $\nabla_{fX} Y = f\nabla_X Y$,

where f is any real valued function on M. Note that if tangent vectors are understood in the usual sense, formula (3) will read $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

The possibility of viewing a connexion as a map from TE to E depends on giving geometric life to the tangent vectors to E.

Definition. If x' is tangent to M, if $Y: M \to E$ is a cross-section and TY its differential then we will denote by $x'Y$ (or at times $x'(Y)$) the element $TY(x')$ of TE.

Thus, if ξ is any curve in M tangent to x' then $x'Y$ is tangent to the curve $Y \circ \xi$ in E, so that $x'Y$ depends only on Y restricted to any neighborhood of the basepoint m of x'.

Further,

$$\pi(x'Y) = Y(m), \qquad Tp(x'Y) = x'.$$

In local coordinates,

$$x'Y \leftrightarrow (m_i, \eta_j(m), a_i, Lx'(\eta_j))$$

when

$$x' \leftrightarrow (m_i, a_i) \quad \text{and} \quad Y \leftrightarrow (x_i, \eta_j).$$

Remark. In the special case that $E = TM$, $x'Y$ can be given a direct meaning as a functional when $T(TM)$ is identified with $(T \otimes T)M$ in the sense of prolongations. Namely, for any function $f: M \to R$, $x'Y(f)$ is the element $x'(Yf)$ in $T \otimes T$.

LEMMA 3. *If $x'' \in TE$ and if $x' = Tp(x'') \neq 0$ then there exists a cross-section $Y: M \to E$ (not unique) such that $x'' = x'Y$.*

Proof. The local coordinate formula for $x'Y$ above shows that if $x'' \leftrightarrow (m_i, v_j, a_i, b_j)$ is given, a desired cross-section $Y \leftrightarrow (x_i, \eta_j)$ can be determined locally by finding functions η_j so that $\eta_j(m) = v_j$ and $Lx'(\eta_j) = b_j$. Choose therefore any function g in a neighborhood of m so that $g(m) = 0$ and $Lx'(g) =$

1, which is possible since we assume $x' = Tp(x'') \neq 0$. Then define $\eta_j = v_j + b_j g$. Thus we obtain Y in a neighborhood of m and we can extend it to all of M by any smoothing function.

Now we show that the lack of uniqueness in the geometric realization of elements of TE does not matter for our purpose. First recall that if ∇ is a connexion in (E, M, p) and if $X(m) = x' \neq 0$ then $\nabla_X Y(m)$ depends only on the value of X at m, not on the rest of X. Hence it makes sense to write $\nabla_{x'} Y$ for $\nabla_X Y(m)$ when $x' \neq 0$.

LEMMA 4. *If two cross-sections Y and Z of M in E satisfy $x'' = x'Y = x'Z$ with $x' \neq 0$, then for any connexion ∇ in (E, M, p)*

$$\nabla_{x'} Y = \nabla_{x'} Z.$$

Proof. Let (x_i) and $(x_i \circ p, y_j)$ be local coordinates in M and E. This means that if $Y \leftrightarrow (x_i, \eta_j)$ there are (pointwise) linearly independent cross-sections Y_j: $M \to E$ such that $Y = \Sigma \eta_j Y_j$ locally.
Then

$$(1) \qquad \nabla_{x'} Y = \sum \eta_j(m) \nabla_{x'} Y_j + \sum Lx'(\eta_j) Y_j(m).$$

Since $x'Y = x'Z$ if and only if $\eta_j(m) = \zeta_j(m)$ and $Lx'(\eta_j) = Lx'(\zeta_j)$ when $Z \leftrightarrow (x_i, \zeta_j)$, Lemma 4 follows from (1).

For any fixed tangent vector $x' \in TM$ the pair of maps $f \rightsquigarrow x'(f)$: $DM \to T$ and $Y \rightsquigarrow x'Y$: $\Gamma(M, E) \to TE$ defines a homomorphism from the DM-module $\Gamma(M, E)$ of cross-sections of M in E to the T-module $Tp^{-1}(x')$, the fiber of TE above x' in the bundle (TE, TM, Tp). We state this more plainly.

LEMMA 5. *For any function f: $M \to R$, tangent vector $x' \in TM$ and cross-sections Y, Z of M in E*

$$x'(Y + Z) = x'Y +_{Tp} x'Z, \qquad x'(fY) = x'(f)_{Tp} x'Y.$$

Proof. Both formulas follow from the fact that the bundle structure on (TE, TM, Tp) is essentially the differential of the bundle structure (E, M, p). Precisely, if L: $R \times E \to E$: $(a, v) \rightsquigarrow av$ is scalar multiplication in E, the corresponding T-multiplication in TE is simply the differential TL: $T \times TE \to TE$ as $TR \approx T$. (Part II, Lemma 2). Then if g: $E \to R$ is any function, $x'(f)_{Tp} x'Y(g) = TL(x'(f), x'Y)(g) = (Tf(x'), TY(x'))(g \circ L) = x'(g \circ L \circ f \times Y \circ \Delta) = x'(g \circ fY) = T(fY)x'(g) = x'(fY)(g)$, proving the second of the formulas above. Here Δ: $M \to M \times M$: $m \rightsquigarrow (m, m)$ is the diagonal map.

The first formula is proved analogously.

Remark. The linearity of the differential in this context translates to

$$(x' + ay')Z = x'Z +_\pi a_\pi y'Z$$

for any real number a.

We now have what we need to prove

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THEOREM 1. *There is a one-to-one correspondence between connexions in the vector bundle* (E, M, p) *and maps* C: $TE \to E$ *satisfying*

(1) C *induces a vector bundle map from* (TE, E, π) *to* (E, M, p);

(2) C *induces a vector bundle map from* (TE, TM, Tp) *to* (E, M, p);

(3) $C \circ \nu$ *is the identity map of* E. (ν *is the natural identification of fibers of* E *with their tangent spaces at* 0.)

Remark. (1) and (2) amount to saying that these diagrams are commutative

$$\begin{array}{ccc} TE & \xrightarrow{C} & E \\ \pi \downarrow & & \downarrow p \\ E & \xrightarrow[p]{} & M \end{array} \qquad \begin{array}{ccc} TE & \xrightarrow{C} & E \\ Tp \downarrow & & \downarrow p \\ TM & \xrightarrow[\pi]{} & M \end{array}$$

and that C is linear on fibers.

COROLLARY. *When* $E = TM$ *we obtain that a linear connexion in the tangent bundle is a map from* T^2M *to* TM, *subject to the three conditions above.*

Before we give the proof of Theorem 1 we show more explicitly how the connexion map C is tied to the T-module structure on (TE, TM, Tp).

LEMMA 6. *Let* C: $TE \to E$ *satisfy the two first conditions of Theorem* 1. *Then the third condition* $C \circ \nu =$ *identity is equivalent to* $C(\tau_{Tp}v') = \pi(v')$ *for all* $v' \in TE$. *Here* τ *is the basis element of* T *such that* $\tau^2 = 0$.

Remark. The second formula of this lemma depends absolutely on a choice of τ as basis for the maximal ideal of T. However, in linking prolongations with real derivations we already had to fix such a choice (see Remark 1, section 2). This choice is implicit also in the definition of ν: $E \to TE$. Namely, if $v \in E$ and $\rho(t) = tv$ then $\nu(v) = \rho'(0)$, where $\rho'(0) = T\rho(D_0)$ and D_t is linked to ordinary differentiation by $D_t(f) = f(t) + f'(t)\tau$ for any function f: $R \to R$.

