THREE PLANE SEXTICS AND THEIR AUTOMORPHISMS

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1. The sextics of the title have five cusps and are particular examples of the curve encountered by Humbert (**5**). The claim to notoriety of Humbert's curve of genus 5 has been that all its abelian integrals of the first kind are linear combinations of five elliptic integrals; it also has (**1**) the striking property that its 120 Weierstrassian points are confluent in threes at only 40 distinct points; whereas a general curve of genus 5 has, after Riemann, 12 moduli, a Humbert curve has merely 2. The three curves to be studied now have no free moduli at all, but although this tempts one to construct period matrices, such transcendental topics will not be handled here. That the quadrics which contain the canonical model C of a Humbert sextic have a common self-polar simplex Σ so that C is invariant for an elementary abelian group E of 16 self-projectivities, seems to have been unremarked until 1951. Thus it is all the more intriguing that, within a year of the printing of Humbert's paper and in utter unawareness of it (still less of any connection with its subject matter) Wiman gave (**6**) the canonical models of the three sextics which we are now to consider. The groups of self-projectivities of Wiman's curves have orders not merely 16 but 64, 96, 160, respectively; it was the quest for canonical curves of genus 5 that admitted self-projectivities, and the more the merrier, that conducted Wiman to his goal.

By far the greater part of the paper is devoted to the Humbert sextics that are projections of Wiman's curves from one of their tangents; but three matters concerning the general curve C invariant under E are treated, knowledge of these being necessary for handling the special cases.

(a) A canonical form for C, never previously exploited if indeed noticed at all, is given in (2.1).

(b) In §§ 6 and 7 we assemble the "machinery" for projecting C from a tangent into a 5-cusped plane sextic. This yields, by a completely different process, some of the equations found in (**1**); but the algebra developed here is perhaps better adapted to handle Wiman's curves.

(c) In §§ 11 and 12 we describe how each of the five harmonic perspectivities under which C is invariant implies a Cremona transformation of the plane sextic into itself, the homaloids being nodal cubics; and how equations for this transformation can be found.

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In the Appendix we describe a plane sextic with five nodes that admits a group of 96 Cremona self-transformations and whose equation is symmetric in the coordinates.

2. Any canonical curve of genus 5 which does not carry a rational pencil of either duads or triads of points is the complete intersection of three quadrics in [4] (that is, in projective 4-space). It has been pointed out in (**1**) that the quadrics which so determine the canonical model C of the plane sextic $\mathscr{H}$ encountered by Humbert in 1894 (**5**) have a common self-polar simplex Σ; C is therefore invariant under an elementary abelian group E of 16 projectivities whose operations answer to changes in sign of coordinates referred to Σ. $\mathscr{H}$ admits an isomorphic group of 16 birational self-transformations which, as will be seen, are subordinate to Cremona transformations of its plane. One can, as has recently been remarked (**2**), span the net N of quadrics containing C by

$$\sum x_j^2 = 0, \qquad \sum a_j x_j^2 = 0, \qquad \sum a_j^2 x_j^2 = 0, \tag{2.1}$$

where, here and hereafter, the summations run over $j = 1, 2, 3, 4, 5$. Since one assumes that no cone of N has a plane for vertex, no two of the five a_j are equal. The reason why (2.1) is available as a standard form for C is, simply, that if C were defined by

$$\sum \alpha_j x_j^2 = \sum \beta_j x_j^2 = \sum \gamma_j x_j^2 = 0,$$

the conic, in a plane where homogeneous coordinates are (α, β, γ), through the five points $(\alpha_j, \beta_j, \gamma_j)$ could, by appropriate choice of the reference system, be taken as $\beta^2 = \alpha\gamma$ with parametric representation $(t^2, t, 1)$. It thus appears that whereas a curve of genus 5 has, in general, 12 moduli (this being the value of $3p - 3$ for $p = 5$), a Humbert curve depends only on the two projective invariants of the five numbers a_j.

Now it so happens that three types of such curves, still further restricted so that no moduli remain free, were discovered by Wiman in his search for curves of genera 4, 5, 6 possessing finite groups of birational self-transformations; equations for them may be found in (**6**, p. 39). These curves in [4] being invariant under groups of projectivities of orders (not merely 16 but) 64, 96, 160, we name W^{64}, W^{96}, W^{160}.

3. The equations for W^{64} are in precisely the form (2.1) with $a_1 = 1$, $a_2 = i$, $a_3 = -1$, $a_4 = -i$, $a_5 = 0$. The projective characterization of this pentad of numbers is "a regular sextuple lacking one of its six members", a regular sextuple (**3**, p. 299) consisting of three pairs each harmonic to both the others. Every quadric containing W^{64} is invariant under c, the cyclic permutation $(x_1 x_2 x_3 x_4)$. This, in common with all 64 projectivities of the group, can be imposed by a matrix of determinant $+1$, indeed by

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}.$$

The harmonic inversion h_j in the solid $x_j = 0$ and the opposite vertex X_j of the simplex of reference is imposed by a diagonal matrix: all its diagonal elements are -1 save for a single $+1$ on row j. Since $ch_5 = h_5c$, while

$$ch_1 = h_4c, \quad ch_2 = h_1c, \quad ch_3 = h_2c, \quad ch_4 = h_3c, \tag{3.1}$$

the whole group can be generated by c and any one h_i other than h_5; for instance,

$$h_5 = h_4h_3h_2h_1 = ch_1c^{-1}\cdot c^2h_1c^{-2}\cdot c^3h_3c^{-3}\cdot c^4h_1 = (ch_1)^4.$$

4. Wiman's equations for W^{160} are, with $\epsilon \neq 1$, a fifth root of unity,

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= 0, \\ x_1^2 + \epsilon x_2^2 + \epsilon^2x_3^2 + \epsilon^3x_4^2 + \epsilon^4x_5^2 &= 0, \\ \epsilon^4x_1^2 + \epsilon^3x_2^2 + \epsilon^2x_3^2 + \epsilon x_4^2 + x_5^2 &= 0. \end{aligned}$$

