BULL. AUSTRAL. MATH. SOC. 68Q25, 55M20
VOL. 56 (1997) [87–94]

COMPUTING THE TOPOLOGICAL DEGREE OF POLYNOMIAL MAPS

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Let C be a cube in $\mathbf{R}^{n+1}$ and let $F = (f_1, \cdots, f_{n+1})$ be a polynomial vector field. In this note we propose a recursive algorithm for the computation of the degree of F on C. The main idea of the algorithm is that the degree of F is equal to the algebraic sum of the degrees of the map $\left(f_1, f_2, \cdots, f_{i-1}, \widehat{f_i}, f_{i+1}, \cdots, f_{n+1}\right)$ over all sides of C, thereby reducing an $(n+1)$–dimensional problem to an n–dimensional one.

1. INTRODUCTION

Let $(x_1, x_2, \cdots, x_k)$ be a point in $\mathbf{R}^k$, $k \geqslant 1$. In the sequel we shall denote such a point by x. Let

$$F(x) := (f_1(x), f_2(x), \cdots, f_{n+1}(x)) : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$$

be a differentiable mapping. A zero of F is a point $x_0 \in \mathbf{R}^{n+1}$ such that $F(x_0) = 0 = (0, 0, \cdots, 0)$. Let $a \in \mathbf{R}^{n+1}$ be a zero of F and suppose that the sphere $S(a, \varepsilon)$ centred at a of radius ε isolates a and it is such that no zero of F lies on $S(a, \varepsilon)$. We then can consider the (Gauss) map

$$G_a : S(a, \varepsilon) \to S^n, \quad G_a(x) = \frac{F(x)}{\|F(x)\|} \tag{1}$$

where S^n is the unit sphere in $\mathbf{R}^{n+1}$. Then the degree, $\deg G_a$, of G_a is an integer which, roughly speaking, tells us the (algebraic) number of times G_a wraps $S(a, \varepsilon)$ around S^n with respect to specified orientations on $S(a, \varepsilon)$ and S^n. We give S^n the following orientation, which we shall call the positive orientation. We think of S^n as the boundary of the unit ball $B(0, 1) = \{x \in \mathbf{R}^{n+1} \mid \|x\| \leqslant 1\}$ centred at 0 of radius 1. Let $e_1, e_2, \cdots, e_{n+1}$ be the standard basis of $\mathbf{R}^{n+1}$. At each point of the open unit ball $Int(B(0, 1))$ the orientation is given by that basis. Therefore, its boundary inherits an orientation by the following rule. Let y be a point on S^n and let v_{n+1} be the unit normal vector to S^n that points towards the origin 0. Let $v_1, v_2, \cdots, v_n$ be a basis for the tangent space of S^n at y. We call this basis positive if and only if $det(v_1, v_2, \cdots, v_n, v_{n+1}) = 1$. By applying this rule to every point y on the sphere we get the positive orientation of S^n. We also give $S(a, \varepsilon)$ the same orientation.

Received 2nd September, 1996.

DEFINITION 1.1: We call $\deg G_a$ the local degree of F at a, and denote $\deg G_a = l.d.F(a)$.

It is well known that the local degree of F at a is independent of the chosen sphere $S(a, \varepsilon)$ as long as the sphere we choose isolates the zero a of F.

We define a cube C (or a box) in $\mathbf{R}^{n+1}$ as the boundary of the set:

$$C = [a_1, b_1] \times \cdots \times [a_{n+1}, b_{n+1}], \ a_i < b_i, \ i = 1, \cdots, n+1.$$

We define the "upper", C_i^+ ("lower", C_i^-) i–th side of C as follows:

$$\begin{aligned} C_i^+ &= [a_1, b_1] \times \cdots \times [b_i] \times \cdots \times [a_{n+1}, b_{n+1}] \\ C_i^- &= [a_1, b_1] \times \cdots \times [a_i] \times \cdots \times [a_{n+1}, b_{n+1}]. \end{aligned} \tag{2}$$

Since C is homeomorphic to S^n we orient each side of C the same way we oriented the sphere S^n, namely, at each side C_i^+ and C_i^- the normal vector points inside the cube C.

Let now $F = (f_1, \cdots, f_{n+1})$ be a polynomial vector field in $\mathbf{R}^{n+1}$ and consider a cube C so that no zero of F lies on C. In that case we can again define the Gauss map, which we call G,

$$G : C \to S^n, \quad G(x) = \frac{F(x)}{\|F(x)\|}.$$

Suppose now that all zeros of F that lie in the interior $Int(C)$ are isolated. Then the following holds:

PROPOSITION 1.1. *For C, F, G as above, we have*

$$\deg G = \sum_a l.d.F(a) \tag{3}$$

where the above sum is taken over all a such that $F(a) = 0$ and $a \in Int(C)$.