Proof of Lemma 6. In local coordinates let $v' \leftrightarrow (m_i, v_j, a_i, b_j)$. Then $\tau_{Tp}v' \leftrightarrow (m_i, 0, a_i, v_j)$. Hence $\tau_{Tp}v' = \nu(v) +_\pi T0_E(x')$ where $v = \pi(v') \leftrightarrow (m_i, v_j)$ and $x' = Tp(v') \leftrightarrow (m_i, a_i)$. The double linearity of C therefore implies $C(\tau_{Tp}v') = C(\nu(\pi(v')))$, proving our lemma since π is surjective.

Proof of Theorem 1. First assume that a connexion ∇ in (E, M, p) is given. We construct a map C: $TE \to E$ by the formula

$$C(x'') = \begin{cases} \nabla_{x'} Y, & \text{when } Tp(x'') = x' \neq 0 \\ \nu^{-1}(u'), & \text{when } Tp(x'') = 0. \end{cases}$$

Here Y: $M \to E$ is any cross-section such that $x'' = x'Y$, which exists by Lemma 3 when $Tp(x'') \neq 0$, and Lemma 4 guarantees $C(x'')$ is well defined in this case. u' is defined uniquely by the equation $x'' = u' +_{Tp} w' \in \nu(E) +_{Tp}$

$0_T(E)$ when $Tp(x'') = 0$ by Lemma 2. Thus C will automatically have Property 3. That C is smooth and a vector bundle map in both structures follows by direct application of local coordinates.

Conversely, if a bundle map C is given with the three listed properties then a connexion operator for (E, M, p) can be retrieved by defining

$$\nabla_X Y(m) = C(X(m)Y) \text{ or more briefly } \nabla_X Y = C \circ TY \circ X$$

for any vector field X in M and any cross-section $Y\colon M \to E$. Lemmas 5 and 6 imply that ∇ has the right properties.

Remark. If we use the same local coordinates as before in M, E and TE, so that D_i are the partial derivative vector fields in M and Y_j are pointwise linearly independent cross-sections in E, and if we define $\Gamma_k{}^{ij} : M \to R$ by

$$C(D_i(m)Y_j) = \sum \Gamma_k{}^{ij}(m) Y_k(m)$$

then the local coordinate expressions for the connexion map $C\colon TE \to E$ are

$$(3)\quad \bar{x}_i \circ C = \bar{x}_{i0}, \quad y_k \circ C = \sum_{i,j} \bar{x}_{i1} y_{j0} (\Gamma_k{}^{ij} \circ p \circ \pi) + y_{k1}.$$

Here $\bar{x}_i = x_i \circ p$.

5. Curvature. The curvature is tied to the connexion map C in a very simple and pleasing way, which depends on the symmetry map of any iterated tangent bundle. The intrinsic reason for the existence of this symmetry map is discussed in Section 2, Part II of this paper. Here all we need to know is that this symmetry map $S(= S_{TT})\colon T^2M \to T^2M$ in the usual induced local coordinates takes

$$x'' \leftrightarrow (m_i, a_i, b_i, c_i) \text{ to } S(x'') \leftrightarrow (m_i, b_i, a_i, c_i),$$

i.e., $x_{i\alpha\beta} \circ S = x_{i\beta\alpha}$. This means that $S = S^{-1}$ and $T\pi \circ S = \pi$. S is in fact a linear isomorphism between the bundles (T^2M, TM, π) and $(T^2M, TM, T\pi)$. In our application S will act on T^2E rather than T^2M. Finally recall that TC is the differential of the map C.

THEOREM 2. *The map $K\colon T^2E \to E$ given by*

$$K = C \circ TC - C \circ TC \circ S$$

is the curvature for the connexion C in the following sense. If R is the usual curvature tensor, if x', y' are tangents to the manifold M at m, and if z is in the fiber over m of the bundle space E, then

$$R(x', y')z = K(x'YZ) = K(T^2Z \circ TY(x')).$$

Here Y is any vector field on M such that $Y(m) = y'$ and Z is any cross-section in E such that $Z(m) = z$. T^2Z is the iterated differential of Z.

Before proving Theorem 2 we inject further geometric life into T^2E, in analogy to what we did for TE. If $Z: M \to E$ is any cross-section and $x'' \in T^2M$ define $x''Z = T^2Z(x'') = x''(TZ)$. If also $x'' = x'Y$ we obtain in this way the element $(x'Y)Z$ in T^2E. On the other hand YZ is a cross-section of M in TE and hence defines an element $x'(YZ)$ of T^2E. Since $x'(YZ) = T(YZ)(x') = T(TZ \circ Y)(x') = T^2Z \circ TY(x') = T^2Z(x'Y) = (x'Y)Z$ the following definition makes good sense.

Definition. Given a tangent vector x' to M, a vector field Y on M and a cross-section Z of M in E. Put

$$x'YZ = T^2Z \circ TY(x').$$

Remark. This notation is meant to be suggestive. YZ is a derivative function which incorporates the change of Z in the direction of any vector of Y. $x'YZ$ is a double derivative incorporating the change of YZ with respect to the vector x'. We now state and prove the analog to Lemma 3 though this is not needed in the sequel.

LEMMA 7. *Given any $z'' \in T^2E$, define $x' = T(p \circ \pi)(z'')$ and $y' = Tp \circ \pi(z'')$. Suppose either*

(a) *x' and y' are linearly independent tangent vectors to M, or*

(b) *$T\pi(z'')$ and $\pi(z'')$ are linearly dependent tangent vectors to E, but $x' \neq 0$ and $y' \neq 0$.*

Then there exist a vector field Y on M and a cross-section Z of M in E (neither of them unique) such that

$$z'' = x'YZ.$$

Proof. In case there did exist Y and Z such that $z'' = x'YZ$ it would follow that $Y(m) = y'$ where m is the basepoint of x', and also that $T\pi(z'') = x'Z$, $\pi(z'') = y'Z$ and $T^2p(z'') = x'Y$. Further, given a local coordinate system in M some computation yields that the induced local coordinates for z'' will be

$$x'YZ \leftrightarrow (m_i, \zeta_k(m), b_i, Ly'(\zeta_k), a_i, Lx'(\zeta_k), Lx'(\eta_i), Lx'(LY(\zeta_k))) \tag{4}$$

when

$$m \leftrightarrow (m_i), x' \leftrightarrow (m_i, a_i), y' \leftrightarrow (m_i, b_i),\ Y \leftrightarrow (x_i, \eta_i), Z \leftrightarrow (x_i, \zeta_k).$$

Now we prove Lemma 7 in case (a). We choose a local coordinate system (x_i) in M centered at m such that $x' = D_1(m)$ and $y' = D_2(m)$ are the first two coordinate tangent vectors at m. Relative to this coordinate system

$$m \leftrightarrow (0), \quad x' \leftrightarrow (0, \delta_{i1}), \quad y' \leftrightarrow (0, \delta_{i2}), \quad \text{and}$$
$$z'' \leftrightarrow (0, A_k, \delta_{i2}, C_k, \delta_{i1}, B_k, E_i, F_k).$$

Comparing this with (4) we see that *if* we can find Y and Z so that the following six equations hold

$$Y(m) = y', \zeta_k(m) = A_k, Lx'(\zeta_k) = B_k, Ly'(\zeta_k) = C_k, Lx'(\eta_i) = E_i,$$
$$Lx'(LY(\zeta_k)) = F_k$$

when $Y \leftrightarrow (x_i, \eta_i)$ and $Z \leftrightarrow (x_i, \zeta_k)$, then $z'' = x'YZ$.

If we now prescribe Y and Z locally (and extend in any smooth way to all of M) by $\eta_i = E_i x_1 + \delta_{i2}$ and

$$\zeta_k = A_k + B_k x_1 + C_k x_2 + (F_k - E_1 B_k - E_2 C_k) x_1 x_2$$

the six equations are indeed satisfied, and hence $z'' = x'YZ$.