As the conic through the five points

$$(1, 1, \epsilon^4), (1, \epsilon, \epsilon^3), (1, \epsilon^2, \epsilon^2), (1, \epsilon^3, \epsilon), (1, \epsilon^4, 1)$$

is $\alpha^2 = \epsilon\beta\gamma$, Wiman's first unit quadric will be our $\sum a_jx_j^2 = 0$. Either of his two others can play the part of our unit quadric. On replacing x_1, x_2, x_3, x_4 by $\epsilon^3x_1, \epsilon x_2, \epsilon^4x_3, \epsilon^2x_4$, Wiman's quadratic forms become

$$\begin{aligned} &\epsilon x_1^2 + \epsilon^2x_2^2 + \epsilon^3x_3^2 + \epsilon^4x_4^2 + x_5^2, \\ &\epsilon^4(\epsilon^2x_1^2 + \epsilon^4x_2^2 + \epsilon x_3^2 + \epsilon^3x_4^2 + x_5^2), \\ &x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2; \end{aligned}$$

equating these to zero yields (2.1) with $a_j = \epsilon^j$. Each of the quadrics is invariant under the cyclic permutation $(x_1x_2x_3x_4x_5)$ while the involution that replaces x_1, x_2, x_3, x_4 by $\epsilon^4x_4, \epsilon^3x_3, \epsilon^2x_2, \epsilon x_1$, respectively, leaves $\sum \epsilon^jx_j^2$ invariant and transposes $\sum x_j^2$ with $\sum \epsilon^{2j}x_j^2$. Here, too, the whole group of projectivities can be imposed by matrices of determinant $+1$; take

$$c = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} \cdot & \cdot & \cdot & \epsilon^4 & \cdot \\ \cdot & \cdot & \epsilon^3 & \cdot & \cdot \\ \cdot & \epsilon^2 & \cdot & \cdot & \cdot \\ \epsilon & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

Then $ch_j = h_{j-1}c$; the harmonic inversions and the cyclic projectivity generate a group of order 80 having E for a normal subgroup. Moreover, while $dh_5 = h_5d$, $dh_1 = h_4d$, $dh_2 = h_3d$, $dh_3 = h_2d$, $dh_4 = h_1d$ so that the involution and the harmonic inversions generate a group of order 32. Finally, the matrices satisfy $dcd = \epsilon^4c^{-1}$, so that the projectivities satisfy $dcd = c^{-1}$ as for a dihedral group.

5. The three quadrics used by Wiman to determine W^{96} are line-cones, so that it is necessary to use linear combinations of them in order to obtain the non-singular quadrics of (2.1). If ω is a complex cube root of unity, Wiman defines W^{96} by

$$(5.1)\quad x_1^2 + x_4^2 + x_5^2 = 0,\ x_2^2 + \omega x_4^2 + \omega^2x_5^2 = 0,\ x_3^2 + \omega^2x_4^2 + \omega x_5^2 = 0.$$

The conic through the five points

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (1, \omega, \omega^2),\ (1, \omega^2, \omega)$$

is $\beta\gamma + \gamma\alpha + \alpha\beta = 0$ and has the parametric form

$$\alpha:\beta:\gamma = t:t^2 - t:1 - t.$$

The chord joining $t = u$ and $t = v$ is

$$(u - 1)(v - 1)\alpha + \beta + uv\gamma = 0$$

and, so long as $u \neq v$, the conic can be referred to this chord and the tangents at $t = u$ and $t = v$. Thus, in order to attain (2.1), it is enough to combine (5.1) linearly using the three sets of multipliers

$$(u - 1)^2, 1, u^2; \quad (u - 1)(v - 1), 1, uv; \quad (v - 1)^2, 1, v^2$$

and then choose the unit point so that either the first or the third of these combinations provides the unit quadratic form. The choice of u and v is free save for the ban on their equality and for the proviso that every x_j must be present in the unit form. Now

$$\begin{aligned}(u - 1)(v - 1)(x_1^2 + x_4^2 + x_5^2) &+ (x_2^2 + \omega x_4^2 + \omega^2x_5^2)\\ + uv(x_3^2 + \omega^2x_4^2 + \omega x_5^2) &\equiv (u - 1)(v - 1)x_1^2 + x_2^2 + uvx_3^2\\ &+ (1 + \omega u)(1 + \omega v)(i\omega x_4)^2 + (1 + \omega^2u)(1 + \omega^2v)(i\omega^2x_5)^2\end{aligned}$$

so that u, v must not both be among $1, 0, -\omega^2, -\omega$. If, then, Wiman's x_j are divided by

$$u - 1,\ 1,\ u,\ (1 + \omega u)i\omega,\ (1 + \omega^2u)i\omega^2,$$

the linear combinations of his three line-cones become (2.1) with

$$a_1 = \frac{u - 1}{v - 1},\quad a_2 = 1,\quad a_3 = \frac{u}{v},\quad a_4 = \frac{1 + \omega u}{1 + \omega v},\quad a_5 = \frac{1 + \omega^2u}{1 + \omega^2v}.$$

These are characterized projectively by a_4 and a_5 being the Hessian duad of the triad a_1, a_2, a_3. If this triad is 0, 1, -1, the duad is $i/\sqrt{3}$, $-i/\sqrt{3}$; one takes u to be 0 or 1 and then v to be 2 or -1 accordingly; the latter alternative yields:

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = -1, \quad a_4 = i/\sqrt{3}, \quad a_5 = -i/\sqrt{3},$$

and W^{96} is determined by

$$\begin{aligned} Q_1 &\equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ Q_2 &\equiv x_2^2 - x_3^2 + (i/\sqrt{3})x_4^2 - (i/\sqrt{3})x_5^2 = 0, \\ Q_3 &\equiv x_2^2 + x_3^2 - \tfrac{1}{3}x_4^2 - \tfrac{1}{3}x_5^2 = 0. \end{aligned}$$

