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KRONECKER COEFFICIENTS FOR (DUAL) SYMMETRIC INVERSE SEMIGROUPS

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Abstract

We study analogues of Kronecker coefficients for symmetric inverse semigroups, for dual symmetric inverse semigroups and for the inverse semigroups of bijections between subquotients of finite sets. In all cases, we reduce the problem of determination of such coefficients to some group-theoretic and combinatorial problems. For symmetric inverse semigroups, we provide an explicit formula in terms of the classical Kronecker and Littlewood–Richardson coefficients for symmetric groups.

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1. Introduction and description of the results

Kronecker and Littlewood–Richardson coefficients are numerical bookkeeping tools that play important roles in the representation theory of symmetric groups and in the theory of symmetric polynomials. While Littlewood–Richardson coefficients have a transparent combinatorial description, to find such a description for Kronecker coefficients is a famous open problem; see [Sa01] for more details.

Symmetric groups have a number of different analogues in semigroup theory. The most obvious one is the full transformation semigroup on a set, that is, the semigroup of all endomorphisms of a set. There are reasons to consider this latter semigroup as ‘too difficult’, for example, in contrast to the group algebra of a (finite) symmetric group over the complex numbers, the semigroup algebra of this semigroup over the complex numbers is not semi-simple; see [St16]. There are two, slightly less obvious but still natural, generalizations of the symmetric group that do have the property that the corresponding semigroup algebras over the complex numbers are semi-simple. These are the symmetric inverse and the dual symmetric inverse semigroups.

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The symmetric inverse semigroup on a set X is the semigroup of all bijections between subsets of X under a natural notion of composition of such bijections; see [GM09]. In the theory of inverse semigroups, it plays a role similar to that the symmetric group plays in group theory. It also appears naturally in other circumstances, for example, in the theory of FI-modules, that is, in the representation theory of the category of injections between finite sets; see [CEF15].

The dual symmetric inverse semigroup on a set X is the semigroup of all bijections between quotients of X under a natural notion of composition of such bijections; see [FL98]. There are certain categorical reasons to use the word dual in this setup, see [FL98, KMaz08a] for details, even if there is no obvious underlying duality. This semigroup also appears naturally in the theory of partition algebras, see [KMaz09b], and in the study of factor powers of symmetric groups; see [Maz09]. The symmetric inverse and the dual symmetric inverse semigroups are connected by a Schur–Weyl duality; see [KMaz09b]. In [J-Z22], one can find another transparent connection between the symmetric inverse and the dual symmetric inverse semigroup.

In this article, we investigate analogues of Kronecker coefficients for both the symmetric inverse semigroup and the dual symmetric inverse semigroup. In both cases, the problem we address strictly contains the original problem of determination of Kronecker coefficients for symmetric groups. This is because the symmetric group is the group of units (that is, invertible elements) both for the symmetric inverse and the dual symmetric inverse semigroups. Therefore, our ambition is not to provide a complete combinatorial solution, which seems to be too difficult, but rather to reduce the problem to some group-theoretic problem. One could argue that it is natural to consider a semigroup-theoretic problem 'solved' if it is reduced to a group-theoretic problem. Our interest in the Kronecker problem for symmetric inverse semigroups originates from our attempts to understand the techniques developed and used in [BV22] during our work on [MazSr24].

The result we obtain in the case of symmetric inverse semigroups is the most complete one. In this case, the formula for Kronecker coefficients looks as follows; see Theorem 3.1:

$$\mathbf{k}^{\nu}_{\lambda,\mu} = \sum_{\alpha\vdash a}\sum_{\beta\vdash b}\sum_{\gamma\vdash c}\sum_{\delta\vdash b}\sum_{\epsilon\vdash b}\sum_{\kappa\vdash a+b} \mathbf{c}^{\lambda}_{\alpha,\beta}\mathbf{c}^{\mu}_{\gamma,\delta}\mathbf{g}^{\epsilon}_{\beta,\delta}\mathbf{c}^{\kappa}_{\alpha,\epsilon}\mathbf{c}^{\nu}_{\kappa,\gamma}.$$

Here, $\alpha \vdash a$ means, as usual, that α is a partition of a, $\mathbf{g}^{\epsilon}_{\beta,\delta}$ denotes the classical Kronecker coefficient for symmetric groups, that is, the multiplicity of the Specht module $\mathbf{S}^{\epsilon}$ in the tensor product $\mathbf{S}^{\beta} \otimes_{\Bbbk} \mathbf{S}^{\delta}$, and $\mathbf{c}^{\nu}_{\kappa,\gamma}$ denotes the classical Littlewood–Richardson coefficients for symmetric groups, that is, the multiplicity of the module $\mathbf{S}^{\nu}$ in $\mathrm{Ind}^{\Bbbk[S_{|\kappa|+|\gamma|}]}_{\Bbbk[S_{|\kappa|}\times S_{|\gamma|}]}(\mathbf{S}^{\kappa} \otimes_{\Bbbk} \mathbf{S}^{\gamma})$. The case of symmetric inverse semigroups is considered and this formula is proved in Section 3. Along the way, we also show that the tensor product of cell modules for symmetric inverse semigroups has a cell filtration and determine the corresponding multiplicities; see Corollary 3.3.

In Section 4, we treat the case of the dual symmetric inverse semigroup. Here, the result we obtain, see Corollary 4.5, is much less explicit and relies on two inputs, one

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combinatorial and one group-theoretic. The combinatorial input is the set of certain rectangular 0/1-matrices, up to permutation of rows and columns. To each element of this set, we associate a group-theoretic problem in terms of induction and restriction between certain subgroups of symmetric groups; see Proposition 4.4. Our answer is given by adding up all solutions. There are a number of interesting problems that appear as special cases of our description; see Examples 4.6, 4.7, 4.8 and 4.9. One of them is related to partition algebras and is discussed in more detail in Section 6.

In Section 5, we generalize the results of Section 4 to the inverse semigroup PI_n^* of all bijections between subquotients of a set with n elements. Note that both I_n^* and PI_n^* appear as Schur–Weyl duals of the symmetric inverse semigroup; see [KMaz09b, Theorems 1 and 2]. The semigroup PI_n^* is, in some sense, a mixture between IS_n and I_n^*. Here, our results are very similar to results we obtain in Section 4. The main difference is that the combinatorial input for the solution becomes more involved and consists of 0/1-matrices with a fixed special column and a fixed special row.

Finally, in Section 6, we look in more detail into one specific situation that pops up in Section 4; see Example 4.6. The original problem here is to understand restriction coefficients for the restriction from S_{kl} to $S_k \times S_l$, where the latter is naturally embedded into the former as the group of independent permutations of rows and columns for a Young tableau of the shape (k^l) (or of the shape (l^k)). We use a Schur–Weyl duality to interpret these restriction coefficients as certain Kronecker-type coefficients for partition algebras (with different parameters). We use this interpretation to establish, in Theorem 6.6, a stability phenomenon for these coefficients (which is also obtained in [Ry21, Theorem 4.2] by completely different methods). In the final subsection, we show that, unlike the cases we considered in the previous sections, the tensor product of cell modules over partition algebras does not have a cell filtration, in general. However, it always has a filtration by standard modules; see Proposition 6.2.

2. Preliminaries

Throughout the paper, we fix an algebraically closed field $\Bbbk$ of characteristic zero. For $n \in \mathbb{Z}_{\geq 0}$, we denote by $\underline{n}$ the set $\{1, 2, \ldots, n\}$. Note that $\underline{0} := \varnothing$, the empty set.

2.1. Symmetric groups. For $n \geq 0$, consider the symmetric group S_n on $\underline{n}$. For $\lambda \vdash n$, denote by $\mathbf{S}^\lambda$ the *Specht* $\Bbbk[S_n]$*-module* corresponding to λ. We refer the reader to [Sa01, Section 2.3] for an explicit definition of Specht modules; we do not need it. We only recall that $\{\mathbf{S}^\lambda : \lambda \vdash n\}$ is a complete and irredundant list of representatives of the isomorphism classes of simple $\Bbbk[S_n]$-modules. For $\lambda, \mu, \nu \vdash n$, we denote by $\mathbf{g}_{\lambda,\mu}^{\nu}$ the corresponding *Kronecker coefficient*, that is, the multiplicity of $\mathbf{S}^\nu$ in $\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu$. The latter is a $\Bbbk[S_n]$-module via the diagonal action of S_n.

For $\lambda \vdash n$, $\mu \vdash m$ and $\nu \vdash n + m$, we denote by $\mathbf{c}_{\lambda,\mu}^{\nu}$ the corresponding *Littlewood–Richardson coefficient*, that is, the multiplicity of $\mathbf{S}^\nu$ in $\mathrm{Ind}_{\Bbbk[S_n \times S_m]}^{\Bbbk[S_{n+m}]}(\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu)$, where S_n is embedded into S_{n+m} using the canonical embedding of $\underline{n}$ into $\underline{n+m}$ while S_m is embedded into S_{n+m} via the $+n$-shift of the canonical embedding of $\underline{m}$ into $\underline{n+m}$.

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We refer to [Sa01] for all details related to symmetric groups and their representations.

2.2. Induction and bimodules. For $k + m = n$, consider the group $G = S_n \times S_k \times S_m$ and its subgroup H defined as the image of the embedding of $S_k \times S_m$ into $S_n \times S_k \times S_m$ constructed as follows: each $\sigma \in S_k$ is sent to $(\tilde{\sigma}, \sigma, e)$, where $\tilde{\cdot} : S_k \to S_n$ is the natural embedding of S_k into S_n with respect to the first k elements; furthermore, each $\sigma \in S_m$ is sent to $(\hat{\sigma}, e, \sigma)$, where $\hat{\cdot} : S_m \to S_n$ is the natural embedding of S_m into S_n with respect to the last m elements.

The group G acts naturally on G/H making $\Bbbk[G/H]$ into a G-module. We can view this G-module as a $\Bbbk[S_n]$-$\Bbbk[S_k \times S_m]$-bimodule using the canonical anti-involution $\sigma \mapsto \sigma^{-1}$ on $S_k \times S_m$. Since $\Bbbk[S_k \times S_m] \cong \Bbbk[S_k] \otimes_{\Bbbk} \Bbbk[S_m]$, given an S_k-module V and an S_m-module W, we have the S_n-module

$$\Bbbk[G/H] \bigotimes_{\Bbbk[S_k \times S_m]} (V \otimes_{\Bbbk} W). \tag{2-1}$$

It is easy to see that mapping H to e gives rise to an isomorphism between the $\Bbbk[S_n]$-$\Bbbk[S_k \times S_m]$-bimodule $\Bbbk[G/H]$ and the $\Bbbk[S_n]$-$\Bbbk[S_k \times S_m]$-bimodule $\Bbbk[S_n]$. Therefore, the S_n-module in (2-1) is isomorphic to $\mathrm{Ind}^{\Bbbk[S_n]}_{\Bbbk[S_k \times S_m]}(V \otimes_{\Bbbk} W)$.

2.3. Tensor product and bimodules. Consider the group $G = S_n \times S_n \times S_n$ and its subgroup H given by the diagonal embedding of S_n, that is, $\sigma \mapsto (\sigma, \sigma, \sigma)$. Then G acts naturally on G/H making $\Bbbk[G/H]$ into a G-module. We can view this G-module as a $\Bbbk[S_n]$-$\Bbbk[S_n \times S_n]$-bimodule using the canonical anti-involution $\sigma \mapsto \sigma^{-1}$ on $S_n \times S_n$ (the last two factors). As $\Bbbk[S_n \times S_n] \cong \Bbbk[S_n] \otimes_{\Bbbk} \Bbbk[S_n]$, given two S_n-modules V and W, we have the S_n-module

$$\Bbbk[G/H] \bigotimes_{\Bbbk[S_n \times S_n]} V \otimes_{\Bbbk} W. \tag{2-2}$$

It is easy to see that mapping H to (e, e) gives rise to an isomorphism between the $\Bbbk[S_n]$-$\Bbbk[S_n \times S_n]$-bimodule $\Bbbk[G/H]$ and the $\Bbbk[S_n]$-$\Bbbk[S_n \times S_n]$-bimodule $\Bbbk[S_n \times S_n]$, where the left action of S_n is diagonal. Therefore, the S_n-module in (2-2) is isomorphic to $V \otimes_{\Bbbk} W$ with the usual (diagonal) action of S_n.