In case (b) the assumptions that $T\pi(z'')$ and $\pi(z'')$ are linearly dependent together with $x' \neq 0$ and $y' \neq 0$ imply that $\pi(z'') = aT\pi(z'')$ and $y' = ax'$ for some non-zero constant a. We choose a local coordinate system centered at m such that $x' = D_1(m)$. Then

$$z'' \leftrightarrow (0, A_k, a\delta_{i1}, aB_k, \delta_{i1}, B_k, E_i, F_k).$$

Again, if we can choose $Y \leftrightarrow (x_i, \eta_i)$ and $Z \leftrightarrow (x_i, \zeta_k)$ such that

$$Y(m) = ax', \zeta_k(m) = A_k, Lx'(\zeta_k) = B_k, Lx'(\eta_i) = E_i, Lx'(LY(\zeta_k)) = F_k$$

then $z'' = x'YZ$. But this is the case when

$$\eta_i = E_i x_1 + a\delta_{i1} \text{ and } \zeta_k = A_k + B_k x_1 + (1/2a)(F_k - E_1 B_k) x_1^2.$$

The following four lemmas will help us avoid much routine computation in local coordinates to prove Theorem 2, and additionally provide more insight.

LEMMA 8. *If X and Y are vector fields on M with $X(m) = x'$ and $Y(m) = y'$ then*

$$S(x'Y) -_\pi y'X = \nu([X, Y](m)) +_{T\pi} 0_{x'}.$$

Here S: $T^2M \to T^2M$ is the symmetry map mentioned above. ν: $TM \to T^2M$ is the usual identification which maps a vector to the tangent at zero of the ray it determines. $0_{x'}$ is the zero tangent vector to x' in T^2M.

Proof. In the usual local coordinates $S(x'Y) -_\pi y'X \leftrightarrow (m_i, a_i, 0, Lx'(\eta_i) - Ly'(\xi_i))$ when $x' \leftrightarrow (m_i, a_i)$, $X \leftrightarrow (x_i, \xi_i)$ and $Y \leftrightarrow (x_i, \eta_i)$. Since $[X, Y](m) \leftrightarrow (m_i, Lx'(\eta_i) - Ly'(\xi_i))$ so that

$$\nu([X, Y](m)) \leftrightarrow (m_i, 0, 0, Lx'(\eta_i) - Ly'(\xi_i)),$$

and since $0_{x'} \leftrightarrow (m_i, a_i, 0, 0)$ the lemma follows by the explicit coordinate formula for addition in the fibers of the bundle $(T^2M, TM, T\pi)$.

LEMMA 9. *If φ: $M \to N$ is any map, then $S \circ T^2\varphi = T^2\varphi \circ S$.*

Proof. For any local algebras A and B the corresponding iterated functor AB is isomorphic to BA. S is this isomorphism when $A = B = T$. (See Part II, Lemma 7).

LEMMA 10. *If C: $TE \to E$ is a connexion map and if $u'', v'' \in T^2E$ such that $T\pi(u'') = T\pi(v'')$ then $TC(u'' +_{T\pi} v'') = TC(u'') +_{Tp} TC(v'')$.*

Proof. $+_{T\pi}$ refers to addition in fibers of the bundle $(T^2E, TE, T\pi)$ and $+_{Tp}$ to addition in the fibers of (TE, TM, Tp). Lemma 2 of Part II implies that the addition $+_{T\pi}$ is the differential $T(+_\pi)$ of addition in fibers of (TE, E, π). Since C is a connexion map it is linear on the fibers of (TE, E, π), that is $C \circ +_\pi = +_p \circ (C \times C)$. Hence

$$TC(u'' +_{T\pi} v'') = TC \circ (+_{T\pi})(u'', v'') = TC \circ T(+_\pi)(u'', v'')$$
$$= T(+_p) \circ (TC \times TC)(u'', v'') = TC(u'') +_{Tp} TC(v'').$$

LEMMA 11. *Let φ: $E \to F$ be a vector bundle map from (E, M, p) to (F, N, q). Then*

$$T\varphi \circ \nu = \nu \circ \varphi.$$

Proof. We have called both the identification maps ν. Let ρ_z be the ray in E determined by z. Because φ is linear on the fibers of E, $\varphi \circ \rho_z = \rho_{\varphi(z)} =$ the ray determined by $\varphi(z)$ in F. Hence $T\varphi \circ \nu(z) = T\varphi(\rho_z'(0)) = \rho_{\varphi(z)}'(0) = \nu \circ \varphi(z)$.

COROLLARY. *If C: $TE \to E$ is a connexion map, then*

$$TC \circ \nu = \nu \circ C.$$

Proof. C is a vector bundle map from (TE, E, π) to (E, M, p).

Proof of Theorem 2. Choose any vector fields X and Y on M and cross-section Z of M in E so that $X(m) = x'$, $Y(m) = y'$ and $Z(m) = z$. In terms of covariant derivatives the standard expression for the curvature tensor is

$$R(x', y')z = (R(X, Y)Z)(m) = \{\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\}(m).$$

Since covariant differentiation is linked to the connexion map via

$$\nabla_X Z = C(X(Z)) = C \circ TZ \circ X,$$

and since therefore

$$(\nabla_X\nabla_Y Z)(m) = C \circ T(C \circ TZ \circ Y)(x') = C \circ TC \circ T^2 Z \circ TY(x')$$
$$= C \circ TC(x'YZ),$$

we obtain for the curvature tensor

$$R(x', y')z = C \circ TC(x'YZ) - C \circ TC(y'XZ) - C([X, Y](m)Z).$$

We intend to show that the difference

$$R(x', y')z - K(x'YZ) = C \circ TC \circ S(x'YZ) - C \circ TC(y'XZ)$$
$$- C([X, Y](m)Z)$$

is zero in E. The two first terms in this expression are

$$\begin{aligned}\delta &= C \circ TC \circ S(x'YZ) - C \circ TC(y'XZ)\\ &= C \circ TC(S(x'Y)Z) - C \circ TC(y'XZ) \quad \text{(by Lemma 9)}\\ &= C \circ TC(S(x'Y)Z -_\pi y'XZ)\end{aligned}$$

because both C and TC are linear with respect to the ordinary tangent bundle structures and because both $S(x'Y)Z$ and $y'XZ$ have the same base point $x'Z$.

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Further the linearity of $T^2Z = T(TZ)$ with respect to $+_\pi$ implies that

$$\delta = C \circ TC((S(x'Y) -_\pi y'X)Z) = C \circ TC((\nu(t') +_{T\pi} 0_{x'})Z)$$

by Lemma 8 when $t' = [X, Y](m)$.