It is manifestly invariant under the involution that transposes, simultaneously, x_2 with x_3 and x_4 with x_5. It is, less obviously, also invariant under the projectivity of period 3 that replaces

$$x_1, x_2, x_3, x_4, x_5$$

by

$$-2x_2, \ -x_3, \ \tfrac{1}{2}x_1, \ \omega^2 x_4, \ \omega x_5,$$

since the effect of this substitution turns Q_1, Q_2, Q_3 into

$$(1/4)Q_1 + (3/2)Q_2 + (9/4)Q_3, \qquad -(1/4)Q_1 - (1/2)Q_2 + (3/4)Q_3,$$
$$(1/4)Q_1 - (1/2)Q_2 + (1/4)Q_3.$$

The matrix of these fractional coefficients is also of period 3; its latent roots are 1, ω, ω^2 and the corresponding latent row-vectors are

$$(1, 0, 3), \ (1, 2\omega^2 - 2\omega, -3), \ (1, 2\omega - 2\omega^2, -3),$$

respectively, so that those quadrics which contain W^{96} and are themselves invariant under this projectivity of period 3 are

$$\left\{ \begin{aligned} &x_1^2 + 4(x_2^2 + x_3^2) = 0, \\ &x_1^2 + 4\omega^2 x_2^2 + 4\omega x_3^2 + 4x_4^2 = 0, \\ &x_1^2 + 4\omega x_2^2 + 4\omega^2 x_3^2 + 4x_5^2 = 0. \end{aligned} \right. \tag{5.2}$$

These quadrics will be alluded to again in an Appendix; it will appear that W^{96} can be projected from a chord into a five-nodal plane sextic whose equation is completely symmetrical in three suitably chosen homogeneous coordinates.

6. The special five-nodal or five-cusped plane sextics that emerge from these discoveries by Wiman are projections of W from chords, the plane curve being cusped when the chord of W is a tangent. First, a few paragraphs will be in place concerning the projection from a tangent of the general curve C given by (2.1). If, as in (**1**),

$$\begin{aligned} f(\vartheta) &\equiv (\vartheta - a_1)(\vartheta - a_2)(\vartheta - a_3)(\vartheta - a_4)(\vartheta - a_5) \\ &\equiv \vartheta^5 - e_1\vartheta^4 + e_2\vartheta^3 - e_3\vartheta^2 + e_4\vartheta - e_5, \end{aligned}$$

a tangent of C is the line λ, given parametrically in terms of t by

$$x_j\sqrt{f'(a_j)} = t + a_j;$$

the different signs of the five square roots produce the 16 lines on the cyclide F whose equations are

$$\sum x_j^2 = \sum a_j x_j^2 = 0. \tag{6.1}$$

For, again as in (**1**), if $s_k = \sum a_j^k/\sqrt{f'(a_j)}$, then

$$s_0 = s_1 = s_2 = s_3 = 0, \qquad s_4 = 1, \tag{6.2}$$

so that λ touches C where $t = \infty$, or $x_j = 1/\sqrt{f'(a_j)}$. None of the $f'(a_j)$ is zero because the five roots a_j of $f(\vartheta) = 0$ are presumed distinct.

The equations of λ are

$$\sum \frac{a_j^2 x_j}{\sqrt{f'(a_j)}} = \sum \frac{a_j x_j}{\sqrt{f'(a_j)}} = \sum \frac{x_j}{\sqrt{f'(a_j)}} = 0$$

or, say

$$X = Y = Z = 0. \tag{6.3}$$

Thus, since no two a_j are equal, the plane $\tilde{\omega}$ given by

$$\sum \frac{a_j^4 x_j}{\sqrt{f'(a_j)}} = \sum \frac{a_j^3 x_j}{\sqrt{f'(a_j)}} = 0 \tag{6.4}$$

is skew to λ and can serve as the plane of the cusped sextic $\mathscr{H}$. The conic through the cusps is, it will be remembered (**1**, p. 488), the intersection Γ of $\tilde{\omega}$ with the quadric line-cone of tangent planes to F at the points of λ. But the tangent solids to the quadrics (6.1) at the point on λ with parameter t are $tZ + Y = 0$ and $tY + X = 0$, so that the cone of tangent planes is $Y^2 = ZX$. The cusps of $\mathscr{H}$ are the intersections of $\tilde{\omega}$ with the tangent planes to F at the points where λ meets the solids $x_j = 0$; the relevant values of t are $-a_j$ so that, in terms of coordinates in $\tilde{\omega}$, the five cusps are $(a_j^2, a_j, 1)$. It is, moreover, a notable property (**5**, p. 139) of $\mathscr{H}$, proved anew by projection in (**1**), that its five cuspidal tangents all meet Γ again at $(1, 0, 0)$.

The vertices of the triangle of reference in $\tilde{\omega}$ are found by dropping, in turn, each one of the equations (6.3) and then solving the remaining two determinantally with (6.4). Or, alternatively, their coordinates are within the range of surmise on appealing to the recurrence relation

$$s_{k+5} - e_1 s_{k+4} + e_2 s_{k+3} - e_3 s_{k+2} + e_4 s_{k+1} - e_5 s_k = 0 \tag{6.5}$$

with its initial conditions (6.2). The outcome is

$$Y = Z = 0\colon x_j\sqrt{f'(a_j)} = e_2 - e_1 a_j + a_j^2,$$
$$Z = X = 0\colon x_j\sqrt{f'(a_j)} = e_3 - e_2 a_j + e_1 a_j^2 - a_j^3,$$
$$X = Y = 0\colon x_j\sqrt{f'(a_j)} = e_4 - e_3 a_j + e_2 a_j^2 - e_1 a_j^3 + a_j^4.$$