3. Symmetric inverse semigroups

3.1. The semigroup IS_n. Consider the symmetric inverse semigroup IS_n of all bijections between subsets of $\underline{n}$. The semigroup operation in IS_n is a natural analogue of composition for partially defined maps. Here is an example for $n = 5$, where composition is from right to left and the elements are presented using the two-row notation for (partial) functions:

$$\begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}.$$

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For $\sigma \in IS_n$, we denote by $\mathrm{dom}(\sigma)$ and $\mathrm{cod}(\sigma)$ the *domain* and the *codomain* of σ, respectively. Recall, see for example [GM09, Section 4.4], the following description of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D} = \mathcal{J}$ for IS_n:

- $\sigma\mathcal{L}\pi$ if and only if $\mathrm{dom}(\sigma) = \mathrm{dom}(\pi)$;
- $\sigma\mathcal{R}\pi$ if and only if $\mathrm{cod}(\sigma) = \mathrm{cod}(\pi)$;
- $\sigma\mathcal{H}\pi$ if and only if $\mathrm{dom}(\sigma) = \mathrm{dom}(\pi)$ and $\mathrm{cod}(\sigma) = \mathrm{cod}(\pi)$;
- $\sigma\mathcal{J}\pi$ if and only if $|\mathrm{dom}(\sigma)| = |\mathrm{dom}(\pi)|$.

The cardinality of $\mathrm{dom}(\pi)$ is called the *rank* of π and denoted $\mathrm{rank}(\pi)$.

For a subset $X \subset \underline{n}$, we denote by ε_X the identity on X. Then $\{\varepsilon_X : X \subset \underline{n}\}$ is the set of all idempotents in IS_n. For $X, Y \subset \underline{n}$, we have $\varepsilon_X\varepsilon_Y = \varepsilon_{X\cap Y}$.

3.2. Cells and simple $\Bbbk[IS_n]$-modules. Let $\mathbf{L}$ be a left cell (that is, an $\mathcal{L}$-equivalence class) in IS_n. The linearization $\Bbbk[\mathbf{L}]$ is naturally a left $\Bbbk[IS_n]$-module with the action given as follows, for $\sigma \in IS_n$ and $\pi \in \mathbf{L}$:

$$\sigma \cdot \pi = \begin{cases} \sigma\pi & \text{if } \sigma\pi \in \mathbf{L}, \\ 0 & \text{otherwise.} \end{cases} \tag{3-1}$$

This is the *cell* $\Bbbk[IS_n]$*-module* corresponding to $\mathbf{L}$.

The set $\mathbf{L}$ contains a unique idempotent, namely ε_X, where X is the common domain for all elements in $\mathbf{L}$. Let $G(\mathbf{L})$ be the corresponding maximal subgroup in IS_n, that is, the $\mathcal{H}$-equivalence class of ε_X. Set $k := |X|$. Then $G(\mathbf{L}) = S_X \cong S_k$. We have the vector space decomposition

$$\Bbbk[\mathbf{L}] \cong \bigoplus_{Y\subset\underline{n},|Y|=k} \varepsilon_Y\Bbbk[\mathbf{L}]. \tag{3-2}$$

The group $G(\mathbf{L}) \cong S_k$ acts on $\mathbf{L}$ by multiplication on the right. This equips $\Bbbk[\mathbf{L}]$ with the natural structure of a $\Bbbk[IS_n]$-$\Bbbk[S_k]$-bimodule. In fact, each $\varepsilon_Y\Bbbk[\mathbf{L}]$ in (3-2) is a free right $\Bbbk[S_k]$-module of rank one. Up to isomorphism, the bimodule $\Bbbk[\mathbf{L}]$ does not depend on the choice of $\mathbf{L}$ inside its $\mathcal{J}$-cell (that is, for the fixed k).

For any $\lambda \vdash k$, the $\Bbbk[IS_n]$-module

$$\mathbf{N}^\lambda := \Bbbk[\mathbf{L}] \bigotimes_{\Bbbk[S_k]} \mathbf{S}^\lambda \tag{3-3}$$

is simple. By the above, up to isomorphism, it does not depend on the choice of $\mathbf{L}$. Moreover, up to isomorphism, each simple $\Bbbk[IS_n]$-module has such a form for some k and λ as above. Putting all this together, we see that the isomorphism classes of simple $\Bbbk[IS_n]$-modules are in bijection with partitions $\lambda \vdash k$, where $k \leq n$.

We refer to [GM09, Ch. 11] and [St16] for further details.

3.3. Kronecker coefficients for IS_n. Being a semigroup algebra, $\Bbbk[IS_n]$ has the canonical comultiplication given by the diagonal map $\sigma \mapsto \sigma \otimes \sigma$. Therefore, for two

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$\Bbbk[IS_n]$-modules M and N, the tensor product $M \otimes_{\Bbbk} N$ has the natural structure of a $\Bbbk[IS_n]$-module.

For a fixed n, let $k, l, m \leq n$, $\lambda \vdash k$, $\mu \vdash l$ and $\nu \vdash m$. We denote by $\mathbf{k}^{\nu}_{\lambda,\mu}$ the multiplicity of $\mathbf{N}^{\nu}$ in $\mathbf{N}^{\lambda} \otimes_{\Bbbk} \mathbf{N}^{\mu}$. It is natural to call these $\mathbf{k}^{\nu}_{\lambda,\mu}$ the *Kronecker coefficients for IS_n*. Our main result in this section is the following theorem.

THEOREM 3.1.

(a) *If $m < \max\{k, l\}$ or $m > k + l$, then $\mathbf{k}^{\nu}_{\lambda,\mu} = 0$.*

(b) *If $\max\{k, l\} \leq m \leq k + l$, set $b := k + l - m$, $a := k - b$ and $c := l - b$. Then*

$$\mathbf{k}^{\nu}_{\lambda,\mu} = \sum_{\alpha \vdash a} \sum_{\beta \vdash b} \sum_{\gamma \vdash c} \sum_{\delta \vdash b} \sum_{\epsilon \vdash b} \sum_{\kappa \vdash a+b} \mathbf{c}^{\lambda}_{\alpha,\beta} \mathbf{c}^{\mu}_{\gamma,\delta} \mathbf{g}^{\epsilon}_{\beta,\delta} \mathbf{c}^{\kappa}_{\alpha,\epsilon} \mathbf{c}^{\nu}_{\kappa,\gamma}. \tag{3-4}$$

We note that the Kronecker coefficients for IS_n appear in the theory of FI-modules, that is, modules over the category of finite sets and injections, and especially [CEF15, (17)], for details. For $\lambda = (1)$, the computation of $\mathbf{k}^{\nu}_{\lambda,\mu}$ can be obtained from [MS21, Corollary 2.7].

3.4. Tensor product of cell modules. Before we can prove Theorem 3.1, we need to consider a simpler decomposition problem, namely, that for the tensor product of cell modules. For $0 \leq r \leq n$, we denote by $\mathbf{L}_r$ the left cell in IS_n containing $\varepsilon_{\underline{r}}$.

Let k, l and m be as in Theorem 3.1. Consider the IS_n-module $\Bbbk[\mathbf{L}_k] \otimes_{\Bbbk} \Bbbk[\mathbf{L}_l]$. We show that this module has a filtration whose subquotients are isomorphic to cell modules and we also determine the multiplicity of $\Bbbk[\mathbf{L}_m]$ as a subquotient in that filtration. The fact that this multiplicity is well defined (that is, does not depend on the choice of a filtration with cell subquotients) is clear since nonisomorphic cell modules do not have any common nonzero simple summands.

The elements in $\mathbf{L}_k$ are given by pairs (X, σ), where X is a cardinality k subset of $\underline{n}$ and σ is a bijection from $\underline{k}$ to X. Similarly, the elements in $\mathbf{L}_l$ are given by pairs (Y, π), where Y is a cardinality l subset of $\underline{n}$ and π is a bijection from $\underline{l}$ to Y. For such a $\xi = (X, \sigma)$, the rank of $\varepsilon_{\underline{m}}\xi$ equals k if and only if $X \subset \underline{m}$. Similarly, for such an $\eta = (Y, \pi)$, the rank of $\varepsilon_{\underline{m}}\eta$ equals l if and only if $Y \subset \underline{m}$. Hence, $\varepsilon_{\underline{m}}$ does not kill $\xi \otimes \eta$ if and only if $X \subset \underline{m}$ and $Y \subset \underline{m}$ (in particular, $m \geq \max\{k, l\}$).

If we assume that $X \cup Y \subsetneq \underline{m}$, then $\xi \otimes \eta$ is not killed by the idempotent $\varepsilon_{X \cup Y}$ whose rank is strictly smaller than m. This means that such $\xi \otimes \eta$ cannot contribute to the cell module $\Bbbk[\mathbf{L}_m]$. Therefore, we only need to consider the case $X \cup Y = \underline{m}$ (which implies $\max\{k, l\} \leq m \leq k + l$). In this case, $|X \cap Y| = k + l - m = b$.

Denote by Q the set of all $\xi \otimes \eta$ as in the previous paragraph. The group S_m, identified as the maximal subgroup $G(\mathbf{L}_m)$, for the idempotent $\varepsilon_{\underline{m}}$, acts on $\xi \otimes \eta \in Q$ naturally by left multiplication. The group $S_k \times S_l$ acts on $\xi \otimes \eta \in Q$ on the right, where S_k acts on σ by right multiplication while S_l acts on π by right multiplication. The two actions obviously commute.

Let us determine the cardinality of Q. We can choose $X \cap Y$ in $\binom{m}{b}$ different ways and then the rest of X in $\binom{m-b}{a} = \binom{m-b}{c}$ different ways. As σ and π are arbitrary and

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$a+b+c=m$, there are

$$\binom{a+b+c}{b}\binom{a+c}{a}(a+b)!\,(b+c)! = \frac{(a+b+c)!\,(a+b)!\,(b+c)!}{a!\,b!\,c!}$$

different choices in total.

Now we can perform the following construction, similar to those presented in Sections 2.2 and 2.3. Consider the group $G = S_{a+b+c} \times S_{a+b} \times S_{c+b}$. We split the $a+b+c$ elements on which S_{a+b+c} acts into a apricot, b blue and c crimson elements. Similarly, we split the $a+b$ elements on which S_{a+b} acts into a apricot and b blue elements. Finally, we split the $b+c$ elements on which S_{b+c} acts into b blue and c crimson elements. Fix some identification of apricot elements for S_{a+b+c} with apricot elements for S_{a+b}, and similarly for blue and crimson elements.

Let H be the subgroup of G generated by the following elements:

- all (ρ, ρ, e), where ρ is a permutation of apricot elements;
- all (ρ, ρ, ρ), where ρ is a permutation of blue elements;
- all (ρ, e, ρ), where ρ is a permutation of crimson elements.