Lemma 2 of Part II tells us that $+_{T\pi} = T(+_\pi)$ so that

$$\begin{aligned}\delta &= C \circ TC \circ T^2Z \circ T(+_\pi)(\nu(t'), 0_{x'})\\ &= C \circ TC \circ T(TZ \circ +_\pi)(\nu(t'), 0_{x'})\\ &= C \circ TC \circ T(+_\pi \circ (TZ \times TZ))(\nu(t'), 0_{x'})\\ &= C \circ T(C \circ +_\pi \circ (TZ \times TZ))(\nu(t'), 0_{x'})\\ &= C \circ T(+_p \circ (C \times C) \circ (TZ \times TZ))(\nu(t'), 0_{x'}), \quad \text{because } C \text{ is}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{linear with respect to } +_\pi\\ &= C(\nu(t')(C \circ TZ) +_{Tp} 0_{x'}(C \circ TZ))\\ &= C(\nu(t')(C \circ TZ)) +_p C(0_{x'}(C \circ TZ)), \quad \text{because } C \text{ is}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{linear with respect to } +_{Tp}.\end{aligned}$$

But $0_{x'}(C \circ TZ) = T(C \circ TZ)(0_{x'}) = 0_{C \circ TZ(x')}$. Hence $C(0_{x'}(C \circ TZ)) = 0$ in E since C is linear with respect to (TE, E, π). As $+_p$ means addition in the bundle space E we have therefore shown that

$$\begin{aligned}\delta &= C \circ TC \circ T^2Z(\nu(t')) = C \circ TC \circ \nu \circ TZ(t') \quad \text{by Lemma 11}\\ &= C \circ \nu \circ C \circ TZ(t') \quad \text{by Corollary to Lemma 11}\\ &= C \circ TZ(t') \quad \text{since } C \text{ is a connexion map.}\end{aligned}$$

This means that indeed

$$R(x', y')z - K(x'YZ) = \delta - C([X, Y](m)Z) = C(t'Z) - C(t'Z) = 0 \text{ in } E.$$

6. Torsion and geodesics. In case the vector bundle (E, M, p) is the tangent bundle (TM, M, π) we can also give a nice formula for the torsion of any linear connexion in terms of the symmetry map S: $T^2M \to T^2M$.

THEOREM 3. *Let C: $T^2M \to TM$ be the connexion map of a linear connexion in the manifold M. Then the map*

$$\Theta = C - C \circ S \colon T^2M \to TM$$

is the torsion of the connexion in the following sense. If θ is the usual torsion tensor and if x' and y' are tangents to the manifold M at m, then

$$\theta(x', y') = \Theta(x'Y) = \Theta(TY(x'))$$

for any vector field Y on M such that $Y(m) = y'$.

Proof. Choose another vector field X so that $X(m) = x'$. Recall that in terms of covariant derivatives the torsion is

$$\begin{aligned}\theta(x', y') &= (\nabla_X Y - \nabla_Y X - [X, Y])(m)\\ &= C(x'Y) - C(y'X) - [X, Y](m)\\ &= C(x'Y) - C\circ S(x'Y) + C\circ S(x'Y) - C(y'X) - [X, Y](m)\\ &= \Theta(x'Y) + C(\nu([X, Y](m)) +_{T\pi} 0_{x'}) - [X, Y](m) \text{ by Lemma 8}\\ &= \Theta(x'Y) \quad \text{because } C \text{ is a connexion map.}\end{aligned}$$

Remark. The fixed points of the symmetry map S form a submanifold $\mathscr{T}M$ of T^2M characterized by $T\pi|\mathscr{T}M = \pi|\mathscr{T}M$. The fibers of this submanifold are not linear subspaces of the fibers in either of the bundle structures $(T^2M, TM, T\pi)$ or (T^2M, TM, π). But, any connexion map is uniquely determined modulo torsion once it is known on this submanifold $\mathscr{T}M$. Physically this means that if we know how to compute accelerations we also know how to compute changes in any vector fields along a curve (not merely velocity vector fields) provided we know the torsion (and in particular, if the torsion is zero). The reasons are firstly, that $\mathscr{T}M$ consists essentially of elements realizable geometrically as $x'X$ and never includes any elements of the form $x'Y$ with $Y(m) \neq x'$; secondly, that every argument except that of the torsion term in the right side of the formula

$$\begin{aligned}2C(x'Y) = C((y' - x')(Y - X)) &+ C(\nu([X, Y](m)))\\ &- C(x'X) - C(y'Y) + \Theta(y'X)\end{aligned}$$

lies in $\mathscr{T}M$.

Finally we point out that ξ is a geodesic of the linear connexion C if and only if

$$C\circ \xi'' = 0\circ\xi$$

where $0 = 0_T$ is the zero cross-section of M in TM.

II. Prolongations of Connexions

1. Introduction. In this part we utilize the functorial approach to prolongations taken by A. Weil [**8**].

The main result is that any given linear connexion in a manifold prolongs in a natural way to a linear connexion in any of the prolongations of the manifold (Theorem 2).

This means for example that a linear connexion in the manifold M gives rise to a linear connexion in the space of k-jets from R^n to M with source 0, including in particular the well known "tangent" connexion [**3**; **4**] in the tangent bundle space TM.

We take advantage of the view that a connexion is merely a certain map, established in Part I. Hence we may expect to prolong a given connexion only by applying the appropriate prolongation functor to the connexion map. For example, to get the tangent connexion we may expect only to apply the tangent functor to (in effect take the differential of) the connexion map. But this is not quite enough, and for good formal reasons we must make some "corrections" via the symmetries which link iterated prolongation functors. In the case of the tangent connexion the symmetry which occurs is the usual one of T^2M with itself.

In Theorem 4 we give curvature and torsion of the prolonged linear connexion, which turn out to be nearly, but not quite, the prolongations (for the tangent functor: differential) of the curvature and torsion of the given linear connexion.

We want to emphasize that we are dealing with the concept of connexion in its standard sense only. (For generalizations to higher order differential geometry, compare P. Libermann [**6**], W. Pohl [**7**], S. Kobayashi [**5**].) As in Part I, it seems easiest to use the context of connexions in vector bundles. First we show that any such connexion extends to a connexion in the prolonged vector bundle (Theorem 1). Then, if the starting connexion is linear, this is modified by symmetries to yield the prolonged linear connexion.

2. Notation and review of "prolongations". We continue the conventions of Part I. Let A be a local, real commutative algebra of finite dimension with unity. That is to say A is a direct sum $R + I$ of the real numbers and a maximal ideal I, so that A is essentially a quotient of a ring of formal real power series in several variables by a co-finite ideal. The prolongation-space AM is a fiberspace over the manifold M. Its fiber over a point m is the collection of all algebra homomorphisms x' from the differentiable functions on M, DM, to the algebra A, subject to the "local" condition:

For any function f in DM the real component of $x'(f)$ in A is $f(m)$.

If $A = A_n^k$ is the formal power series in n variables modulo terms of order greater than k then the fiber of A_n^kM over m consists of the k-jets from R^n to M with source 0 and target m. The fibers of AM are *not* generally vector spaces; in fact they carry a natural linear structure if and only if $A = A_n^1$ which is to say if and only if AM is the n-fold Whitney sum of the tangent bundle TM
in fact they carry a natural linear structure if and only if $A = A_n^1$ which is to say if and only if AM is the n-fold Whitney sum of the tangent bundle TM
$(A_1^1 = T)$.

The dimension of the bundle space AM is the product of the dimensions of the algebra A and the manifold M.

The map π: $AM \to M$ which collapses fibers will be referred to as the base-point projection. There is also an opposite cross-section map 0: $M \to AM$ in analogy to the zero cross-section for the tangent bundle. Namely, for any $m \in M$ and $f \in DM$ $0(m)(f) = f(m)$, which is to say the component of $0(m)(f)$ along the maximal ideal I in A is 0. By this cross-section 0 we can regard M as a submanifold of AM, or AM as a prolongation of M. But it is slightly misleading to refer to this cross-section 0 as "zero" in the general case when the fibers of AM are not vector spaces.

If the maps i: $M \to M'$ and j: $N \to N'$ are imbeddings of M and N as submanifolds of M' and N' and if F: $M \to N$ is any map, then a prolongation or extension of F is any map φ: $M' \to N'$ making the following diagram commutative

$$\begin{array}{ccc} M' & \xrightarrow{\varphi} & N' \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ M & \xrightarrow[F]{} & N \end{array}$$

In this sense every map F: $M \to N$ between manifolds prolongs to a bundle map AF: $AM \to AN$ by the formula $AF(x')(g) = x'(g \circ F)$ for any $x' \in AM$ and $g \in DN$, when the cross-sections 0 imbed M and N as submanifolds of AM and AN:

$$\begin{array}{ccc} AM & \xrightarrow{AF} & AN \\ {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle 0} \\ M & \xrightarrow[F]{} & N \end{array}$$

The assignment (M, F) into (AM, AF) is a functor from the category of C^∞ manifolds and maps to the category of C^∞ bundles and bundle maps. As is already apparent in our notation we call this functor A too, so that in what follows context will decide when we mean the functor A or the algebra A.