When each of these three expressions for x_j is substituted in X, Y, Z, respectively, the results are 1, -1, 1; so one takes, as a point of general position in [4],

$$(6.6)\quad x_j\sqrt{f'(a_j)} = p + qa_j + x(e_2 - e_1a_j + a_j^2) - y(e_3 - e_2a_j + e_1a_j^2 - a_j^3) + z(e_4 - e_3a_j + e_2a_j^2 - e_1a_j^3 + a_j^4).$$

If one now substitutes for x_j from (6.6) in $\sum x_j^2 = 0$ and $\sum a_jx_j^2 = 0$, the resulting equations are both linear in p and q; the calculations need not be given. One applies (6.5) repeatedly to express the s_k as polynomials in the elementary symmetric functions e_j, and the two equations appear as

$$(6.7)\quad \begin{cases} 2(pz + qy) + x^2 - 2e_1xy + e_2(y^2 + 2zx) - 2e_3yz + e_4z^2 = 0, \\ 2(py + qx) - e_1x^2 + 2e_2xy - e_3y^2 + e_5z^2 = 0. \end{cases}$$

When these are solved for p and q and the solution inserted in the result, namely

$$(6.8)\qquad q^2 + 2px + e_2x^2 - e_4y^2 + 2e_5yz = 0,$$

of substituting from (6.6) in $\sum a_j^2x_j^2 = 0$, the outcome is the equation of $\mathscr{H}$, an equation already on record in (**1**) where, in the notation there used, it was $\mathbf{C}_2 = 0$. The ternary sextic $\mathbf{C}_2$ is "normalized" by assigning to y^2x^4 the coefficient $+5$.

7. The projections from λ of the sections of F by the solids $x_j = 0$ are those adjoint cubics ϕ_j that cut, apart from the five cusps, octads of Weierstrassian points on $\mathscr{H}$. Their equations $\Phi_j = 0$ were found in (**1**) by using the fact that ϕ_j is determined by its behaviour at the pentad of cusps of $\mathscr{H}$. Here they are found by remarking that, just as elimination between (6.7) and (6.8) produces the equation of $\mathscr{H}$, so elimination between (6.7) and the outcome of using (6.6) for $x_j = 0$ produces the equation of ϕ_j; this latter elimination is achieved simply by equating a three-rowed determinant to zero. To accord with the earlier results one normalizes Φ_j so that x^3 has coefficient $+1$; this is feasible because no ϕ_j passes through (1, 0, 0), all its intersections with Γ being accounted for by four of the cusps and a contact at the remaining one. And there is an identity

$$\mathbf{C}_2 \equiv \sum a_j^2\Phi_j^2/f'(a_j).$$

8. The procedure for W^{64} runs easily. Here

$$(8.1)\quad a_1 = 1,\quad a_2 = i,\quad a_3 = -1,\quad a_4 = -i,\quad a_5 = 0; \qquad f(\vartheta) \equiv \vartheta^5 - \vartheta,\qquad f'(\vartheta) \equiv 5\vartheta^4 - 1;$$

$$(8.2)\quad f'(a_1) = f'(a_2) = f'(a_3) = f'(a_4) = 4,\qquad f'(a_5) = -1; \qquad e_1 = e_2 = e_3 = 0 = e_5,\qquad e_4 = -1,$$

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and the five relations (6.6) are

$$(8.3)\qquad \begin{aligned} 2x_1 &= p + q + x + y, & 2x_2 &= p + iq - x - iy,\\ 2x_3 &= p - q + x - y, & 2x_4 &= p - iq - x + iy,\end{aligned}$$
$$2x_5 = -2ip + 2iz.$$

Then

$$(8.4)\qquad \begin{cases} \sum x_j^2 = 2pz + 2qy + x^2 - z^2 = 0,\\ \sum a_jx_j^2 = 2py + 2qx = 0,\end{cases}$$

giving

$$p:q:z^2 - x^2 = x:-y:2(zx - y^2)$$

and hence, since $\sum a_j^2x_j^2 = q^2 + 2px + y^2$, the equation of $\mathscr{H}^{64}$ is

$$(8.5)\qquad 4y^6 - 8y^4zx + y^2(z^4 - 2z^2x^2 + 5x^4) + 4zx^3(z^2 - x^2) = 0.$$

And this is, precisely, the third column of the table in (**1**, p. 493) when the values (8.2), with $e_0 = 1$, are inserted. $\mathscr{H}^{64}$ is, clearly, unaffected when x, y, z are replaced by $ix, y, -iz$.

Since the equation $x_5 = 0$ is, here, $p = z$, ϕ_5 is

$$\begin{vmatrix} 1 & \cdot & -z\\ 2z & 2y & x^2 - z^2\\ 2y & 2x & \cdot \end{vmatrix} = 0,$$

so that $\Phi_5 \equiv x^3 + xz^2 - 2y^2z$.

ϕ_5, like every adjoint curve, contains all the cusps of $\mathscr{H}^{64}$; but it touches Γ at $(a_5^2, a_5, 1)$, i.e. at $(0, 0, 1)$, and its tangents at the other four cusps of $\mathscr{H}^{64}$ all meet Γ again at this point. Thus these four concurrent tangents of ϕ_5 form a harmonic pencil and ϕ_5 is a harmonic cubic; it is, like $\mathscr{H}^{64}$ itself, invariant when x, y, z are replaced by $ix, y, -iz$. This replacement permutes $\phi_1, \phi_2, \phi_3, \phi_4$ cyclically. Since ϕ_1 is the projection from λ of the section of F by $x_1 = 0$, its equation is

$$\begin{vmatrix} 1 & 1 & x + y\\ 2z & 2y & x^2 - z^2\\ 2y & 2x & \cdot \end{vmatrix} = 0$$

and $\Phi_1 \equiv (x^2 - z^2)(x - y) - 2(x + y)(zx - y^2)$. On replacing x, y, z by $x, -iy, -z$ (thereby ensuring that the coefficient of x^3 remains $+1$) we find

$$\begin{aligned} \Phi_2 &\equiv (x^2 - z^2)(x + iy) + 2(x - iy)(zx - y^2),\\ \Phi_3 &\equiv (x^2 - z^2)(x + y) - 2(x - y)(zx - y^2),\\ \Phi_4 &\equiv (x^2 - z^2)(x - iy) + 2(x + iy)(zx - y^2).\end{aligned}$$

Then $\frac{1}{4}(\Phi_1^2 - \Phi_2^2 + \Phi_3^2 - \Phi_4^2)$ agrees in every detail with the left-hand side of (8.5).