Clearly, H is isomorphic to $S_a \times S_b \times S_c$. Consider the linearization $\Bbbk[G/H]$, which is naturally a G-module. In particular, it is a $\Bbbk[S_{a+b+c}]$-$\Bbbk[S_{a+b} \times S_{c+b}]$-bimodule, using the canonical anti-involution $\sigma \mapsto \sigma^{-1}$ on $S_{a+b} \times S_{c+b}$. Now we can compare the $\Bbbk[S_{a+b+c}]$-$\Bbbk[S_{a+b} \times S_{c+b}]$-bimodules $\Bbbk[G/H]$ and $\Bbbk Q$.

LEMMA 3.2. *The $\Bbbk[S_{a+b+c}]$-$\Bbbk[S_{a+b} \times S_{c+b}]$-bimodules $\Bbbk[G/H]$ and $\Bbbk Q$ are isomorphic.*

PROOF. Consider $\eta = (\{a+1, \ldots, a+l\}, \tau)$, where τ is the order preserving bijection from $\underline{l}$ to $\{a+1, \ldots, a+l\}$. Then it is easy to see that the G-stabilizer of $\varepsilon_{\underline{k}} \otimes \eta \in Q$ equals H. Therefore, sending H to $\varepsilon_{\underline{k}} \otimes \eta$ gives rise to a homomorphism from $\Bbbk[G/H]$ to $\Bbbk Q$. This homomorphism is easily seen to be an isomorphism since the two spaces have the same dimension and the element $\varepsilon_{\underline{k}} \otimes \eta$ generates the space $\Bbbk Q$ as a $\Bbbk[S_{a+b+c}]$-$\Bbbk[S_{a+b} \times S_{c+b}]$-bimodule. □

From the definitions, it is easy to see that for any conjugate H' of H in G, we have $H' \cap S_{a+b+c} = e$. This implies that as an S_{a+b+c}-module, the module $\Bbbk[G/H]$ is free. By Lemma 3.2, we thus also have that $\Bbbk Q$ is free, as an S_{a+b+c}-module. In particular, Q is just a disjoint union of copies of regular S_m-orbits. Combining the definitions with decomposition (3-2), we have that each such regular S_m-orbit determines in $\Bbbk[\mathbf{L}_k] \otimes_{\Bbbk} \Bbbk[\mathbf{L}_l]$ a subquotient isomorphic to $\Bbbk[\mathbf{L}_m]$. This, on the one hand, implies that $\Bbbk[\mathbf{L}_k] \otimes_{\Bbbk} \Bbbk[\mathbf{L}_l]$ has a filtration whose subquotients are cell modules. On the other hand, it also says that the multiplicity of $\Bbbk[\mathbf{L}_m]$ in $\Bbbk[\mathbf{L}_k] \otimes_{\Bbbk} \Bbbk[\mathbf{L}_l]$ equals the number of S_m-orbits on Q. Summing everything up, we obtain the following corollary.

COROLLARY 3.3. *For $0 \le k, l \le n$, the module $\Bbbk[\mathbf{L}_k] \otimes_{\Bbbk} \Bbbk[\mathbf{L}_l]$ has a filtration whose subquotients are isomorphic to cell modules. Moreover, for $\max\{k, l\} \le m \le k+l$, the multiplicity of $\Bbbk[\mathbf{L}_m]$ in $\Bbbk[\mathbf{L}_k] \otimes_{\Bbbk} \Bbbk[\mathbf{L}_l]$ equals $(a+b)!\,(b+c)!/a!\,b!\,c!$, where $b = k+l-m$, $a = k-b$ and $c = l-b$.*

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3.5. Proof of Theorem 3.1. Claim (a) of Theorem 3.1 follows from the two estimates $m \geq \max\{k, l\}$ and $m \leq k + l$ obtained in the previous subsection.

In the previous subsection, we considered the bimodule $\Bbbk Q$ that describes the part of the tensor product $\Bbbk[\mathbf{L}_k] \otimes_{\Bbbk} \Bbbk[\mathbf{L}_l]$ that determines the $\Bbbk[\mathbf{L}_m]$-summands of that tensor product. In particular, combining the definition of Q with (3-3), it follows that

$$[\mathbf{N}^\lambda \otimes_{\Bbbk} \mathbf{N}^\mu : \mathbf{N}^\nu] = [\Bbbk Q \otimes_{\Bbbk[S_k] \otimes_{\Bbbk} \Bbbk[S_l]} (\mathbf{S}^\lambda \otimes_{\Bbbk} \mathbf{S}^\mu) : \mathbf{S}^\nu]. \tag{3-5}$$

Therefore, due to Lemma 3.2, to prove Claim (b), we need to show that (3-4) gives the multiplicity

$$[\Bbbk[G/H] \otimes_{\Bbbk[S_k] \otimes_{\Bbbk} \Bbbk[S_l]} (\mathbf{S}^\lambda \otimes_{\Bbbk} \mathbf{S}^\mu) : \mathbf{S}^\nu].$$

Define an additive functor $\mathcal{F}$ from $\Bbbk[S_{a+b} \times S_{b+c}]$-mod to $\Bbbk[S_{a+b+c}]$-mod for two modules $V \in S_{a+b}$-mod and $W \in S_{b+c}$-mod, as follows.

- First, restrict V to $S_a \times S_b$ and W to $S_b \times S_c$.
- Next, consider $V \otimes_{\Bbbk} W$ as an $S_a \times S_b \times S_c$-module with respect to the obvious actions of S_a and S_c on the first and on the second components, respectively, and the diagonal action of S_b.
- Finally, induce the obtained $S_a \times S_b \times S_c$-module up to S_{a+b+c}.

Let us now compute $\mathcal{F}(\mathbf{S}^\lambda \otimes_{\Bbbk} \mathbf{S}^\mu)$. To start with,

$$\mathrm{Res}^{\Bbbk[S_{a+b}]}_{\Bbbk[S_a \times S_b]} \mathbf{S}^\lambda \cong \bigoplus_{\alpha \vdash a} \bigoplus_{\beta \vdash b} (\mathbf{S}^\alpha \otimes_{\Bbbk} \mathbf{S}^\beta)^{\oplus \mathbf{c}^\lambda_{\alpha,\beta}}$$

and

$$\mathrm{Res}^{\Bbbk[S_{b+c}]}_{\Bbbk[S_b \times S_c]} \mathbf{S}^\mu \cong \bigoplus_{\gamma \vdash c} \bigoplus_{\delta \vdash b} (\mathbf{S}^\delta \otimes_{\Bbbk} \mathbf{S}^\gamma)^{\oplus \mathbf{c}^\mu_{\delta,\gamma}}.$$

At the next step,

$$(\mathbf{S}^\alpha \otimes_{\Bbbk} \mathbf{S}^\beta) \otimes_{\Bbbk} (\mathbf{S}^\delta \otimes_{\Bbbk} \mathbf{S}^\gamma) \cong \bigoplus_{\epsilon \vdash b} (\mathbf{S}^\alpha \otimes_{\Bbbk} \mathbf{S}^\epsilon \otimes_{\Bbbk} \mathbf{S}^\gamma)^{\oplus \mathbf{g}^\epsilon_{\beta,\delta}}.$$

Finally, at the last step, which we realize as the composition of first inducing from $S_a \times S_b \times S_c$ to $S_{a+b} \times S_c$ and then from the latter to S_{a+b+c},

$$\mathrm{Ind}^{S_{a+b+c}}_{S_a \times S_b \times S_c} \mathbf{S}^\alpha \otimes_{\Bbbk} \mathbf{S}^\epsilon \otimes_{\Bbbk} \mathbf{S}^\gamma \cong \bigoplus_{\nu \vdash a+b+c} \bigoplus_{\kappa \vdash a+b} (\mathbf{S}^\nu)^{\oplus \mathbf{c}^\kappa_{\alpha,\epsilon} \cdot \mathbf{c}^\nu_{\kappa,\gamma}}.$$

Collecting all these formulae together, we see that the multiplicity of $\mathbf{S}^\nu$ in $\mathcal{F}(\mathbf{S}^\lambda \otimes_{\Bbbk} \mathbf{S}^\mu)$ is given by (3-4).

Let us now try to understand the $\Bbbk[S_{a+b+c}]$-$\Bbbk[S_{a+b} \times S_{b+c}]$-bimodule that realizes $\mathcal{F}$. Taking into account the constructions presented in Sections 2.2 and 2.3, the combination of the first two steps is given by

$$\Bbbk[S_{a+b}] \otimes_{\Bbbk} \Bbbk[S_{b+c}],$$

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where the right action is regular, the left actions of S_a and S_c are the obvious ones, and the left action of S_b is the diagonal action. At the last step, we just tensor with the regular $\Bbbk[S_{a+b+c}]$-$\Bbbk[S_{a+b+c}]$-bimodule, where the right action is restricted to $S_a \times S_b \times S_c$.

Directly from the definitions, we see that the stabilizer (in G) of the element $e \otimes (e \otimes e)$ from

$$\Bbbk[S_{a+b+c}] \otimes_{\Bbbk[S_a \times S_b \times S_c]} (\Bbbk[S_{a+b}] \otimes_{\Bbbk} \Bbbk[S_{b+c}]) \tag{3-6}$$

contains H. Thus, sending the element H to the element $e \otimes (e \otimes e)$, gives rise to a $\Bbbk[S_{a+b+c}]$-$\Bbbk[S_{a+b} \times S_{b+c}]$-bimodule homomorphism from $\Bbbk[G/H]$ to the bimodule in (3-6). As $e \otimes (e \otimes e)$ generates the latter, this homomorphism is surjective. By comparing the dimensions, we see that it is bijective. This completes the proof of Claim (b) in Theorem 3.1.

3.6. Advance remark. We complete this section with a remark about symmetric inverse semigroups that is inspired by Example 4.6 below (this example is also studied in more detail in Section 6).

If we interpret the elements in IS_n as rook 0/1-matrices, then the Kronecker product of matrices defines a homomorphism from $IS_n \times IS_m$ to IS_{nm}. Note that this homomorphism is not injective (as tensoring anything with the zero matrix outputs the zero matrix). Nevertheless, it gives rise to a homomorphism of unital associative algebras $\Bbbk[IS_n] \otimes_{\Bbbk} \Bbbk[IS_m] \to \Bbbk[IS_{nm}]$. In particular, we have the corresponding pullback functor, which allows us to consider each IS_{nm}-module as an $IS_n \times IS_m$-module.

Cell modules over symmetric inverse semigroups (and their direct products) have a special feature: the cell module for a left cell $\mathbf{L}$ is the linearization of the obvious transitive action of our semigroup on $\mathbf{L}$ by partial transformations given by left multiplication; compare with (3-1). Classification of transitive acts of inverse semigroups is a classical part of the theory of inverse semigroups, see [La20, Theorem 5.4], and it is usually formulated in terms of closed inverse subsemigroups, which serve as stabilizers of points. Each such closed inverse subsemigroup has a distinguished idempotent and contains some subgroup H of the maximal subgroup corresponding to this idempotent. Abusing terminology, we can call this H the *apex* of the stabilizer. A left cell act corresponds exactly to the case when H is trivial, that is, consists only of the identity (that is, only of the corresponding idempotent).

If we now take a left cell IS_{nm}-act and pull it back to $IS_n \times IS_m$, we can consider a natural filtration of the latter act whose subquotients are transitive subquotient acts. The property that the apex of each point of the original act is trivial is clearly inherited under pullback. Consequently, the pullback of any cell IS_{nm}-module has a filtration whose subquotients are cell $IS_n \times IS_m$-modules. One can compare this with the results of Section 3.4.