Whereas it looks like a shortcoming that prolongation spaces in general are not vector bundles with respect to their base manifolds, repeated application or iteration of prolongations is particularly simple. If A and B are two local algebras then so is their tensor product $A \otimes B$, and for any manifold M the iterated prolongation space $A(BM)$ is naturally isomorphic to $(A \otimes B)M$ (A. Weil [**8**]). This implies in particular that there is a natural isomorphism or symmetry map S_{AB} from $A(BM)$ to $B(AM)$. When $A = B = T$ this symmetry map becomes a well-known endomorphism of the iterated tangent bundle space $T^2M = T(M)$ (Kobayashi [**4**]).

For the purpose of describing local coordinates in AM we must choose a linear basis for the algebra A. It is often convenient to do this by first selecting $\tau_1, \tau_2, \ldots, \tau_n$ in the maximal ideal I of A so they form a linear basis of I modulo I^2. Together with unity these elements generate the algebra A. Since $I^{k+1} = \{0\}$ for some least positive integer k = order of A, we then fix among all ordered n-tuples $\alpha = (i_1, i_2, \ldots, i_n)$ of non-negative integers a finite subset $\alpha_1, \alpha_2, \ldots, \alpha_a$ such that $\tau^{\alpha_1}, \tau^{\alpha_2}, \ldots, \tau^{\alpha_a}$ form a linear basis for A. Here τ^α means $\tau_1{}^{i_1}\tau_2{}^{i_2} \ldots \tau_n{}^{i_n}$ with the understanding that $\tau_i{}^0 = 1$.

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Now suppose $x_1, x_2, \ldots, x_d$ are a system of local coordinate functions in the manifold M. The appropriate version of Taylor's theorem implies that any $x' \in AM$ is determined if and only if $x'(x_1), x'(x_2), \ldots, x'(x_d)$ are known. Therefore we obtain a system of local coordinate functions $x_{i\alpha_j}(i = 1, \ldots, d, j = 1, \ldots, a)$ by defining

$$x_{i\alpha_j}(x') = \text{component of } x'(x_i) \text{ along } \tau^{\alpha_j} \text{ in } A.$$

Note that the basepoint projection $\pi: AM \to M$ satisfies

$$x_i \circ \pi = x_{i0,0,\ldots,0}.$$

Assume that bases τ^{α_j} and σ^{β_k} have been chosen for the algebras A and B respectively. In the corresponding induced local coordinates the symmetry map $S_{AB}: A(BM) \to B(AM)$ amounts to

$$x_{i\alpha_j\beta_k} \circ S_{AB} = x_{i\beta_k\alpha_j}.$$

For the case $A = B = T =$ the two-dimensional algebra $\{1, \tau\}$ this means that if $x'' \leftrightarrow (m_i, a_i, b_i, c_i)$ in T^2M then $S_{TT}(x'') = S(x'') \leftrightarrow (m_i, b_i, a_i, c_i)$.

For any local algebra A the corresponding prolongation functor conserves all sorts of pleasant properties. For example, if $M \times N$ is a product manifold then so is $A(M \times N)$ because it is naturally isomorphic to $AM \times AN$ and will be identified with $AM \times AN$ in the following. We will also identify AR with the local algebra A, because any $x' \in AR$ is completely determined by its value $x'(i)$ on the identity map of the reals.

The functor A prolongs vector spaces, algebras and vector bundles to structures of the same kind. We state this precisely in Lemmas 1 and 2 below without giving the routine proofs. Note that when $A = T$ which produces the tangent functor, Lemma 2 gives a little more structure to the tangent bundle of a vector bundle than may be obvious in other contexts, namely that the fibers are T-modules.

LEMMA 1. *Let V be any finite-dimensional real vector space and A any local algebra as above.*

(1) *Then the prolongation space AV becomes an A-module, when addition and module multiplication in AV are merely the A-prolongations of addition and scalar multiplication in V.*

(2) *There is a canonical isomorphism $n: AV \to A \otimes V$. If $\{v^\alpha\}$ is a basis for V and $\{f_\alpha\}$ a dual basis, then*

$$n(v') = \sum v'(f_\alpha) \otimes v^\alpha \quad \textit{for any } v' \in AV.$$

(3) *If $\mu: V \times V \to V$ is an algebra structure on V, then $A\mu: AV \times AV \to AV$ is an algebra structure on AV. In this case the isomorphism $n: AV \to A \otimes V$ becomes an algebra isomorphism when $A \otimes V$ is given the algebra structure induced by $(a \otimes v)(b \otimes w) = (ab) \otimes (\mu(v, w))$.*

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(4) *If* j: $V \to A \otimes V$ *is the injection which takes* v *to* $1 \otimes v$ *then the following diagram is commutative*:

$$\begin{tikzcd} AV \arrow[rr, "n"] & & A \otimes V \\ & V \arrow[ul, "O"] \arrow[ur, "j"'] & \end{tikzcd}$$

Remark. Lemma 1 implies that if $\mathscr{M}(q)$ is the algebra of $q \times q$ real matrices then $A\mathscr{M}(q)$ is isomorphic to the algebra of $q \times q$ matrices with entries in A. In particular $AGL(q)$ is the set of $q \times q$ matrices with entries in A such that their real component matrices (= base points) are non-singular.

LEMMA 2. *Let* A *be a local algebra of linear dimension* a *and* (E, M, p) *a vector bundle with fiber* R^q *and group* $GL(q)$.

(1) *Then* (AE, AM, Ap) *is an* A*-module bundle with fiber* AR^q $(\approx A \otimes R^q)$ *and group* $AGL(q)(\subset GL(aq))$.

(2) (AE, AM, Ap) *is a prolongation of* (E, M, p) *in the sense that*

$$\begin{tikzcd} AE \arrow[d, "Ap"'] & E \arrow[l, "0"'] \arrow[d, "p"] \\ AM & M \arrow[l, "0"] \end{tikzcd}$$

is commutative.

(3) *If* (E^+, M, p^+) *denotes the Whitney sum of the bundle* (E, M, p) *with itself, then* (AE^+, AM, Ap^+) *is the Whitney sum of* (AE, AM, Ap) *with itself.*

(4) *If addition and scalar multiplication in the fibers of* E *are viewed as maps* $+$: $E^+ \to E$ *and* L: $R \times E \to E$, *then addition and* A*-module multiplication are merely the prolongations*

$$A+: AE^+ \to AE \quad \text{and} \quad AL: A \times AE \to AE.$$

We list below a series of technical lemmas needed en route to our goals. Part of the purpose in giving the simple proofs is to show how it is possible to operate computationally in this formalism.

LEMMA 3. *Let* 0 *denote either of the zero cross-sections in the vector bundles* (E, M, p) *or* (AE, AM, Ap). *Then* $A0 = 0$.

Remark. This means that the zero cross-section of AM in AE is the A-prolongation of the zero cross-section of M in E, so that the diagram

$$\begin{tikzcd} AM \arrow[r, "0"] & AE \\ M \arrow[u, "0"] \arrow[r, "0"'] & E \arrow[u, "0"'] \end{tikzcd}$$

is commutative.