9. The corresponding discussion of W^{160} stems from

$$(9.1)\qquad a_j = \epsilon^j,\quad f(\vartheta) \equiv \vartheta^5 - 1,\quad f'(\vartheta) = 5\vartheta^4,\quad f'(a_j) = 5\epsilon^{4j};$$
$$e_1 = e_2 = e_3 = e_4 = 0,\qquad e_5 = 1.$$

Corresponding to (6.6) we have

$$\epsilon^{2j}x_j\sqrt{5} = p + q\epsilon^j + x\epsilon^{2j} + y\epsilon^{3j} + z\epsilon^{4j}$$

and so must eliminate p and q between

$$(9.2)\qquad \begin{cases} 5\sum x_j^2 = 5(x^2 + 2qy + 2pz) = 0, \\ 5\sum \epsilon^j x_j^2 = 5(z^2 + 2qx + 2py) = 0, \\ 5\sum \epsilon^{2j} x_j^2 = 5(q^2 + 2px + 2yz) = 0. \end{cases}$$

The first two of these three equations yield

$$(9.3)\qquad 2p(zx - y^2) = yz^2 - x^3, \qquad 2q(zx - y^2) = x^2y - z^3,$$

whereupon, on substitution in the third, the equation of $\mathscr{H}^{160}$ appears as

$$(9.4)\qquad \begin{aligned} &(x^2y - z^3)^2 + 4x(yz^2 - x^3)(zx - y^2) + 8yz(zx - y^2)^2 = 0, \\ &8y^5z - 20y^3z^2x + 5y^2x^4 + 10yz^3x^2 + z(z^5 - 4x^5) = 0. \end{aligned}$$

The left-hand side is $\mathbf{C}_2$, and agrees with the right-hand column in (**1**, p. 493) when (9.1) holds. If x, y, z are replaced by $\epsilon^3x, y, \epsilon^2z$, this ternary sextic is merely multiplied by ϵ^2.

As for the curves ϕ_k, one takes

$$\epsilon^{2k}x_k\sqrt{5} = p + q\epsilon^k + x\epsilon^{2k} + y\epsilon^{3k} + z\epsilon^{4k} = 0$$

and on eliminating p and q between this and the first two of the equations (9.2), one has

$$\begin{vmatrix} 1 & \epsilon^k & x\epsilon^{2k} + y\epsilon^{3k} + z\epsilon^{4k} \\ 2z & 2y & x^2 \\ 2y & 2x & z^2 \end{vmatrix} = 0,$$

$$\Phi_k \equiv x^3 - yz^2 + \epsilon^k(z^3 - x^2y) + 2(x\epsilon^{2k} + y\epsilon^{3k} + z\epsilon^{4k})(y^2 - zx).$$

Then $\sum a_j^2\Phi_j^2/f'(a_j)$, being here $\frac{1}{5}\sum \epsilon^{3j}\Phi_j^2$, is identical with the left-hand side of (9.4).

10. For W^{96}, with

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = -1, \quad a_4 = i/\sqrt{3}, \quad a_5 = -i/\sqrt{3},$$

the relevant data are

$$(10.1)\qquad \begin{aligned} &f(\vartheta) \equiv \vartheta^5 - \tfrac{2}{3}\vartheta^3 - \tfrac{1}{3}\vartheta, \qquad f'(\vartheta) \equiv 5\vartheta^4 - 2\vartheta^2 - \tfrac{1}{3}, \\ &e_1 = e_3 = e_5 = 0, \qquad e_2 = -\tfrac{2}{3}, \qquad e_4 = -\tfrac{1}{3}. \end{aligned}$$

The equations (6.7) and (6.8) become

$$(10.2)\qquad \begin{cases} 2(pz + qy) + x^2 - (2/3)(y^2 + 2zx) - (1/3)z^2 = 0, \\ 2(py + qx) - (4/3)xy = 0, \\ q^2 + 2px - (2/3)x^2 + (1/3)y^2 = 0, \end{cases}$$

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and the elimination of p and q leads to

$$9\mathbf{C}_2 = 16y^6 - 4y^4(3x^2 + 6zx - z^2) + y^2(45x^4 - 24x^3z - 6x^2z^2 + z^4) - 12zx^3(z + 3x)(z - x) = 0,$$

again in agreement with (**1**) when the general form there displayed for $\mathbf{C}_2$ is specialized by (10.1). $\mathscr{H}^{96}$ is thus invariant when the sign of y is changed. One also finds

$$3\Phi_1 \equiv 3x^3 + z^2x - 2y^2(z + x),$$
$$3\Phi_2 \equiv 3x^3 - 6zx^2 - z^2x - y(3x^2 + 2zx - z^2) + 4y^2x + 4y^3,$$
$$3\sqrt{3}\Phi_4 \equiv (3x^3 + 2zx^2 - z^2x)\sqrt{3} - iy(3x^2 - 6zx - z^2) - 4y^2x\sqrt{3} - 4iy^3;$$

Φ_3 is obtained from Φ_2 and Φ_5 from Φ_4 by changing the sign of y. And there is the identity

$$3(\Phi_2^2 + \Phi_3^2 - \Phi_4^2 - \Phi_5^2) \equiv 8\mathbf{C}_2.$$

Since the tangents to ϕ_j at those four cusps of $\mathscr{H}$ where ϕ_j does not touch Γ all pass through the fifth cusp, where ϕ_j does touch Γ, ϕ_4 and ϕ_5 are, for $\mathscr{H}^{96}$, equianharmonic cubics.