4. Dual symmetric inverse semigroups

4.1. The semigroup I_n^*. Consider the dual symmetric inverse semigroup I_n^*, see [FL98]. The elements of I_n^* are all possible bijections between the quotients of $\underline{n}$ and the

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operation is the natural version of composition for such bijections. Here is an example for $n = 5$, where composition is from right to left and the elements are presented using the two-row notation for functions:

$$\begin{pmatrix} \{1\} & \{2,3\} & \{4,5\} \\ \{1,2\} & \{3,4\} & \{5\} \end{pmatrix} \circ \begin{pmatrix} \{1,2\} & \{3\} & \{4,5\} \\ \{1\} & \{2,3,4\} & \{5\} \end{pmatrix} = \begin{pmatrix} \{1,2\} & \{3,4,5\} \\ \{1,2\} & \{3,4,5\} \end{pmatrix}.$$

Each element ξ in I_n^* is thus uniquely given by the following data:

- a set partition ξ_d of $\underline{n}$, called the *domain* of ξ and denoted $\mathrm{dom}(\xi)$;
- a set partition ξ_r of $\underline{n}$ (with the same number of parts as ξ_d), called the *codomain* of ξ and denoted $\mathrm{cod}(\xi)$;
- a bijection $\overline{\xi}$ from $\underline{n}/\xi_d$ to $\underline{n}/\xi_r$.

The cardinality of $\underline{n}/\xi_d$ is called the *rank* of ξ and denoted $\mathrm{rank}(\xi)$. We also call the same number the *rank* of ξ_d.

Let $\xi, \zeta \in I_n^*$. Recall, see for example [FL98, Theorem 2.2], the following description of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D} = \mathcal{J}$ for I_n^*:

- $\xi \mathcal{L} \zeta$ if and only if $\mathrm{dom}(\xi) = \mathrm{dom}(\zeta)$;
- $\xi \mathcal{R} \zeta$ if and only if $\mathrm{cod}(\xi) = \mathrm{cod}(\zeta)$;
- $\xi \mathcal{H} \zeta$ if and only if $\mathrm{dom}(\xi) = \mathrm{dom}(\zeta)$ and $\mathrm{cod}(\xi) = \mathrm{cod}(\zeta)$;
- $\xi \mathcal{J} \zeta$ if and only if $\mathrm{rank}(\xi) = \mathrm{rank}(\zeta)$.

The idempotents of I_n^* are naturally identified with equivalence relations on $\underline{n}$: given an equivalence relation ρ on $\underline{n}$, the corresponding idempotent ε_ρ is the identity map on $\underline{n}/\rho$. The maximal subgroup of I_n^* corresponding to ρ is the symmetric group S_ρ on the equivalence classes of ρ.

The semigroup I_k^* appears naturally in the context of the Schur–Weyl duality for the kth tensor power of the natural representation of IS_n; see [KMaz09b, Theorem 1].

4.2. Cells and simple I_n^*-modules. Let ρ be an equivalence relation on $\underline{n}$. Let $\mathbf{L}_\rho$ be the left cell (that is, an $\mathcal{L}$-equivalence class) in I_n^* containing ε_ρ.

The linearization $\Bbbk[\mathbf{L}_\rho]$ is naturally a left I_n^*-module, where the action is given as follows for $\xi \in I_n^*$ and $\zeta \in \mathbf{L}_\rho$:

$$\xi \cdot \zeta = \begin{cases} \xi\zeta & \text{if } \xi\zeta \in \mathbf{L}_\rho, \\ 0 & \text{otherwise.} \end{cases}$$

This is the *cell I_n^*-module* corresponding to $\mathbf{L}_\rho$.

The maximal subgroup S_ρ acts on $\mathbf{L}_\rho$ by multiplication on the right. This equips $\Bbbk[\mathbf{L}_\rho]$ with the natural structure of a $\Bbbk[I_n^*]$-$\Bbbk[S_\rho]$-bimodule.

For any $\lambda \vdash \mathrm{rank}(\varepsilon_\rho)$, the $\Bbbk[I_n^*]$-module

$$\mathbf{N}^\lambda := \Bbbk[\mathbf{L}_\rho] \bigotimes_{\Bbbk[S_\rho]} \mathbf{S}^\lambda$$

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is simple. Up to isomorphism, it does not depend on the choice of $\mathbf{L}_\rho$ inside its $\mathcal{J}$-cell (that is, for the fixed rank of ε_ρ). Moreover, up to isomorphism, each simple $\Bbbk[I_n^*]$-module has such a form for some ρ and λ. We refer to [FL98, St16] for details.

4.3. Tensor product of cell modules. Let ρ and σ be two set partitions (also known as equivalence relations) of $\underline{n}$. The tensor product

$$\Bbbk[\mathbf{L}_\rho] \otimes_\Bbbk \Bbbk[\mathbf{L}_\sigma]$$

is an I_n^*-module via the diagonal action of I_n^*. In this subsection, we show that this module has a filtration whose subquotients are isomorphic to cell modules. Also, we describe a combinatorial object that determines the corresponding multiplicities (just like in the case of IS_n, the fact that these multiplicities are well defined follows directly from the semi-simplicity of the semigroup algebra of I_n^* combined with the fact that two nonisomorphic cell modules do not have common isomorphic simple summands).

Set $k = \operatorname{rank}(\varepsilon_\rho)$ and $l = \operatorname{rank}(\varepsilon_\sigma)$. Let $\tilde{\rho}$ be a set partition of $\underline{n}$ of rank k and $\tilde{\sigma}$ be a set partition of $\underline{n}$ of rank l. Consider the set partition τ of $\underline{n}$ whose parts are exactly the nonempty intersections of a part of $\tilde{\rho}$ with a part of $\tilde{\sigma}$. We denote this τ by $\tilde{\rho} \cap \tilde{\sigma}$.

EXAMPLE 4.1. If $n = 4$, $\tilde{\rho} = \{1,2\} \cup \{3,4\}$ and $\tilde{\sigma} = \{1,2,3\} \cup \{4\}$, then we have $\tau = \{1,2\} \cup \{3\} \cup \{4\}$.

Let $\preceq$ be the natural order on set partitions: we have $\alpha \preceq \beta$ if each part of α is a subset of some part of β. Let $\xi \in \mathbf{L}_\rho$ and $\zeta \in \mathbf{L}_\sigma$ be such that $\operatorname{cod}(\xi) = \tilde{\rho}$ and $\operatorname{cod}(\zeta) = \tilde{\sigma}$. Then τ is the maximum element in the set of all set partitions ω of $\underline{n}$ such that $\varepsilon_\omega \xi \in \mathbf{L}_\rho$ and $\varepsilon_\omega \zeta \in \mathbf{L}_\sigma$. Note also that the action of S_τ on $\xi \otimes \zeta$ is, clearly, free.

All this means exactly that the element $\xi \otimes \zeta$ contributes to a cell subquotient $\Bbbk[\mathbf{L}_\tau]$ of $\Bbbk[\mathbf{L}_\rho] \otimes_\Bbbk \Bbbk[\mathbf{L}_\sigma]$.

To provide a combinatorial object for counting the multiplicities, we need to invert our problem. Given some τ, we need to find the number of all possible $\tilde{\rho}$ and $\tilde{\sigma}$ as above that are connected to τ also in the above sense.

Set $m := \operatorname{rank}(\varepsilon_\tau)$. Let $Q_{k,l}^m$ be the set of all $k \times l$ matrices with $0/1$-entries satisfying the following conditions:

- exactly m entries are equal to 1;
- all rows are nonzero;
- all columns are nonzero.

The group $G := S_k \times S_l$ acts on $Q_{k,l}^m$ by (independent) permutations of rows and columns. Consider the corresponding orbit set $Q_{k,l}^m/G$, and set $\mathbf{p}_{k,l}^m := |Q_{k,l}^m|$ and $\mathbf{q}_{k,l}^m := |Q_{k,l}^m/G|$. Note that for $\mathbf{p}_{k,l}^m$ or $\mathbf{q}_{k,l}^m$ to be nonzero, we must have the inequalities $\max\{k,l\} \le m \le kl$. The cardinalities of stabilizers in G of elements of a fixed orbit are equal (as these stabilizers are conjugate) and, for an orbit $O \in Q_{k,l}^m/G$, we denote this cardinality by $|\mathrm{stab}_O|$.

PROPOSITION 4.2. *The I_n^*-module $\Bbbk[\mathbf{L}_\rho] \otimes_\Bbbk \Bbbk[\mathbf{L}_\sigma]$ has a filtration whose subquotients are isomorphic to cell modules. Moreover, the multiplicity of $\Bbbk[\mathbf{L}_\tau]$ in $\Bbbk[\mathbf{L}_\rho] \otimes_\Bbbk \Bbbk[\mathbf{L}_\sigma]$*

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equals

$$\mathbf{p}_{k,l}^{m} = \sum_{i=0}^{k}\sum_{j=0}^{l}(-1)^{k+l-i-j}\binom{k}{i}\binom{l}{j}\binom{ij}{m}. \tag{4-1}$$

PROOF. Similarly to Section 3.4, the existence of the filtration follows from the fact that the action of S_τ on the elements of the form $\xi\otimes\zeta$ described above is free.

To compute the multiplicity, for a fixed τ, we need to count the number of the elements of the form $\xi\otimes\zeta$ as above, with $\mathrm{cod}(\xi)=\tilde{\rho}$ and $\mathrm{cod}(\zeta)=\tilde{\sigma}$, for which $\tilde{\rho}\cap\tilde{\sigma}=\tau$.

Given $M\in Q_{k,l}^{m}$, there are $m!$ bijections between the parts of τ and the '1'-entries in M. If we fix such a bijection ψ, we can read off the parts of $\tilde{\rho}$ by uniting the parts of τ along the rows of M. Also, we can read off the parts of $\tilde{\sigma}$ by uniting the parts of τ along the columns of M. This determines the codomains of ξ and ζ, respectively (note that there is no fixed order on the parts of these codomains, which explains the equivalence with respect to the action of G). As ρ and σ are fixed, to determine ξ and ζ, it remains to choose $\overline{\xi}$, in $k!$ different ways, and $\overline{\zeta}$, in $l!$ different ways.

Now let $\mathcal{Y}(M,\psi)$ be the set of all $\xi\otimes\zeta$ described above. Let $\mathcal{Y}(M)$ be the union of $\mathcal{Y}(M,\psi)$ over all bijections ψ between the parts of τ and the '1'-entries of M. If M and N belong to the same G-orbit O, then $\mathcal{Y}(M)=\mathcal{Y}(N)$; we denote this set by $\mathcal{Y}_O$. The cardinality of $\mathcal{Y}_O$ is $m!\cdot k!\cdot l!/|\mathrm{stab}_O|$. Let $\mathcal{Y}$ be union of $\mathcal{Y}_O$ over the orbits O in $Q_{k,l}^{m}/G$. Note that the set $\mathcal{Y}$ consists of $\xi\otimes\zeta$ with $\mathrm{cod}(\xi)\cap\mathrm{cod}(\zeta)=\tau$ with $\xi\in\mathbf{L}_\rho$ and $\zeta\in\mathbf{L}_\sigma$, and it has the cardinality:

$$m!\cdot\sum_{O\in Q_{k,l}^{m}/G}\frac{k!\cdot l!}{|\mathrm{stab}_O|}.$$

Since the action of S_τ on $\mathcal{Y}$ is free, dividing by $|S_\tau|=m!$ gives

$$\sum_{O\in Q_{k,l}^{m}/G}\frac{k!\cdot l!}{|\mathrm{stab}_O|}=\mathbf{p}_{k,l}^{m}.$$

The right-hand side of (4-1) is obtained using the inclusion-exclusion principle with respect to the requirement that all rows and all columns of elements in $Q_{k,l}^{m}$ should be nonzero. □

It would be interesting to have a combinatorial formula for $\mathbf{q}_{k,l}^{m}$. Although it is easy to determine $\mathbf{q}_{k,l}^{m}$ in some special cases (for instance, $\mathbf{q}_{k,l}^{kl}=1$ and $\mathbf{q}_{k,l}^{\max\{k,l\}}$ is the number of partitions of $\max\{k,l\}$ with exactly $\min\{k,l\}$ parts), we do not know the answer in general.