Proof. The zero vectors in E are uniquely characterized by

$$z + 0 = z$$

for all z in the same fiber as 0. This can be formulated as a functional equation

$$+ \circ (i \times 0 \circ p) \circ \Delta = i$$

where i is the identity map of E, Δ the diagonal map of E in $E \times E$ and $+$ the addition in E conceived as a map from the Whitney sum E^+ to E. Applying the functor A we get

$$(A+) \circ (Ai \times A0 \circ Ap) \circ A\Delta = Ai.$$

Lemma 1 implies that the addition in (AE, AM, Ap), $+_A$, is $A+$, and since $A\Delta = \Delta$ and $Ai = i$ we get

$$+_A \circ (i \times A0 \circ Ap) \circ \Delta = i$$

which means that $A0$ satisfies the same functional equation which uniquely characterizes the zero cross-section in AE. Hence $A0 = 0$.

LEMMA 4. *If φ: $E \to F$ is a vector bundle map from (E, M, p) to (F, N, q) then $A\varphi$: $AE \to AF$ is an A-module bundle map from (AE, AM, Ap) to (AF, AN, Aq).*

Proof. This is similar in principle to Lemma 2 and in fact to most of our arguments regarding prolongations. Namely, the given properties are stated functionally and then the A-functor is applied. For example, if L_E denotes scalar multiplication in E then φ being a vector bundle map implies in particular the commutativity of the diagram

$$\begin{tikzcd} R \times E \arrow[r, "i \times \varphi"] \arrow[d, "L_E"'] & R \times F \arrow[d, "L_F"] \\ E \arrow[r, "\varphi"'] & F \end{tikzcd}.$$

Applying the functor A we obtain commutativity of

$$\begin{tikzcd} AR \times AE \arrow[r, "Ai \times A\varphi"] \arrow[d, "AL_E"'] & AR \times AF \arrow[d, "AL_F"] \\ AE \arrow[r, "A\varphi"'] & AF \end{tikzcd}.$$

Since $AR \approx A$, $Ai = i$ and since $AL_E = L_{AE}$ by definition, this tells us that $A\varphi$ commutes with scalars from the local algebra A. The proof that $A\varphi$ is linear on fibers of AE is similar, with the aid of Lemma 1.

LEMMA 5. *For any local algebras A and B the symmetry map S_{AB}: $ABM \to BAM$* satisfies

$$A\pi \circ S_{BA} = \pi \quad \textit{and} \quad S_{BA} \circ 0 = A0.$$

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Here π is the usual base point projection, while 0 is the standard imbedding of a manifold into its prolongation space.

Proof. Our claim is that for any manifold M these are commutative diagrams:

$$\begin{tikzcd} BAM \arrow[rr, "S_{BA}"] \arrow[dr, "\pi"'] & & ABM \arrow[dl, "A\pi"] \\ & AM & \end{tikzcd} \qquad \begin{tikzcd} BAM \arrow[rr, "S_{BA}"] & & ABM \\ & AM \arrow[ul, "0"] \arrow[ur, "A0"'] & \end{tikzcd}$$

The equations are quickly verified in local coordinates.

LEMMA 6. *The symmetry map S_{TA}: $TAM \rightarrow ATM$ is a vector bundle isomorphism from (TAM, AM, π) to $(ATM, AM, A\pi)$.*

Proof. Let $(\tilde{x}_i, y_k)$ be local coordinates in the vector bundle (E, M, p) such that $\tilde{x}_i = x_i \circ p$ and x_i are local coordinates in M. Lemma 1 then implies the following local coordinate formulas for addition and A-module multiplication in the prolonged bundle (AE, AM, Ap):

$$\tilde{x}_{i\alpha} \circ +_{Ap} = \tilde{x}_{i\alpha} \circ \pi_1, \quad y_{k\alpha} \circ +_{Ap} = y_{k\alpha} \circ \pi_1 + y_{k\alpha} \circ \pi_2,$$
$$\tilde{x}_{i\alpha}(s \cdot_{Ap} z') = \tilde{x}_{i\alpha}(z'), \quad y_{k\alpha}(s \cdot_{Ap} z') = [s(z'(y_k))]_\alpha.$$

Here π_1: $AE \times AE \rightarrow AE$ is the projection on the first factor, $s \in A$ and $z' \in AE$. Note that if $s \in R$ then $y_{k\alpha}(s \cdot_{Ap} z') = sy_{k\alpha}(z')$.

We apply these formulas when (E, M, p) is the tangent bundle (TM, M, π). Thus local coordinates in TM will be $(x_{i\sigma}) = (x_{i0}, x_{i1})$, in ATM $(x_{i\sigma\alpha})$, in AM $(\bar{x}_{i\alpha})$ and in TAM $(\bar{x}_{i\alpha\sigma})$. Then for any x'', y'' tangent to $x' \in AM$ and any real number s,

$$x_{i0\alpha} \circ S_{TA}(x'' +_\pi s_\pi y'') = \bar{x}_{i\alpha 0}(x'' +_\pi s_\pi y'') = \bar{x}_{i\alpha 0}(x''),$$
$$x_{i1\alpha} \circ S_{TA}(x'' +_\pi s_\pi y'') = \bar{x}_{i\alpha 1}(x'' +_\pi s_\pi y'') = \bar{x}_{i\alpha 1}(x'') + s\bar{x}_{i\alpha 1}(y'').$$

On the other hand

$$x_{i0\alpha}(S_{TA}x'' +_{A\pi} S_{TA}y'') = x_{i0\alpha}(S_{TA}x'') = \bar{x}_{i\alpha 0}(x''),$$
$$x_{i1\alpha}(S_{TA}x'' +_{A\pi} s \cdot_{A\pi} S_{TA}y'') = x_{i1\alpha}(S_{TA}x'') + x_{i1\alpha}(s \cdot_{A\pi} S_{TA}y'')$$
$$= \bar{x}_{i\alpha 1}(x'') + s\bar{x}_{i\alpha 1}(y'') \text{ because } s \text{ is real.}$$

We have thus proved that

$$S_{TA}(x'' +_\pi s_\pi y'') = S_{TA}x'' +_{A\pi} s \cdot_{A\pi} S_{TA}y''$$

which is the claim of Lemma 4.

LEMMA 7. *If A and B are local algebras, denote also by S_{AB} the algebra isomorphism from AB to BA induced by the isomorphism n: $AB \rightarrow A \otimes B$ and the symmetry of the tensor product. Let (E, M, p) be a vector bundle. Then*:

(1) *(ABE, ABM, ABp) is an AB-module bundle.*

(2) *The symmetry maps S_{AB} induce an isomorphism from the AB-module bundle (ABE, ABM, ABp) to the BA-module bundle (BAE, BAM, BAp) in the sense that*

$$S_{AB}(z'' +_{ABp} w'') = S_{AB}(z'') +_{BAp} S_{AB}(w'')$$
$$S_{AB}(b' \cdot_{ABp} z'') = S_{AB}(b') \cdot_{BAp} S_{AB}(z'')$$

for any z'', w'' in the same fiber of ABE and any $b' \in AB$.

(3) *In the same sense the AB-module bundle (ABE, ABM, ABp) is isomorphic to the $A \otimes B$-module bundle $(A \otimes BE, A \otimes BM, A \otimes Bp)$.*

(4) *(ABE, ABM, ABp) may in particular be considered a B-module bundle when B is viewed as a subalgebra of AB via the cross-scction 0: $B \to AB$. Then S_{AB} is a B-module bundle isomorphism from (ABE, ABM, ABp) to (BAE, BAM, BAp).*

Proof. (1) and (2) depend on the fact that addition and module multiplication in the fibers of (ABE, ABM, ABp) are just the iterated prolongations of addition and scalar multiplication in E. (3) is essentially Weil's observation that iterating the functors corresponding to the local algebras A and B is equivalent to applying the functor corresponding to $A \otimes B$. (4) depends on the commutativity of the diagram

$$\begin{tikzcd} AB \arrow[r, "S_{AB}"] & BA \\ B \arrow[u, "0"] \arrow[r, phantom, "\approx"] & BR \arrow[u, "Bi"'] \end{tikzcd}$$

Here $\approx$ is the isomorphism identifying B and BR, and i: $R \to A$ is the inclusion map.