11. Every Humbert sextic is invariant under 15 transformations of period 2 corresponding to harmonic inversions of C in [4] and, since these inversions all leave F invariant, projection from λ induces Cremona transformations, necessarily also of period 2, in $\tilde{\omega}$.

The join of λ to a line l of general position in $\tilde{\omega}$ is a solid λl whose residual intersection with F is a twisted cubic c which, since λl meets the quadric cone of tangent planes to F at the points of λ in two planes, has λ for a chord. The five lines on F that meet λ are skew to c, their intersections with λl being not on c but on λ; but each of the ten lines on F that are skew to λ meets λl, and so c, once. The central harmonic inversion, h_k, in $x_k = 0$ and the opposite vertex of Σ, transforms λ into λ_k, that line on F which meets λ on $x_k = 0$; the transform c_k of c by h_k has λ_k as a chord. But c_k meets the other four lines on F which intersect λ each once, because they are transforms by h_k of lines on F skew to λ and so meeting c once. Hence the Cremona transformation $\mathscr{C}_k$ in $\tilde{\omega}$ that corresponds to h_k replaces l by a plane cubic with a node at $A_k(a_k^2, a_k, 1)$ and passing through the remaining four A_j. Were l to touch Γ, this cubic would have a cusp at A_k.

Since the join of each pair of points that correspond in h_k passes through X_k, the join of each pair of points that correspond in $\mathscr{C}_k$ passes through A_k; and since, on F, the branch curve of h_k is the section by $x_k = 0$, every individual point (apart from fundamental points of $\mathscr{C}_k$) of ϕ_k is invariant under $\mathscr{C}_k$.

If l passes through A_k, the solid λl contains λ_k, as well as λ, and meets F further in a conic γ that intersects both λ and λ_k; it is γ of which l is now the projection, and just as the points of γ are paired in involution by h_k so the points of l are (when it passes through A_k) paired in involution by $\mathscr{C}_k$. The

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foci of this involution on l are its two intersections, other than A_k, with ϕ_k, these being the projections from λ of the two points on γ where $x_k = 0$. The harmonic conjugate of A_k with respect to these two points on ϕ_k is on the polar conic of A_k with respect to ϕ_k, in other words, on Γ. A_k is, as contemporary writers sometimes say, dilated by $\mathscr{C}_k$ into Γ. $\mathscr{C}_k$ is an instance of a central involution (**4**, pp. 47, 108, 116); its properties are manifest when it is derived as a projection of h_k.

The five cubic Cremona transformations $\mathscr{C}_k$, like the five harmonic inversions h_k, generate an elementary abelian group of order 16; their ten products in pairs are quadratic transformations. For $h_k h_j$ transposes λ with λ_{jk}, which is skew to λ, so that, as c meets λ_{jk} once, its transform c_{jk} meets λ once and is projected into a conic in $\tilde{\omega}$; λ_j and λ_k, being transposed, are skew to c_{jk} as they are to c, but the other three lines on F that meet λ are transforms of lines skew to λ and so each meet c_{jk} once. $\mathscr{C}_j\mathscr{C}_k \equiv \mathscr{C}_k\mathscr{C}_j$ thus turns the lines of $\tilde{\omega}$ into conics through the three cusps of $\mathscr{H}$ other than A_j and A_k.

12. Each line in $\tilde{\omega}$ is its intersection with a solid

$$\sum (la_j^2 + ma_j + n)x_j/\sqrt{f'(a_j)} = 0 \tag{12.1}$$

through λ. The image of this solid under h_k is

$$\sum (la_j^2 + ma_j + n)x_j/\sqrt{f'(a_j)} = 2(la_k^2 + ma_k + n)x_k/\sqrt{f'(a_k)}. \tag{12.2}$$

On substituting here from (6.6), the left-hand side collapses to $lx + my + nz$. The right-hand side is linear in p and q which may therefore be eliminated determinantally between this equation and (6.7). The eliminant is the nodal (or cuspidal) plane cubic that is the transform of the line $\tilde{\omega}$ under $\mathscr{C}_k$.

13. Now apply this process to obtain $\mathscr{C}_5$ for $\mathscr{H}^{64}$. By (12.2), (8.1), and (8.3), we have:

$$lx + my + nz = 2nx_5/i = -2n(p - z),$$

and hence, eliminating p and q between this and (8.4),

$$\begin{vmatrix} n & \cdot & lx + my - nz \\ z & y & x^2 - z^2 \\ y & x & \cdot \end{vmatrix} = 0,$$

$$(lx + my)(zx - y^2) + n(y^2 z - x^3) = 0.$$

Thus the equations for $\mathscr{C}_5$ are

$$\xi = x(zx - y^2), \qquad \eta = y(zx - y^2), \qquad \zeta = y^2 z - x^3.$$

The analogous process for $\mathscr{C}_1$ yields:

$$\begin{aligned} lx + my + nz &= 2(l+m+n)x_1/2,\\ 2(lx+my+nz) &= (l+m+n)(p+q+x+y),\\ \xi &= (x-y)\{z^2 - x^2 - 2(zx-y^2)\},\\ \eta &= (x-y)\{z^2-x^2+2(zx-y^2)\},\\ \zeta &= (x-y)(z^2-x^2) + 2(x+y-2z)(zx-y^2). \end{aligned}$$

The product, in either order, of $\mathscr{C}_1$ and $\mathscr{C}_5$ is found to be the quadratic transformation

$$\begin{aligned} \xi &= z^2 - x^2 - 2(zx-y^2),\\ \eta &= z^2 - x^2 + 2(zx-y^2),\\ \zeta &= (z+x)(z+3x+4y) - 2(zx-y^2). \end{aligned}$$

The quadratic forms on the right are linearly independent and have the common zeros

$$(1, -1, 1), \quad (-1, i, 1), \quad (-1, -i, 1);$$

i.e. the cusps of $\mathscr{H}^{64}$ other than (1, 1, 1) and (0, 0, 1).