EXAMPLE 4.3. Consider the case $k=l=m=n$. In this case, $Q_{n,n}^{n}$ is just the set of all permutation matrices and hence, $Q_{n,n}^{n}/G$ is a singleton (as any permutation matrix can be reduced to the identity matrix by permutations of rows or columns). This means that we have $\mathbf{q}_{n,n}^{n}=1$, and, moreover, $\mathbf{p}_{n,n}^{n}=n!$ is exactly the multiplicity of $\Bbbk[S_n]$ in $\Bbbk[S_n]\otimes_{\Bbbk}\Bbbk[S_n]$.

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In general it is easy to see that if $\mathbf{q}^m_{k,l} = 1$, then the multiplicity of $\Bbbk[\mathbf{L}_\tau]$ in $\Bbbk[\mathbf{L}_\rho] \otimes \Bbbk[\mathbf{L}_\sigma]$ is the cardinality of the set $Q^m_{k,l}$.

4.4. Tensor product of simple modules. The results of Section 4.3 suggest what should be done to understand tensor products of simple I^*_n-modules. This seems to be a rather difficult problem in the general case. Below we outline how to apply Section 4.3 to this problem in more detail.

For $0 \le k, l, m \le n$, consider the corresponding set $Q^m_{k,l}$. Fix:

- a set partition ρ of $\underline{n}$ with exactly k parts;
- a set partition σ of $\underline{n}$ with exactly l parts;
- a set partition τ of $\underline{n}$ with exactly m parts.

Recall the set $\mathcal{Y}$ from the proof of Proposition 4.2. Note that for $\xi \otimes \zeta \in \mathcal{Y}$, we have $\varepsilon_\tau(\xi \otimes \zeta) = \xi \otimes \zeta$ and, in fact, ε_τ is the maximum (with respect to $\preceq$) element in the set of all idempotents in I^*_n that do not annihilate $\xi \otimes \zeta$. Also, note that $\Bbbk\mathcal{Y}$ has the natural structure of a $\Bbbk[S_m]$-$\Bbbk[S_k \times S_l]$-bimodule, where the group S_m acts on the left by permuting the parts of τ, the group S_k acts on the right by permuting the parts of ρ and the group S_l acts on the right by permuting the parts of σ.

Similarly to (3-5), for $\lambda \vdash k$, $\mu \vdash l$ and $\nu \vdash m$,

$$[\mathbf{N}^\lambda \otimes_\Bbbk \mathbf{N}^\mu : \mathbf{N}^\nu] = [\Bbbk\mathcal{Y} \otimes_{\Bbbk[S_k \times S_l]} (\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu) : \mathbf{S}^\nu].$$

By construction, the bimodule $\Bbbk\mathcal{Y}$ splits into a direct sum of subbimodules given by varying M from the previous paragraph inside a fixed $S_k \times S_l$-orbit in $Q^m_{k,l}$. For a fixed such orbit O, we denote by $\Bbbk\mathcal{Y}_O$ the corresponding direct summand of $\Bbbk\mathcal{Y}$. Therefore, the real problem is to determine the multiplicities

$$[\Bbbk\mathcal{Y}_O \otimes_{\Bbbk[S_k \times S_l]} (\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu) : \mathbf{S}^\nu]$$

for each $O \in Q^m_{k,l}/_{S_k \times S_l}$. Below we give a more group-theoretic reformulation of this problem and discuss various examples of different orbits. An explicit solution in all cases seems to be quite difficult.

In the above construction, assume that $O = (S_k \times S_l) \cdot M$ and let H be the stabilizer of M in $S_k \times S_l$. Consider the set A of all possible bijections between the elements of $\underline{m}$ and the '1'-entries in M. Then each element of H gives rise to a permutation of the '1'-entries in M, defining a homomorphism from H to S_m. Moreover, a permutation in S_m corresponding to a nontrivial element of H is, obviously, nontrivial. Therefore, this homomorphism is injective and thus we can identify H as a subgroup of S_m.

PROPOSITION 4.4. *We have*

$$[\Bbbk\mathcal{Y}_O \otimes_{\Bbbk[S_k \times S_l]} (\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu) : \mathbf{S}^\nu] = [\mathrm{Ind}^{S_m}_H \mathrm{Res}^{S_k \times S_l}_H (\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu) : \mathbf{S}^\nu].$$

PROOF. For $G = S_k \times S_l$, the $\Bbbk[S_m]$-$\Bbbk[G]$-bimodule corresponding to the functor $\mathrm{Ind}^{S_m}_H \mathrm{Res}^G_H$ is given by

$$B := \Bbbk[S_m] \otimes_{\Bbbk[H]} \Bbbk[G].$$

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Note that the dimension of this bimodule is $|S_m||G|/|H|$ and that this coincides with the dimension of $\Bbbk\mathcal{Y}_O$. From our definition of H, it follows that the right action of H on $\mathcal{Y}_O$ (that is, the action in terms of G) coincides with the left action of H on $\mathcal{Y}_O$ (that is, the action in terms of S_m).

Therefore, sending $e \otimes e \in B$ to the element in $\mathcal{Y}_O$ corresponding to (M, ψ) gives rise to a homomorphism of $\Bbbk[S_m]$-$\Bbbk[G]$-bimodules from B to $\Bbbk\mathcal{Y}_O$. Since the action of $S_m \times G$ on $\mathcal{Y}_O$ is transitive, this homomorphism is surjective. By comparing the dimensions, we obtain that this homomorphism is an isomorphism. □

As an immediate corollary of Propositions 4.2 and 4.4, we have the following corollary.

COROLLARY 4.5. *The multiplicity of* $\mathbf{N}^\nu$ *in* $\mathbf{N}^\lambda \otimes_\Bbbk \mathbf{N}^\mu$ *is given by the sum of the multiplicities in Proposition 4.4, taken along some cross-section of* $Q^m_{k,l}/_{S_k \times S_l}$.

EXAMPLE 4.6. Consider the extreme case $m = kl$. In this case, $G = H$ is a subgroup of S_m and hence we are interested in the multiplicities

$$[\mathrm{Ind}^{S_m}_{S_k \times S_l}(\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu) : \mathbf{S}^\nu].$$

Note that these are not the Littlewood–Richardson coefficients as our $S_k \times S_l$ is not a Young subgroup of S_m. This example appears in [Ry21] and is studied in more detail in Section 6. The extreme example of this case is $k = 1$, when we get $H = G = S_m$.

EXAMPLE 4.7. Consider the other extreme case $m = l > k$. Then we can assume that the i th row of M has exactly x_i entries equal to 1 and $x_1 \geq x_2 \geq \cdots \geq x_k$ with $x_1 + x_2 + \cdots + x_k = l$. In other words, we have a partition $\mathbf{x}$ of l with exactly k parts. Note that each column of M has exactly one entry equal to 1, since $m = l$. Assume that exactly y_1 parts of this partition are equal to $z_1 = x_1$, then exactly y_2 parts are equal to the next size z_2 of the parts in $\mathbf{x}$ and so on. Then the group H is isomorphic to the direct product $(S_{y_1} \wr S_{z_1}) \times (S_{y_2} \wr S_{z_2}) \times \cdots$ of wreath products of symmetric groups, (recall that $S_a \wr S_b$ can be defined as the subgroup of S_{ab} consisting of all elements σ such that for each $i \in \{0, 1, \ldots, b-1\}$, there is $j \in \{0, 1, \ldots, b-1\}$ having the property that for each $s \in \{1, 2, \ldots, a\}$, there is $t \in \{1, 2, \ldots, a\}$ such that $\sigma(ia + s) = ja + t$; the group $S_a \wr S_b$ has cardinality $b! \cdot (a!)^b$; see [JK, Section 4.1]). The extreme example of this case is again $k = 1$, when we get $H = G = S_m$.

EXAMPLE 4.8. Consider the case $m = (l-1) + (k-1)$ and take M such that the first row of M is $(0, 1, 1, \ldots, 1)$ and the first column of M is $(0, 1, 1, \ldots, 1)^t$. Then the remaining entries of M are zero. In this case, $H \cong S_{k-1} \times S_{l-1}$ is both a Young subgroup of G and a Young subgroup of S_m. Let us denote by $\to$ the usual branching relation on partitions (that is, $\xi \to \eta$ means that ξ is obtained from η by removing a removable node in the Young diagram). Then the multiplicities described in Proposition 4.4 are given by

$$\sum_{\alpha \to \lambda} \sum_{\beta \to \mu} \mathbf{c}^\nu_{\alpha,\beta}.$$

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EXAMPLE 4.9. Consider the case $l = k$, $m = l(l+1)/2$ and take M such that it is upper triangular. In this case, H is the trivial group (a singleton). Then the multiplicities described in Proposition 4.4 are given by the formula $\dim(\mathbf{S}^\lambda) \cdot \dim(\mathbf{S}^\mu) \cdot \dim(\mathbf{S}^\nu)$.

5. The inverse semigroup of all bijections between subquotients of a finite set

5.1. The semigroup PI_n^*. Consider the semigroup PI_n^* defined in [KMal11, Section 2.2]. The elements of PI_n^* are all possible bijections between quotients of subsets of $\underline{n}$ (equivalently, between subsets of quotients of $\underline{n}$) and the operation is the natural version of composition for such bijections. Here is an example for $n = 8$, where composition is from right to left and the elements are presented using the two-row notation for functions:

$$\begin{pmatrix} \{1\} & \{3,5\} & \{6,7\} & \{8\} \\ \{1,2\} & \{3,4\} & \{5\} & \{6,8\} \end{pmatrix} \circ \begin{pmatrix} \{2,3\} & \{4\} & \{5,6\} & \{7,8\} \\ \{1,2\} & \{3\} & \{5\} & \{6,7,8\} \end{pmatrix} = \begin{pmatrix} \{4,5,6\} & \{7,8\} \\ \{3,4\} & \{5,6,8\} \end{pmatrix}.$$

The main point in this multiplication is that as soon some element x in the product hits an undefined part of the other element, all elements connected to x become undefined. In the above example, this happens for $x = 2$ in the right factor. We do refer the reader to [KMal11, Section 2.2] for the full definition which is very technical. The semigroup PI_n^* is, in some sense, a mixture between IS_n and I_n^*.

Each element ξ in PI_n^* is thus uniquely given by the following data:

- a set partition ξ_d of some subset X of $\underline{n}$, called the *domain* of ξ;
- a set partition ξ_r of some subset Y of $\underline{n}$ (with the same number of parts as ξ_d), called the *codomain* of ξ;
- a bijection $\bar{\xi}$ from X/ξ_d to Y/ξ_r.

The cardinality of X/ξ_d is called the *rank* of ξ and denoted $\mathrm{rank}(\xi)$. We also call the same number the *rank* of ξ_d.

Let $\xi, \zeta \in PI_n^*$. Recall, see for example [KMal11, Proposition 3], the following description of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D} = \mathcal{J}$ for PI_n^*:

- $\xi \mathcal{L} \zeta$ if and only if $\mathrm{dom}(\xi) = \mathrm{dom}(\zeta)$;
- $\xi \mathcal{R} \zeta$ if and only if $\mathrm{cod}(\xi) = \mathrm{cod}(\zeta)$;
- $\xi \mathcal{H} \zeta$ if and only if $\mathrm{dom}(\xi) = \mathrm{dom}(\zeta)$ and $\mathrm{cod}(\xi) = \mathrm{cod}(\zeta)$;
- $\xi \mathcal{J} \zeta$ if and only if $\mathrm{rank}(\xi) = \mathrm{rank}(\zeta)$.