LEMMA 8. *If (E, M, p) is a vector bundle then*

$$S_{TA} \circ \nu = A\nu.$$

Proof. ν is the identification which maps any element of a vector bundle to the tangent to zero of the ray it determines. We use local coordinates in TAE and ATE as in the proof of Lemma 6. Then Lemma 1, Part I, implies that

$$x_{i\sigma\alpha} \circ S_{TA} \circ \nu = \bar{x}_{i\alpha\sigma} \circ \nu = \begin{cases} \bar{x}_{i\alpha}, & \sigma = 0 \\ 0, & \sigma = 1 \end{cases}$$

$$y_{k\sigma\alpha} \circ S_{TA} \circ \nu = \bar{y}_{k\alpha\sigma} \circ \nu = \begin{cases} 0, & \sigma = 0 \\ \bar{y}_{k\alpha}, & \sigma = 1. \end{cases}$$

On the other hand

$$x_{i\sigma\alpha} \circ A\nu(x') = [x'(x_{i\sigma} \circ \nu)]_\alpha = \begin{cases} [x'(x_i)]_\alpha, & \sigma = 0 \\ [x'(0)]_\alpha, & \sigma = 1 \end{cases}$$

$$= \begin{cases} \bar{x}_{i\alpha}(x'), & \sigma = 0 \\ 0, & \sigma = 1. \end{cases}$$

Hence

$$x_{t\sigma\alpha} \circ A\nu = \begin{cases} \bar{x}_{t\alpha}, & \sigma = 0 \\ 0, & \sigma = 1. \end{cases}$$

Similarly

$$y_{k\sigma\alpha} \circ A\nu = \begin{cases} 0, & \sigma = 0 \\ \bar{y}_{k\alpha}, & \sigma = 1. \end{cases}$$

Thus the coordinate components of the maps $S_{A\,T} \circ \nu$ and $A\nu$ are equal which proves Lemma 8.

3. Connexions.

THEOREM 1. *Suppose that a connexion C: $TE \to E$ is given in the vector bundle (E, M, p). Then the map C_A: $TAE \to AE$ defined as*

$$C_A = AC \circ S_{TA}$$

is a connexion in the A-module bundle (AE, AM, Ap). C_A' is a prolongation of the connexion C in the sense that the diagram

$$\begin{tikzcd} TAE \arrow[r, "C_A"] & AE \\ TE \arrow[u, "TO"] \arrow[r, "C"'] & E \arrow[u, "0"'] \end{tikzcd}$$

is commutative. Here S_{TA}: $TAE \to ATE$ is the symmetry map, and 0: $E \to AE$ is the "zero"-cross-section (imbedding) of E in AE.

Remark 1. By stating that C_A is a connexion in the A-module bundle (AE, AM, Ap) we mean that not only does C_A-parallel-translation along curves in AM preserve the linear structure of the fibers of AE, but indeed the A-module structure of these fibers. This is to say that C_A is A-linear on the fibers of (TAE, TAM, TAp).

Remark 2. The facts that C_A is a connexion map and a prolongation in the sense stated imply the commutativity of these diagrams:

$$\begin{tikzcd} TAE \arrow[rrr, "C_A"] \arrow[ddd, "\pi"'] & & & AE \arrow[ddd, "Ap"] \\ & TE \arrow[ul, "TO"'] \arrow[r, "C"] \arrow[d, "\pi"'] & E \arrow[ur, "O"] \arrow[d, "p"] & \\ & E \arrow[r] \arrow[dl, "O"'] \arrow[r, phantom, shift right=4, "p"] & M \arrow[dr, "O"] & \\ AE \arrow[rrr, "Ap"'] & & & AM \end{tikzcd} \qquad \begin{tikzcd} TAE \arrow[rrr, "C_A"] \arrow[ddd, "TAp"'] & & & AE \arrow[ddd, "Ap"] \\ & TE \arrow[ul, "TO"'] \arrow[r, "C"] \arrow[d, "Tp"'] & E \arrow[ur, "O"] \arrow[d, "p"] & \\ & TM \arrow[r, "\pi"'] \arrow[dl, "TO"'] & M \arrow[dr, "O"] & \\ TAM \arrow[rrr, "\pi"'] & & & AM \end{tikzcd}$$

Proof of Theorem 1. The following two diagrams are commutative by Lemma 5 and because C is a connexion map.

$$
\begin{tikzcd}
TAE \arrow[rr, "C_A"] \arrow[dr, "S_{TA}"] \arrow[dd, "\pi"'] & & AE \arrow[dd, "Ap"] \\
& ATE \arrow[ur, "AC"'] \arrow[dl, "A\pi"] & \\
AE \arrow[rr, "Ap"'] & & AM
\end{tikzcd}
\qquad
\begin{tikzcd}
TAE \arrow[rr, "C_A"] \arrow[dr, "S_{TA}"] \arrow[ddd, "TAp"'] & & AE \arrow[ddd, "Ap"] \\
& ATE \arrow[ur, "AC"'] \arrow[d, "ATp"] & \\
& ATM \arrow[dr, "A\pi"] & \\
TAM \arrow[ur, "S_{TA}"] \arrow[rr, "\pi"'] & & AM
\end{tikzcd}
$$

Thus C_A takes fibers in the bundles (TAE, AE, π) and (TAE, TAM, TAp) to fibers in the bundle (AE, AM, Ap). Next we show

(1) C_A is linear on the fibers of (TAE, AE, π): Referring to the left diagram above we see that this claim holds because AC is a vector bundle map by Lemma 4 and S_{TA} is a vector bundle map by Lemma 6.

(2) C_A is A-linear on fibers of (TAE, TAM, TAp): Lemma 7 (4) shows how (TAE, TAM, TAp) is an A-module bundle isomorphic to the A-module bundle (ATE, ATM, ATp). By Lemma 4, AC is an A-module bundle map from the latter to (AE, AM, Ap), and hence the right commutative diagram above implies that C_A is A-linear.

(3) $C_A \circ \nu = i$ when i is the identity map: By Lemma 8 and the fact that C is a connexion map we get

$$C_A \circ \nu = AC \circ S_{TA} \circ \nu = AC \circ A\nu = A(C \circ \nu) = A(i) = i.$$

Together, these three facts imply that C_A is a connexion map.

Finally Lemma 5 implies commutativity of

$$
\begin{tikzcd}
TAE \arrow[rr, "C_A"] \arrow[dr, "S_{TA}"] & & AE \\
& ATE \arrow[ur, "AC"'] & \\
& TE \arrow[uul, "TO"] \arrow[u, "O"] \arrow[r, "C"'] & E \arrow[uu, "O"']
\end{tikzcd}
$$

which in turn shows that C_A is a prolongation in the sense of Theorem 1.

We wish to make a slight simplification of notation. It is the symmetry of the tensorproduct $B \otimes A$ with $A \otimes B$ induced by $b \otimes a \rightsquigarrow a \otimes b$ which is responsible for the functorial symmetry S_{BA} of iterated prolongations. Similarly the symmetry $B \otimes B \otimes A$ with $A \otimes B \otimes B$ induced by $b_1 \otimes b_2 \otimes a \rightsquigarrow a \otimes b_1 \otimes b_2$ is responsible for the functorial symmetry $S_{BA} \circ BS_{BA}$. Therefore it is sensible to call the latter S_{B^2A}.