Of course this quadratic transformation can be obtained directly by projection from [4]. The image of (12.1) under h_1h_5 is

$$\sum (la_j^2 + ma_j + n)x_j/\sqrt{f'}(a_j) = 2(la_1^2 + ma_1 + n)x_1/\sqrt{f'}(a_1) + 2(la_5^2 + ma_5 + n)x_5/\sqrt{f'}(a_5)$$

which, for the special curve W^{64}, is, by § 8,

$$2(lx+my+nz) = (l+m+n)(p+q+x+y) - 2n(p-z).$$

The three-rowed determinant that is the eliminant of this equation and (8.4) has $x - y$ as a factor; the remaining factor provides the above quadratic transformation. The line $x = y$ is the join of those two cusps of $\mathscr{H}^{64}$ that are not base points of the homaloidal net of conics.

The group of 64 Cremona transformations that leave $\mathscr{H}^{64}$ invariant is generated by $\mathscr{C}_1$ and

$$\xi = ix, \qquad \eta = y, \qquad \zeta = -iz.$$

14. The equation (12.2) for W^{160} is

$$\sum (l\epsilon^{2j} + m\epsilon^j + n)x_j/\epsilon^{2j}\sqrt{5} = 2(l\epsilon^{2k} + m\epsilon^k + n)x_k/\epsilon^{2k}\sqrt{5}$$

which, on using (6.6), yields:

$$5(lx+my+nz) = 2(l\epsilon^{3k} + m\epsilon^{2k} + n\epsilon^k)(p + q\epsilon^k + x\epsilon^{2k} + y\epsilon^{3k} + z\epsilon^{4k}).$$

Elimination of p and q between this and (9.3) produces a determinant wherein the coefficients of l, m, n provide the equations for $\mathscr{C}_k$. One finds

$$\xi = \epsilon^{3k}(yz^2 - x^3) + \epsilon^{4k}(x^2y - z^3) + (zx-y^2)(2\epsilon^k y + 2\epsilon^{2k}z - 3x),$$

and similar expressions for η, ζ. That the transformation is central, with centre $(\epsilon^{2k}, \epsilon^k, 1)$, is seen from the relations

$$\frac{\xi - \epsilon^k\eta}{x - \epsilon^k y} = \frac{\eta - \epsilon^k\zeta}{y - \epsilon^k z} = 5(y^2 - zx).$$

There is a group of 80 Cremona transformations, generated by the projectivity

$$\xi = \epsilon^3 x, \qquad \eta = y, \qquad \zeta = \epsilon^2 z$$

and any one of the five $\mathscr{C}_k$, for which $\mathscr{H}^{160}$ is invariant.

15. Wiman's group of 160 projectivities mentioned in § 4 included an involution that transposed the quadrics $\sum x_j^2 = 0$ and $\sum \epsilon^{2j}x_j^2 = 0$, and hence transformed F into another surface. Since F is not invariant, the corresponding involution in $\tilde{\omega}$ is not a Cremona transformation; it need only be a Riemann transformation operating merely on $\mathscr{H}^{160}$, not on the whole of $\tilde{\omega}$. Equations (9.2) show that the interchange of $\sum x_j^2 = 0$ with $\sum \epsilon^{2j}x_j^2 = 0$, with $\sum \epsilon^j x_j^2 = 0$ unaffected, is achieved by transposing x with q and y with p; thus, in virtue of (9.3), the corresponding transformation of $\mathscr{H}^{160}$ occurs when z is left unaltered, x replaced by $\frac{1}{2}(x^2y - z^3)/(zx - y^2)$ and y by $\frac{1}{2}(yz^2 - x^3)/(zx - y^2)$. This transformation is given by

$$\xi = x^2y - z^3, \qquad \eta = yz^2 - x^3, \qquad \zeta = 2z(zx - y^2),$$

equations which may also be written as

$$\xi = \sum \epsilon^{4j}\Phi_j, \qquad \eta = \sum \Phi_j, \qquad \zeta = \sum \epsilon^j\Phi_j.$$

Since it appears, on calculation, that

$$\begin{aligned} \xi^2\eta - \zeta^3 &= yz^2\mathbf{C}_2 - x(3z^3x - 2z^2y^2 - x^3y)^2, \\ \eta\zeta^2 - \xi^3 &= z^3\mathbf{C}_2 - y(3z^3x - 2z^2y^2 - x^3y)^2, \\ 2\zeta(\zeta\xi - \eta^2) &= zx^2\mathbf{C}_2 - z(3z^3x - 2z^2y^2 - x^3y)^2, \end{aligned}$$

the transformation has period 2 so far as the points on $\mathbf{C}_2 = 0$ are concerned.

16. The same methods serve to find equations for the Cremona self-transformations of $\mathscr{H}^{96}$, and it will suffice to give $\mathscr{C}_2$. The appropriate form of (12.2) leads to

$$3(l + m + n)(p + q) + x(m + n - 3l) + y(n + l - 3m) - 4nz = 0;$$

elimination of p and q between this equation and (10.2) yields:

$$\begin{aligned} &(l + m + n)(z^2x - z^2y + 6zx^2 + 2xyz - 3x^3 + 3x^2y - 4xy^2 - 4y^3) \\ &\qquad = 8(lx + my + nz)(zx - y^2). \end{aligned}$$

Hence the equations for $\mathscr{C}_2$, thrown into a form so that (1, 1, 1) is seen to be a double point on all the cubic curves $l\xi + m\eta + n\zeta = 0$, are

$$\xi = (x - y)\{(z - x)^2 - 4(x^2 - y^2)\},$$
$$\eta = (x - y)\{(z - x)(z + 7x) + 4(x^2 - y^2)\},$$
$$\zeta = (x - y)(z - x)(z + 3x) + 4(x + y - 2z)(zx - y^2).$$

These cubics all pass through the other four cusps of $\mathscr{H}^{96}$, namely the points $(t^2, t, 1)$ with $t = 0, -1, i/\sqrt{3}, -i/\sqrt{3}$.