The idempotents of PI_n^* are naturally identified with equivalence relations on subsets of $\underline{n}$: given an equivalence relation ρ on some subset X of $\underline{n}$, the corresponding idempotent ε_ρ is the identity map on X/ρ. The maximal subgroup of PI_n^* corresponding to ρ is the symmetric group S_ρ on the equivalence classes of ρ.

The semigroup PI_k^* appears in the Schur–Weyl duality for the kth tensor power of the direct sum of the natural and the trivial representations of IS_n; see [KMaz09b, Theorem 2].

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5.2. Cells and simple PI_n^*-modules. Let ρ be an equivalence relation on some subset X of $\underline{n}$. Let $\mathbf{L}_\rho$ be a left cell (that is, an $\mathcal{L}$-equivalence class) in PI_n^* containing ε_ρ.

The linearization $\mathbb{k}[\mathbf{L}_\rho]$ is naturally a left PI_n^*-module with the action given as follows, for $\xi \in PI_n^*$ and $\zeta \in \mathbf{L}_\rho$:

$$\xi \cdot \zeta = \begin{cases} \xi\zeta & \text{if } \xi\zeta \in \mathbf{L}_\rho, \\ 0 & \text{otherwise.} \end{cases}$$

This is the cell PI_n^*-module corresponding to $\mathbf{L}_\rho$.

The maximal subgroup S_ρ acts on $\mathbf{L}_\rho$ by multiplication on the right. This equips $\mathbb{k}[\mathbf{L}_\rho]$ with the natural structure of a $\mathbb{k}[PI_n^*]$-$\mathbb{k}[S_\rho]$-bimodule.

For any $\lambda \vdash \operatorname{rank}(\varepsilon_\rho)$, the $\mathbb{k}[PI_n^*]$-module

$$\mathbf{N}^\lambda := \mathbb{k}[\mathbf{L}_\rho] \bigotimes_{\mathbb{k}[S_\rho]} \mathbf{S}^\lambda$$

is simple. Up to isomorphism, it does not depend on the choice of $\mathbf{L}_\rho$ inside its $\mathcal{J}$-cell (that is, for the fixed rank of ε_ρ). Moreover, up to isomorphism, each simple $\mathbb{k}[PI_n^*]$-module has such a form for some ρ and λ. We refer to [St16] for details.

5.3. Tensor product of cell modules. Let ρ and σ be two set partitions (also known as equivalence relations) of two subsets X and Y of $\underline{n}$, respectively. The tensor product

$$\mathbb{k}[\mathbf{L}_\rho] \otimes_{\mathbb{k}} \mathbb{k}[\mathbf{L}_\sigma]$$

is a PI_n^*-module via the diagonal action of PI_n^*. In this subsection, we show that this module has a filtration whose subquotients are isomorphic to cell modules. Also, we describe a combinatorial object that determines the corresponding multiplicities (just like in the cases of IS_n and I_n^*, the fact that these multiplicities are well defined follows directly from the semi-simplicity of the semigroup algebra of PI_n^*).

Set $k = \operatorname{rank}(\varepsilon_\rho)$, $l = \operatorname{rank}(\varepsilon_\sigma)$. Let U be a subset of $\underline{n}$ of cardinality at least k, and $\tilde{\rho}$ a set partition of U of the same rank as ρ. Let V be a subset of $\underline{n}$ of cardinality at least l, and $\tilde{\sigma}$ a set partition of V of the same rank as σ. Consider the set partition τ of $U \cup V$ whose parts are exactly the following sets:

- all nonempty intersections of a part of $\tilde{\rho}$ with a part of $\tilde{\sigma}$;
- all nonempty intersections of a part of $\tilde{\rho}$ with $\underline{n} \setminus V$;
- all nonempty intersections of a part of $\tilde{\sigma}$ with $\underline{n} \setminus U$.

Let $\xi \in \mathbf{L}_\rho$ and $\zeta \in \mathbf{L}_\sigma$ be such that $\operatorname{cod}(\xi) = \tilde{\rho}$ and $\operatorname{cod}(\zeta) = \tilde{\sigma}$. Then τ is the maximum element in the set of all set partitions ω of $U \cup V$ such that $\varepsilon_\omega\xi \in \mathbf{L}_\rho$ and $\varepsilon_\omega\zeta \in \mathbf{L}_\sigma$. Note also that the action of S_τ on $\xi \otimes \zeta$ is, clearly, free. All this means exactly that the element $\xi \otimes \zeta$ contributes to a cell subquotient $\mathbb{k}[\mathbf{L}_\tau]$ of $\mathbb{k}[\mathbf{L}_\rho] \otimes_{\mathbb{k}} \mathbb{k}[\mathbf{L}_\sigma]$.

To provide a combinatorial object for counting the multiplicities, we need to invert our problem. Given some τ, we need to find the number of all possible $\tilde{\rho}$ and $\tilde{\sigma}$ as above, which are connected to τ also in the above sense.

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Set $m := \mathrm{rank}(\varepsilon_\tau)$. Let $\tilde{Q}^m_{k,l}$ be the set of all 0/1-matrices whose rows are indexed by the elements in $\{0, 1, 2, \dots, k\}$ and whose columns are indexes by the elements in $\{0, 1, 2, \dots, l\}$ satisfying the following conditions:

- the $(0, 0)$-entry is 0;
- exactly m entries are equal to 1;
- all rows with positive indices are nonzero;
- all columns with positive indices are nonzero.

The group $G := S_k \times S_l$ acts on $\tilde{Q}^m_{k,l}$ by (independent) permutation of rows and columns with positive indices. Consider the corresponding orbit set $\tilde{Q}^m_{k,l}/G$, and set $\tilde{\mathbf{p}}^m_{k,l} := |\tilde{Q}^m_{k,l}|$ and $\tilde{\mathbf{q}}^m_{k,l} := |\tilde{Q}^m_{k,l}/G|$. Note that for $\tilde{\mathbf{q}}^m_{k,l}$ to be nonzero, we must have the inequality $\max\{k, l\} \leq m \leq (k + 1)(l + 1) - 1$. It would be interesting to have a combinatorial formula for $\tilde{\mathbf{q}}^m_{k,l}$.

PROPOSITION 5.1. *The module $\Bbbk[\mathbf{L}_\rho] \otimes_\Bbbk \Bbbk[\mathbf{L}_\sigma]$ has a filtration whose subquotients are isomorphic to cell modules. Moreover, the multiplicity of $\Bbbk[\mathbf{L}_\tau]$ in $\Bbbk[\mathbf{L}_\rho] \otimes_\Bbbk \Bbbk[\mathbf{L}_\sigma]$ equals*

$$\tilde{\mathbf{p}}^m_{k,l} = \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{k+l-i-j} \binom{k}{i} \binom{l}{j} \binom{(i + 1)(j + 1) - 1}{m}.$$

PROOF. *Mutatis mutandis* the proof of Proposition 4.2. □

EXAMPLE 5.2. Consider the case $k = m = l = n$. In this case, $\tilde{Q}^n_{n,n}$ is just the set of all permutation matrices on positive indices (and both the 0 th row and the 0 th column are zero). Hence, $\tilde{Q}^n_{n,n}/G$ is a singleton and thus $\tilde{\mathbf{q}}^n_{n,n} = 1$. We also have $\tilde{\mathbf{p}}^n_{n,n} = n!$, which is exactly the multiplicity of $\Bbbk[S_n]$ in $\Bbbk[S_n] \otimes_\Bbbk \Bbbk[S_n]$.

5.4. Tensor product of simple modules. Similarly to the case of the dual symmetric inverse monoid, the results of Section 5.3 suggest what should be done to understand tensor products of simple PI^*_n-modules. This seems to be a rather difficult problem. Below we outline how to apply Section 5.3 to this problem in more detail.

Let ρ be a set partition of $X \subset \underline{n}$ with k parts, and σ a set partition of $Y \subset \underline{n}$ with l parts. Let τ be the set partition of $X \cup Y$ given by intersecting the parts of ρ with the parts of σ and, additionally, the parts of ρ with $\underline{n} \setminus Y$ and the parts of σ with $\underline{n} \setminus X$. Then we have $\varepsilon_\tau(\varepsilon_\rho \otimes \varepsilon_\sigma) = \varepsilon_\rho \otimes \varepsilon_\sigma$ and, in fact, ε_τ is the maximum (with respect to $\preceq$) element in the set of all idempotents in PI^*_n that do not annihilate $\varepsilon_\rho \otimes \varepsilon_\sigma$. Assume that τ has exactly m parts.

Fix some linear orderings for the parts of ρ and for the parts of σ. Then we have the matrix $M \in \tilde{Q}^m_{k,l}$ defined as follows:

- for positive i and j, the (i, j) th entry of M is 1 if and only if the intersection of the i th part of ρ with the j th part of σ is not empty;
- the $(0, j)$ th entry of M is 1 if and only if the intersection of the j th part of σ with $\underline{n} \setminus X$ is not empty;

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- the $(i, 0)$ th entry of M is 1 if and only if the intersection of the i th part of ρ with $\underline{n} \setminus Y$ is not empty.

Consider the G-orbit $G \cdot M$ of M inside $\tilde{Q}^m_{k,l}$ and denote by $\mathbf{M}$ the set of all possible bijective decorations of the '1'-entries of the elements in $G \cdot M$ by the elements of $\underline{m}$. Clearly, both S_m and G act on $\mathbf{M}$ and these two actions commute. Hence, the linearization $\Bbbk[\mathbf{M}]$ of $\mathbf{M}$ becomes an S_m-G-bimodule.

Let $\lambda \vdash k$, $\mu \vdash l$ and $\nu \vdash m$. To understand tensor products of simple PI^*_n-modules, Proposition 5.1 (more precisely, it's proof based on the proof of Proposition 4.2) implies that we need to understand the multiplicity of $\mathbf{S}^\nu$ in the module

$$\Bbbk[\mathbf{M}] \otimes_{\Bbbk[G]} (\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu).$$

Let H be the stabilizer of M in G. Consider the set A of all possible bijections between the elements of $\underline{m}$ and the '1'-entries in M. Then each element of H gives rise to a permutation of the '1'-entries in M, defining a homomorphism from H to S_m. Moreover, the permutation in S_m corresponding to a nontrivial element of H is, obviously, nontrivial. Therefore, this homomorphism is injective and thus we can canonically identify H as a subgroup of S_m.

PROPOSITION 5.3. *We have*

$$[\Bbbk[\mathbf{M}] \otimes_{\Bbbk[G]} (\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu) : \mathbf{S}^\nu] = [\mathrm{Ind}^{S_m}_H \mathrm{Res}^G_H (\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{S}^\mu) : \mathbf{S}^\nu].$$

PROOF. *Mutatis mutandis* the proof of Proposition 4.4. □

COROLLARY 5.4. *The multiplicity of $\mathbf{N}^\nu$ in $\mathbf{N}^\lambda \otimes_\Bbbk \mathbf{N}^\mu$ is given by the sum of the multiplicities in Proposition 5.3, taken along some cross-section of $\tilde{Q}^m_{k,l}/G$.*

EXAMPLE 5.5. Consider the case where $m = k + l$ and M is the matrix in which all '1'-entries are either in the 0 th row or in the 0 th column. In this case, $G = H$ is a Young subgroup of S_m and hence the multiplicities in Proposition 5.3 we are interested in are exactly the Littlewood–Richardson coefficients $\mathbf{c}^\nu_{\lambda,\mu}$.