Definition. For any local algebras A and B and integer $k > 1$ put

$$S_{B^kA} = S_{BA} \circ BS_{B^{k-1}A}$$

with the understanding that $B^1 = B$.

Next we state formally a consequence of the fact that isomorphic vector bundles are "indistinguishable" within the category of C^∞ vector bundles and maps.

LEMMA 9. *If φ: $(E, M, p) \rightarrow (F, N, q)$ is a vector bundle isomorphism, and if C: $TF \rightarrow F$ is a connexion map, then $C^\varphi = \varphi^{-1} \circ C \circ T\varphi$: $TE \rightarrow E$ is a connexion map. In particular the curvature of C^φ is*

$$K^\varphi = \varphi^{-1} \circ K \circ T^2\varphi$$

when K is the curvature of C.

If C defines a linear connexion in the manifold M in the usual, but restricted sense that C is a connexion map for the tangent bundle (TM, M, π), then the prolongation C_A does *not* define a linear connexion in the prolongation manifold AM. That is to say, C_A is a connexion map for the bundle $(ATM, AM, A\pi)$ and not for (TAM, AM, π). As expected a small modification of C_A via the symmetries relating iterated prolongations does yield a linear connexion in AM.

THEOREM 2. *Suppose a linear connexion in the manifold M is given. Let C: $T^2M \rightarrow TM$ be the corresponding connexion map for the tangent bundle (TM, M, π). Then the map C^A: $T^2AM \rightarrow TAM$ defined as*

$$C^A = S_{AT} \circ C_A \circ TS_{TA} = S_{AT} \circ AC \circ S_{T^2A}$$

is a linear connexion in the manifold AM, in effect a connexion map for the tangent bundle (TAM, AM, π). C^A is a prolongation of the connexion C in the sense that

$$\begin{array}{ccc} T^2AM & \xrightarrow{C^A} & TAM \\ {\scriptstyle T^2O}\uparrow & & \uparrow{\scriptstyle TO} \\ T^2M & \xrightarrow[C]{} & TM \end{array}$$

is commutative.

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Remark. This theorem implies commutativity of the following diagrams, which further illustrate the sense in which C^A is a prolongation.

$$\begin{tikzcd}
T^2AM \arrow[rrr, "C^A"] \arrow[ddd, "\pi"'] & & & TAM \arrow[ddd, "\pi"] \\
& T^2M \arrow[ul, "T^2O"'] \arrow[r, "C"] \arrow[d, "\pi"'] & TM \arrow[ur, "TO"] \arrow[d, "\pi"] & \\
& TM \arrow[r, "\pi"'] \arrow[dl, "TO"] & M \arrow[dr, "O"] & \\
TAM \arrow[rrr, "\pi"'] & & & AM
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2AM \arrow[rrr, "C^A"] \arrow[ddd, "T\pi"'] & & & TAM \arrow[ddd, "\pi"] \\
& T^2M \arrow[ul, "T^2O"'] \arrow[r, "C"] \arrow[d, "T\pi"'] & TM \arrow[ur, "TO"] \arrow[d, "\pi"] & \\
& TM \arrow[r, "\pi"'] \arrow[dl, "TO"] & M \arrow[dr, "O"] & \\
TAM \arrow[rrr, "\pi"'] & & & AM
\end{tikzcd}$$

Proof of Theorem 2. C_A is a connexion map for the bundle $(ATM, AM, A\pi)$. Lemma 6 says that S_{TA} is a vector bundle isomorphism from (TAM, AM, π) to $(ATM, AM, A\pi)$. Hence Lemma 9 yields that $C^A = S_{AT} \circ C_A \circ TS_{TA}$ is a connexion map.

The following diagram is commutative by Lemma 5 and Theorem 1.

$$\begin{tikzcd}
T^2AM \arrow[rrr, "C^A"] & & & TAM \\
& TATM \arrow[ul, "TS_{AT}"'] \arrow[r, "C_A"] & ATM \arrow[ur, "S_{AT}"] & \\
T^2M \arrow[uu, "T^2O"] \arrow[ur, "TO"] \arrow[rrr, "C"] & & & TM \arrow[uu, "TO"'] \arrow[ul, "O"']
\end{tikzcd}$$

This completes the proof of Theorem 2.

THEOREM 3. *The curvature of the prolonged connexion C_A in the vector bundle (AE, AM, Ap) is*

$$K_A = AK \circ S_{T^2A}$$

where K is the curvature of the connexion map C.

Proof. From the definition of the curvature K (Part I, Theorem 2) it follows that

$$\begin{aligned}
AK \circ S_{T^2A} &= A(C \circ TC - C \circ TC \circ S) \circ S_{T^2A} \\
&= AC \circ ATC \circ S_{T^2A} -_{Ap} AC \circ ATC \circ AS \circ S_{T^2A} \quad \text{by Lemma 4} \\
&= AC \circ ATC \circ S_{TA} \circ TS_{TA} -_{Ap} AC \circ ATC \circ AS \circ S_{T^2A} \\
&= AC \circ S_{TA} \circ TAC \circ TS_{TA} -_{Ap} AC \circ S_{TA} \circ TAC \circ S_{AT} \circ AS \circ S_{T^2A} \\
&= C_A \circ TC_A -_{Ap} C_A \circ TC_A \circ S \\
&= K_A.
\end{aligned}$$

The next to the last step holds true because $S = S_{TT}$ and

$$TS_{TA} \circ S_{TT} = S_{AT} \circ AS \circ S_{T^2A}.$$

THEOREM 4. *If C is a linear connexion in the manifold M with curvature K then the prolonged linear connexion C^A in AM has curvature*

$$K^A = S_{AT} \circ AK \circ S_{T^3A}.$$

Its torsion is

$$\Theta^A = S_{AT} \circ A\Theta \circ TS_{TA}$$

when the torsion of C is Θ.

Proof. By Theorem 3 the curvature of the connexion C^A for the bundle $(ATM, AM, A\pi)$ will be $K_A = AK \circ S_{T^2A}$. By Lemma 9

$$\begin{aligned} K^A &= S_{AT} \circ K_A \circ T^2S_{TA} = S_{AT} \circ AK \circ S_{T^2A} \circ T^2S_{TA} \\ &= S_{AT} \circ AK \circ S_{TA} \circ TS_{TA} \circ T^2S_{TA} \\ &= S_{AT} \circ AK \circ S_{TA} \circ T(S_{TA} \circ TS_{TA}) = S_{AT} \circ AK \circ S_{TA} \circ T(S_{T^2A}) \\ &= S_{AT} \circ AK \circ S_{T^3A}. \end{aligned}$$

The torsion formula is really a consequence of the same reasoning which led to Lemma 9.

Remark. The theorems above show that corresponding to a given linear connexion in the manifold M there are two distinct prolongations of it to the tangent manifold TM. One is C_T which is not a linear connexion but nevertheless gives rise to parallel translation of second order tangent vectors along curves in TM because it is a connexion in the vector bundle $(T^2M, TM, T\pi)$. In fact it preserves the T-module structure of this bundle. The other is C^T which is indeed the usual linear "tangent" connexion [**4**]. These two connexions are of course isomorphic in the sense of Lemma 9, so that $C_T = S \circ C^T \circ TS$ where $S = S_{TT}$ is the symmetry of T^2M.

The Jacobi fields of the linear connexion C in M are the geodesics of the tangent connexion C^T in M [**3**]. A nice intrinsic proof of this fact can be made, using our characterization $C \circ \xi'' = 0 \circ \xi$ of geodesics, by regarding Jacobi fields as "infinitesmal variations" (= transverse vector fields generated by one parameter families of geodesics) and then applying the tangent functor T.

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