17. The transformations $\mathscr{C}_k$, combined with the change of sign of y, yield a group of 32 Cremona self-transformations of $\mathscr{H}^{96}$. But the operation of period 3 that was described in § 5 transforms F into another cyclide, and the associated (1, 1) correspondence between the points of $\mathscr{H}^{96}$ will not be subordinate to a Cremona transformation. Yet there is a plane Humbert sextic $\mathscr{H}_n$ that admits 96 Cremona transformations and it is entitled to a brief mention. As the curves studied in this paper are cuspidal, this nodal curve $\mathscr{H}_n$, and the way to obtain it by projection from a chord of W^{96}, may be relegated to an appendix.

Appendix. The quadrics (5.2) were found as those three containing W^{96} which are invariant under a projectivity of period 3. The involutory projectivity of § 5 leaves the first of the three invariant and transposes the other two; hence the cyclide Ψ common to these two is invariant under the whole group of 96 projectivities, so that the projection of W^{96}, from any one of the 16 lines μ on Ψ onto a plane π skew to μ, is invariant under a group of 96 Cremona transformations of π. One line μ is

$$x_1 + 2(x_2 + x_3) = x_4 + \omega^2 x_2 - \omega x_3 = x_5 + \omega x_2 - \omega^2 x_3 = 0.$$

This is skew to the plane $x_4 = x_5 = 0$ which will serve as π. The quadric cone generated by the tangent planes of Ψ at the points of μ is

$$2x_2x_3 + x_3x_1 + x_1x_2 = 0$$

which meets π in the conic Γ whose equation is

$$\eta\zeta + \zeta\xi + \xi\eta = 0;$$

here, since it adds much to arithmetical convenience, the coordinates in π are

$$\xi:\eta:\zeta = \tfrac{1}{2}x_1:x_2:x_3.$$

The nodes of $\mathscr{H}_n$ all lie on Γ and are the intersections of π with the five planes μX_i; hence three of the nodes are X_1, X_2, X_3 while, by § 5, the other two are the Hessian duad of this triad on Γ. This duad is the pair of intersections of Γ with $\xi + \eta + \zeta = 0$, the join of the collinear intersections of X_2X_3, X_3X_1, X_1X_2 with the tangents to Γ at X_1, X_2, X_3, respectively. Hence the remaining two nodes N, N' of $\mathscr{H}_n$ are $(1, \omega^2, \omega)$ and $(1, \omega, \omega^2)$.

μ meets $x_4 = 0$ at $(2, \omega, \omega^2, 0, \omega - \omega^2)$ and $x_5 = 0$ at $(2, \omega^2, \omega, \omega^2 - \omega, 0)$; since these are also its intersections with $x_1^2 + 4(x_2^2 + x_3^2) = 0$, they lie on W^{96}. Moreover, as they lie one in each of two bounding solids of Σ they are Weierstrassian points, and the tangents to W^{96} there pass through X_4 and X_5, respectively. Now (**1**, pp. 487–488) when C was projected from λ onto $\tilde{\omega}$ the planes joining λ to the tangents of C at its two intersections with λ met $\tilde{\omega}$ in points T, U on Γ; and every node of the Humbert sextic had its nodal tangents passing one through each of T, U. Thus, for $\mathscr{H}_n$, there is a confluence of T (say) with N and U with N'; NN' is a tangent at both nodes while, at both N and N', the other tangent touches Γ, $\mathscr{H}_n$ having an inflection on this latter branch; N and N' are flecnodes of $\mathscr{H}_n$.

The plane joining μ to $(2\xi, \eta, \zeta, 0, 0)$ is spanned by this latter point and the intersections of μ with $x_2 = 0$ and $x_3 = 0$; it is therefore obtained by varying p and q in

$$(2\xi + 2p - 2q, \eta - p, \zeta + q, p\omega^2 + q\omega, p\omega + q\omega^2). \tag{A.1}$$

It meets Ψ, in general, in a single point not on μ. When the coordinates (A.1) are substituted in the equations of any two quadrics through Ψ, the terms in p^2, pq, q^2 all cancel; the two resulting linear equations for p and q have the solution

$$p:q:1 = (\eta^2 - \zeta\xi)(\zeta + \xi):(\xi\eta - \zeta^2)(\xi + \eta):2(\eta\zeta + \zeta\xi + \xi\eta). \tag{A.2}$$

If the point (A.1) is on W^{96}, its projection $(2\xi, \eta, \zeta, 0, 0)$ from μ is on $\mathscr{H}_n$; this occurs when the coordinates satisfy $x_1^2 + 4(x_2^2 + x_3^2) = 0$, and hence when

$$(\xi + p - q)^2 + (\eta - p)^2 + (\zeta + q)^2 = 0.$$

The equation of $\mathscr{H}_n$ is found on substituting here for p and q from (A.2). The calculations have no intrinsic interest, but their outcome has since it is symmetric in ξ, η, ζ. In terms of monomial symmetric functions it is

$$\sum \xi^4\eta^2 + \sum \xi^4\eta\zeta + \sum \xi^3\eta^3 + 5\sum \xi^3\eta^2\zeta + 12\xi^2\eta^2\zeta^2 = 0;$$

in terms of

$$E_1 \equiv \xi + \eta + \zeta, \qquad E_2 \equiv \eta\zeta + \zeta\xi + \xi\eta, \qquad E_3 \equiv \xi\eta\zeta$$

it is

$$E_1^3E_3 + E_2^3 = E_1^2E_2^2 + 3E_1E_2E_3.$$

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