EXAMPLE 5.6. Consider the case $k = l$ and $m = 3k$, where M is the matrix in which all '1'-entries are either in the 0 th row or in the 0 th column or on the main diagonal. In this case, H is isomorphic to S_k and is diagonally embedded both into G and into the Young subgroup of S_m isomorphic to $S_k \times S_k \times S_k$. Since these embeddings are diagonal, the restriction from G to H and the induction from H to $S_k \times S_k \times S_k$ are computed using Kronecker coefficients. After that, the induction from $S_k \times S_k \times S_k$ to S_m is computed using Littlewood–Richardson coefficients. Consequently, the multiplicities in Proposition 5.3 are given by the following expression:

$$\sum_{\alpha \vdash k} \sum_{\beta \vdash k} \sum_{\gamma \vdash k} \sum_{\delta \vdash k} \sum_{\epsilon \vdash k} \sum_{\kappa \vdash 2k} \mathbf{c}^\nu_{\epsilon,\kappa} \mathbf{c}^\kappa_{\gamma,\delta} \mathbf{g}^\beta_{\delta,\epsilon} \mathbf{g}^\alpha_{\beta,\gamma} \mathbf{g}^\alpha_{\lambda,\mu}.$$

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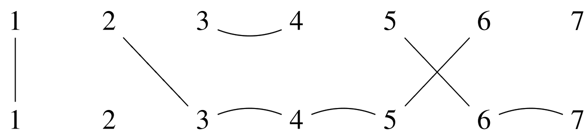
6. Branching related to $S_k \times S_l$ as a subgroup of S_{kl}

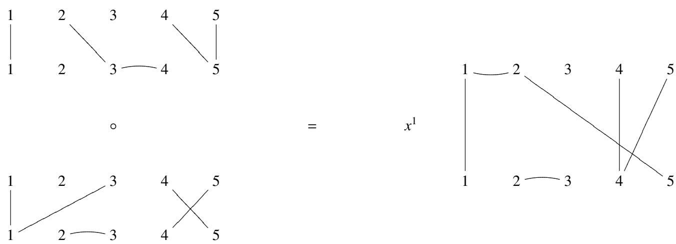
6.1. Setup. In this section, we look at Example 4.6 in more detail. Let $m = kl$ and let M be the $k \times l$ matrix in which all entries are equal to 1. We fix the bijection between $\underline{m}$ and the entries of this matrix that reads the entries left-to-right and then top-to-bottom. The group S_k acts by permuting the rows of M, which identifies it as a subgroup of S_m. The group S_l acts by permuting the columns of M, which identifies it as a subgroup of S_m. Put together, we have identified $S_k \times S_l$ as a subgroup of S_{kl}.

For $\lambda \vdash k$, $\mu \vdash l$ and $\nu \vdash kl$, denote by $\mathbf{b}^{\nu}_{\lambda,\mu}$ the multiplicity of $\mathbf{S}^{\lambda} \otimes_{\Bbbk} \mathbf{S}^{\mu}$ in the restriction of $\mathbf{S}^{\nu}$ from S_{kl} to $S_k \times S_l$.

6.2. Partition algebra. For $n \geq 0$ and $x \in \Bbbk \setminus \{0\}$, consider the corresponding partition algebra $\mathcal{P}_n(x)$, as defined in [Jo, Mar94]. It is a $\Bbbk$-algebra that has a basis consisting of set partitions of two (the upper and the lower) copies of $\underline{n}$. Such set partitions are usually represented by certain diagrams (partition diagrams) whose connected components, viewed as graphs, represent the parts of the partition, for example:



Note that such a diagram representing a partition is not unique. Algebra multiplication is defined in terms of concatenation of such diagrams followed by a certain straightening procedure that results in a new diagram and a nonnegative integer r (the number of parts removed during the straightening procedure). The product is then this new diagram times x^r. Here is an example:



In this example, the part $\{2\}$ in the bottom of the top diagram and the part $\{2\}$ in the top of the bottom diagram together form a unique part that is removed in the straightening procedure. We refer the reader to [HR05] for details.

Given a partition diagram $\mathbf{d}$, the parts that intersect both copies of $\underline{n}$ are called *propagating lines* and the number of such parts is called *rank*. This rank is an integer between 0 and n. For $0 \leq i \leq n$, all partition diagrams of rank at most i span a two-sided ideal in $\mathcal{P}_n(x)$, which we denote by I_i. Given a simple $\mathcal{P}_n(x)$-module L, there is a minimal i such that I_i does not belong to the annihilator of L. Simple $\mathcal{P}_n(x)$-modules with this property are in a natural bijection with partitions of i and are constructed as follows.

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Let $\mathbf{d}$ be a partition diagram of rank i. Consider the set $\mathbf{L_d}$ of all partition diagrams of rank i that can be obtained from $\mathbf{d}$ by left concatenation with other diagrams. Then the linearization $\Bbbk\mathbf{L_d}$ is naturally a $\mathcal{P}_n(x)$-$\Bbbk[S_i]$-bimodule, which is free as a $\Bbbk[S_i]$-module. For $\lambda \vdash i$, the module

$$\mathbf{C}^\lambda := \Bbbk\mathbf{L_d} \bigotimes_{\Bbbk[S_i]} \mathbf{S}^\lambda$$

has a simple top that we denote by $\mathbf{N}^\lambda$. The set

$$\{\mathbf{N}^\lambda \ : \ \lambda \vdash i,\ 0 \le i \le n\}$$

is a full and irredundant set of representatives of isomorphism classes of simple $\mathcal{P}_n(x)$-modules.

We also note that $\mathbf{C}^\lambda \cong \mathbf{N}^\lambda$ provided that $x \notin \{0, 1, 2, \dots, 2n-2\}$ since in this case, the partition algebra is semi-simple; see [MarS94] or [HR05, Theorem 3.27]. Note that for $x \in \{0, 1, 2, \dots, 2n-2\}$, the isomorphism $\mathbf{C}^\lambda \cong \mathbf{N}^\lambda$ might still hold for some λ. In particular, under the assumption $x \notin \{0, 1, 2, \dots, 2n\}$, the module $\mathbf{N}^\lambda$ has a basis that does not depend on x and in which the action of each partition diagram is given by a matrix in which all coefficients are polynomials in x.

We call the modules $\mathbf{C}^\lambda$ *standard*. A *standard filtration* of some $\mathcal{P}_n(x)$-module is a filtration whose subquotients are standard modules. This terminology reflects the fact that the algebra $\mathcal{P}_n(x)$ is quasi-hereditary (here our assumption $x \neq 0$ is important).

6.3. Schur–Weyl dualities for partition algebras. Consider the natural S_k-module $\Bbbk^k$ and its n-fold tensor power $(\Bbbk^k)^{\otimes n}$. It is an S_k-module via the diagonal action of S_k. The algebra $\mathcal{P}_n(k)$ acts naturally on $(\Bbbk^k)^{\otimes n}$, see for example, [HR05, (3.2)], this action commutes with the action of S_k and, moreover, the two commuting actions generate each other's centralizers. This is known as the *Schur–Weyl duality* for partition algebras; see [HR05, Section 3]. As a $\Bbbk[S_k]$-$\mathcal{P}_n(k)$ bimodule, the space $(\Bbbk^k)^{\otimes n}$ decomposes into a direct sum of simple bimodules indexed by those partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of k for which $k - \lambda_1 \le n$. The corresponding simple $\Bbbk[S_k]$-$\mathcal{P}_n(k)$ bimodule is then $\mathbf{S}^\lambda \otimes_\Bbbk \mathbf{N}^{\lambda_{>1}}$, where $\lambda_{>1}$ is the partition of $k - \lambda_1$ obtained from λ by deleting the first part λ_1.

We note that neither the action of $\Bbbk[S_k]$ nor the action of $\mathcal{P}_n(k)$ on $(\Bbbk^k)^{\otimes n}$ is faithful, in general.

Now let $m = kl$. Then we have the natural S_k-module $\Bbbk^k$ and the natural S_l-module $\Bbbk^l$. Their tensor product $\Bbbk^k \otimes_\Bbbk \Bbbk^l$ is isomorphic to the natural S_m-module $\Bbbk^m$. For any $n \ge 0$, we also have the obvious isomorphism

$$(\Bbbk^k \otimes_\Bbbk \Bbbk^l)^{\otimes n} \cong (\Bbbk^k)^{\otimes n} \otimes_\Bbbk (\Bbbk^l)^{\otimes n}.$$

By the Schur–Weyl duality, the S_{kl}-endomorphisms of the left-hand side are given by the image of $\mathcal{P}_n(kl)$. Similarly, the $S_k \times S_l$-endomorphisms of the right-hand side are given by the image of $\mathcal{P}_n(k) \otimes_\Bbbk \mathcal{P}_n(l)$. This suggests that our original problem of restriction from S_{kl} to $S_k \times S_l$ should be interpretable, via the Schur–Weyl duality, in

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terms of some induction problem for partition algebras. This is what we investigate in the next subsections.

6.4. Generalized comultiplication for partition algebras. We do not know of any interesting comultiplication for partition algebras. However, the following proposition describes a generalized version of such comultiplication.

PROPOSITION 6.1. *Let $n \in \mathbb{Z}_{>0}$ and $a, b \in \Bbbk$. Then, mapping a partition diagram* $\mathbf{d}$ *to* $\mathbf{d} \otimes \mathbf{d}$ *defines an algebra homomorphism from $\mathcal{P}_n(ab)$ to $\mathcal{P}_n(a) \otimes_{\Bbbk} \mathcal{P}_n(b)$.*

PROOF. Let $\mathbf{d}$ and $\mathbf{d}'$ be two partition diagrams. To compute their product, we have to compose them and then apply the straightening procedure during which r parts are removed and that outputs a new partition diagram $\mathbf{d}''$. Neither r nor $\mathbf{d}''$ depend on the parameter of the partition algebra.

Hence, for the algebra $\mathcal{P}_n(ab)$, the product of $\mathbf{d}$ and $\mathbf{d}'$ equals $(ab)^r\mathbf{d}''$. At the same time, for the algebra $\mathcal{P}_n(a) \otimes_{\Bbbk} \mathcal{P}_n(b)$, the product of $\mathbf{d} \otimes \mathbf{d}$ and $\mathbf{d}' \otimes \mathbf{d}'$ equals the element $((a^r)\mathbf{d}'') \otimes ((b^r)\mathbf{d}'') = (ab)^r\mathbf{d}'' \otimes \mathbf{d}''$. The claim follows. □

6.5. Partition algebra and $\mathbf{b}^{\nu}_{\lambda,\mu}$. Proposition 6.1 implies that, given a $\mathcal{P}_n(a)$-module X and a $\mathcal{P}_n(b)$-module Y, the tensor product $X \otimes_{\Bbbk} Y$ is, naturally, a $\mathcal{P}_n(ab)$-module.

PROPOSITION 6.2. *Assume that the $\mathcal{P}_n(a)$-module X has a standard filtration and that the $\mathcal{P}_n(b)$-module Y has a standard filtration. Then the $\mathcal{P}_n(ab)$-module $X \otimes_{\Bbbk} Y$ has a standard filtration as well.*

PROOF. By additivity, it is enough to prove the claim under the assumption that both X and Y are cell modules.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two left cells such that $X = \Bbbk[\mathbf{L}_1]$ and $X = \Bbbk[\mathbf{L}_2]$. Then X has the canonical basis consisting of all partition diagrams in $\mathbf{L}_1$ and Y has the canonical basis consisting of all partition diagrams in $\mathbf{L}_2$. Therefore, $X \otimes_{\Bbbk} Y$ has the canonical basis consisting of all elements of the form $\mathbf{d}_1 \otimes \mathbf{d}_2$, where $\mathbf{d}_i \in \mathbf{L}_i$ for $i = 1, 2$.

Consider the oriented graph Γ whose vertices are all these elements in the basis of $X \otimes_{\Bbbk} Y$. For two vertices $\mathbf{d}_1 \otimes \mathbf{d}_2$ and $\mathbf{d}'_1 \otimes \mathbf{d}'_2$, put an oriented edge from $\mathbf{d}_1 \otimes \mathbf{d}_2$ to $\mathbf{d}'_1 \otimes \mathbf{d}'_2$ provided that there is a partition diagram $\tilde{\mathbf{d}}$ which, when applied to $\mathbf{d}_1 \otimes \mathbf{d}_2$, outputs $\mathbf{d}'_1 \otimes \mathbf{d}'_2$, up to a nonzero scalar. Note that Γ does not depend on $x \in \{a, b\}$, since we assume $x \neq 0$. Also note that all vertices have loops as we can choose $\tilde{\mathbf{d}}$ to be the identity partition diagram.

Since Γ is finite, we can choose a filtration of Γ of the form

$$\varnothing = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$$

by nonempty full subgraphs such that the following two conditions are satisfied, for each i:

- if α is an edge from u to v and $u \in \Gamma_i$, then $v \in \Gamma_i$;
- the full subgraph of Γ_i whose vertices are all vertices outside Γ_{i-1} is strongly connected.

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Then the linearization $\Bbbk[\Gamma_i]$ of the vertex set of each Γ_i is a submodule of $X \otimes_{\Bbbk} Y$, so we have the induced filtration of $X \otimes_{\Bbbk} Y$ by submodules

$$0 \subset \Bbbk[\Gamma_1] \subset \Bbbk[\Gamma_2] \subset \cdots \subset X \otimes_{\Bbbk} Y$$

and this filtration, by construction, does not depend on x.

We claim that each $\mathcal{P}_n(ab)$-module $N_i := \Bbbk[\Gamma_i]/\Bbbk[\Gamma_{i-1}]$ has a filtration by standard modules. Let j be the minimal rank for which there is a partition diagram that does not annihilate $\Bbbk[\Gamma_i]/\Bbbk[\Gamma_{i-1}]$. Choose some idempotent partition diagram $\tilde{\mathbf{d}}$ of rank j and fix some $\mathbf{d}_1 \otimes \mathbf{d}_2 \in \Gamma_i \setminus \Gamma_{i-1}$ that is fixed by $\tilde{\mathbf{d}}$ (this exists as $\tilde{\mathbf{d}}$ is an idempotent and it does not annihilate N_i). Applying the left cell of $\tilde{\mathbf{d}}$ to $\mathbf{d}_1 \otimes \mathbf{d}_2$ gives a set of vertices that is invariant under the action of all partition diagrams (modulo Γ_{i-1}). Hence, this must coincide with the whole of $\Gamma_i \setminus \Gamma_{i-1}$ due to the strong connectivity of the latter. In reality, we just proved that N_i is a quotient of $\Bbbk[\mathbf{L}_{\tilde{\mathbf{d}}}]$ since we have a natural surjective homomorphism from the latter to the former that sends $\tilde{\mathbf{d}}' \in \mathbf{L}_{\tilde{\mathbf{d}}}$ to $\tilde{\mathbf{d}}' \cdot (\mathbf{d}_1 \otimes \mathbf{d}_2)$.

Let G be the $\mathcal{H}$-class of $\tilde{\mathbf{d}}$. Then $\mathbf{L}_{\tilde{\mathbf{d}}}$ is a free right G-act. Let H be the stabilizer of $\mathbf{d}_1 \otimes \mathbf{d}_2$ in G. Then the above gives a surjection from the module

$$\Bbbk[\mathbf{L}_{\tilde{\mathbf{d}}}] \otimes_{\Bbbk[G]} \Bbbk[G/H] \tag{6-1}$$

to N_i. Note that the module (6-1) has a filtration by standard modules by construction.

Everything above is independent of x, as long as $x \neq 0$. For generic x, standard modules are simple and hence the surjection in the previous paragraph is, in fact, an isomorphism between the module (6-1) and N_i. However, then the fact that this surjection is an isomorphism does not depend on x, completing the proof. □

REMARK 6.3. Analogues of Proposition 6.2 are true, with the same proof, for other classical diagram subalgebras of the partition algebra, in particular, for the Brauer algebra, the partial Brauer algebra, the Temperley–Lieb algebra and so forth.

REMARK 6.4. For partition algebras, natural analogues of the Littlewood–Richardson coefficient, that is, coefficients that describe multiplicities of standard modules in the induced external tensor product of standard modules, are given by the reduced Kronecker coefficient. This can be found in [BDO15, Theorem 4.3] and in [BV22, Theorem 3.11].

From the theory of quasi-hereditary algebras, we know that the multiplicity of a standard subquotient in a standard filtration of some module that admits a standard filtration does not depend on the choice of such a filtration.

Let us now go back to the setup as described in Section 6.1.

COROLLARY 6.5. *Let $\lambda \vdash k$, $\mu \vdash l$ and $\nu \vdash kl$. Then, for*

$$n \geq \max(kl - \nu_1, k - \lambda_1, l - \mu_1),$$

the number $\mathbf{b}^{\nu}_{\lambda,\mu}$ coincides with the multiplicity of the $\mathcal{P}_n(kl)$-module $\mathbf{N}^{\nu_{>1}}$ in the tensor product of the $\mathcal{P}_n(k)$-module $\mathbf{N}^{\lambda_{>1}}$ and the $\mathcal{P}_n(l)$-module $\mathbf{N}^{\mu_{>1}}$.

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PROOF. By adjunction, the restriction from S_{kl} to $S_k \times S_l$ on the left-hand side of the module $(\Bbbk^k \otimes_{\Bbbk} \Bbbk^l)^{\otimes n}$ corresponds to induction from the tensor product of $\mathcal{P}_n(k)$ and $\mathcal{P}_n(l)$ to $\mathcal{P}_n(kl)$ on the right-hand side. Because of our assumption that $n \geq \max(kl - \nu_1, k - \lambda_1, l - \mu_1)$, we can apply the bimodule decomposition in the Schur–Weyl duality. As the connection from the left- to the right-hand side of the Schur–Weyl duality maps a partition γ to $\gamma_{>1}$, the claim follows. □

6.6. Stability phenomenon for $\mathbf{b}^{\gamma}_{\mu,\nu}$. For a partition $\lambda \vdash n$ and a positive integer a, we denote by $\lambda^{(a)}$ the partition of $n + a$ obtained from λ by increasing λ_1 to $\lambda_1 + a$. The following claim was also proved in [Ry21, Theorem 4.2] by completely different arguments.

THEOREM 6.6. *Let $\lambda \vdash k$, $\mu \vdash l$ and $\nu \vdash kl$. Then, for all $a \gg 0$, the value $\mathbf{b}^{\nu^{(la)}}_{\lambda^{(a)},\mu}$ is constant.*

PROOF. Let $n \geq \max(kl - \nu_1, k - \lambda_1, l - \mu_1)$ and note that

$$\max(kl - \nu_1, k - \lambda_1, l - \mu_1) = \max((kl + la) - \nu^{(la)}_1, (k + a) - \lambda^{(a)}_1, l - \nu_1)$$

for all $a > 0$. This allows us to use the interpretation of $\mathbf{b}^{\nu^{(la)}}_{\lambda^{(a)},\mu}$ given by Corollary 6.5. We have $\nu^{(la)}_{>1} = \nu_{>1}$ and $\lambda^{(a)}_{>1} = \lambda_{>1}$ for all $a > 0$. In particular, on the partition algebra side, all involved modules have exactly the same corresponding indices, regardless of $a > 0$.

Next we note that for all $a \gg 0$, both algebras $\mathcal{P}_n((k + a)l)$ and $\mathcal{P}_n(k + a)$ are semi-simple; moreover, each simple module $\mathbf{N}^{\gamma}$ for each of these algebras coincides with the corresponding $\mathbf{C}^{\gamma}$.

The multiplicity of a simple module over a semi-simple finite dimensional algebra can be computed as the rank of a certain primitive idempotent. Generic primitive idempotents for partition algebras were constructed in [MW99]. Outside of a finite number of values of the parameter, each such idempotent is given as a linear combination of diagrams whose coefficients are rational functions in the parameter of the partition algebra. We can take $a \gg 0$ such that the construction of these primitive idempotents applies (meaning that the denominators are nonzero). If we fix such a primitive idempotent for $\mathcal{P}_n((k + a)l)$, then the action of this idempotent on the tensor product of some $\mathcal{P}_n(k + a)$-module $\mathbf{C}^{\gamma}$ (in which we fix the standard diagram basis) with a fixed $\mathcal{P}_n(l)$-module is given by a matrix whose coefficients are rational functions in a. Hence, this matrix has maximal rank for all but finitely many values of a. The claim follows. □

6.7. Tensor product of cell modules for partition algebras. Unlike the cases of the (dual) symmetric inverse semigroups, the tensor product of cell modules for partition algebras does not have to have a filtration whose subquotients are cell modules. In this subsection, we give an explicit example illustrating how this property fails already for $n = 2$.

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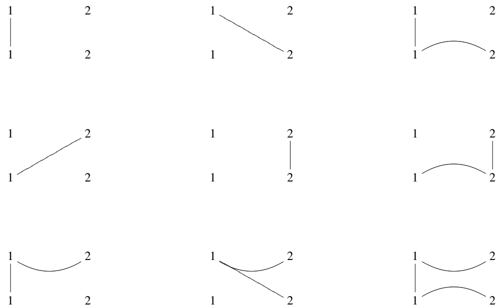
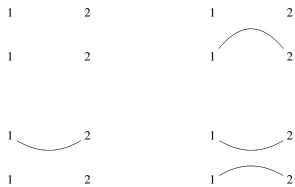
The algebra $\mathcal{P}_2(x)$ has dimension 15 (the bell number $\mathbf{B}_4$). As usual, we assume $x \neq 0$. The possible ranks for partition diagrams are 0, 1 and 2. We have one left cell $\mathcal{L}_2$ of rank two, consisting of the following two diagrams:

We have three left cells $\mathcal{L}_1^{(i)}$, where $i = 1, 2, 3$, of rank one, each containing three elements, given by the following rows:

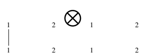
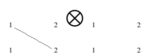
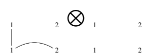


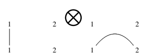
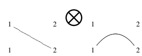
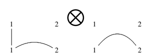
Finally, we have two left cells $\mathcal{L}_0^{(i)}$, where $i = 1, 2$, of rank zero, each containing two elements, given by the following rows:



In particular, we have, up to isomorphism, three cell modules $\Bbbk[\mathcal{L}_2]$, $\Bbbk[\mathcal{L}_1^{(1)}]$ and $\Bbbk[\mathcal{L}_0^{(1)}]$, of respective dimensions 2, 3 and 2.

Now we claim that $\Bbbk[\mathcal{L}_1^{(1)}] \otimes_{\Bbbk} \Bbbk[\mathcal{L}_0^{(1)}]$ does not have a cell filtration. Indeed, this module has dimension six with the following basis:

$v_1 :=$  $v_2 :=$  $v_3 :=$ 

$w_1 :=$  $w_2 :=$  $w_3 :=$ 

It is easy to check that v_1, v_2 and w_3 span in $\Bbbk[\mathcal{L}_1^{(1)}] \otimes_{\Bbbk} \Bbbk[\mathcal{L}_0^{(1)}]$ a submodule isomorphic to $\Bbbk[\mathcal{L}_1^{(1)}]$. In the quotient, w_1 and w_2 span a submodule isomorphic to $\Bbbk[\mathcal{L}_2]$, while v_3 only spans a submodule of $\Bbbk[\mathcal{L}_2]$, as the latter has dimension two and we only have one vector left.

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