DEFINITION 1.2: We call $\deg G$ the degree of F on C, and denote $\deg G = \deg F_C$.

Our aim is to compute $\deg F_C$ given the cube C and the polynomial map F. Various methods of computing this degree have been proposed. O'Neal and Thomas [**4**] use quadrature methods to evaluate the Kronecker integral formula for the degree. Stenger [**6**] has a method, derived from the Kronecker integral formula as well. Kearfortt describes two methods, one that is related to Stenger's method [**1**], and one that is recursive, repeatedly reducing by one the functions to be considered [**2**]. Our method is also recursive and uses the idea of Gröbner basis.

In the next section we describe the method and prove the main result. The last section gives an implementation of the algorithm, as well as examples, in dimension 3.

2. A PROCEDURE FOR COMPUTING $\deg F_C$

We begin by defining some "special" points on S^n that we shall need later:

$$N_i = e_i, \quad S_i = -e_i, \quad i = 1, 2, \cdots, n+1. \tag{4}$$

We shall call the N_i, S_i the i–th "north" and "south" poles of S^n, respectively. It is easy to see that a local orienting basis at N_i (S_i), is the following:

$$\begin{gathered}\left(e_1, e_2, \cdots, e_{i-1}, \widehat{e}_i, e_{i+1}, \cdots, (-1)^{n-i} e_{n+1}, -e_i\right)\\ \left(e_1, e_2, \cdots, e_{i-1}, \widehat{e}_i, e_{i+1}, \cdots, (-1)^{n-i+1} e_{n+1}, e_i\right)\end{gathered}$$

respectively, where$\widehat{\ }$denotes omission.

Let now $U_i = \{(x_1, \cdots, x_{n+1}) \mid 0 < x_i \leqslant 1\}$, and $L_i = \{(x_1, \cdots, x_{n+1}) \mid -1 \leqslant x_i < 0\}$ be the *upper, lower* i–th hemisphere of S^{n+1}, respectively. We can then define the natural charts from these sets to the unit open ball $B(0,1) := \{x \mid \|x\| < 1\} \subset \mathbf{R}^n$, as follows:

$$(5)\qquad \begin{aligned} g_i &: U_i \to \mathbf{R}^n, \quad g_i(x) = (x_1, \cdots, x_{i-1}, \widehat{x}_i, x_{i+1}, \cdots, x_{n+1})\\ h_i &: L_i \to \mathbf{R}^n, \quad h_i(x) = (x_1, \cdots, x_{i-1}, \widehat{x}_i, x_{i+1}, \cdots, x_{n+1}). \end{aligned}$$

These are diffeomorphisms $U_i \overset{g_i}{\approx} B(0,1)$, and $L_i \overset{h_i}{\approx} B(0,1)$. Let $dg_i(y), dh_i(y)$ be the corresponding linear maps that are defined between the tangent planes at a point y of U_i, L_i and $\mathbf{R}^n$, respectively, and let $|dg_i(y)|, |dh_i(y)|$ denote the determinants of these linear maps. It is then easy to see that

$$(6)\qquad sign\,|dg_i(N_i)| = (-1)^{n-i} \quad \text{while} \quad sign\,|dh_i(S_i)| = (-1)^{n-i+1}$$

where *sign* signifies the signature of a real number. Thus, since g_i and h_i are diffeomorphisms, and if y is any point in U_i or L_i, we get that

$$(7)\qquad \begin{aligned} sign\,|dg_i(y)| &= (-1)^{n-i}, \quad \text{for all points } y \in U_i\\ sign\,|dh_i(y)| &= (-1)^{n-i+1}, \quad \text{for all points } y \in L_i. \end{aligned}$$

For a point $b \in \mathbf{R}$ and $x \in \mathbf{R}^n$ we define

$$(8)\qquad x^i(b) = (x_1, \cdots, x_{i-1}, b, x_{i+1}, \cdots, x_n).$$

Let now K_i be the image of the natural projection of the interior of either C_i^+ or C_i^- on $\mathbf{R}^n$. We then may define the lifting maps

$$(9)\qquad \begin{aligned} p_i &: K_i \to Int\left(C_i^+\right), \quad p_i(x) = x^i(b_i)\\ q_i &: K_i \to Int\left(C_i^-\right), \quad q_i(x) = x^i(a_i). \end{aligned}$$

Since the orientation of C_i^+ (C_i^-) is the same as the orientation at N_i (S_i) of the sphere S^n, respectively, we can see from (7) that at each point $y \in K_i$ we have

$$(10)\qquad sign\,|dp_i(y)| = (-1)^{n-i}, \quad \text{while} \quad sign\,|dq_i(y)| = (-1)^{n-i+1}.$$

Recall that we are given a cube C so that no zero of F lies on C. For each N_j, S_j define

$$\begin{aligned} X(N_j)_i^+ &:= G^{-1}(N_j) \cap C_i^+ \quad X(N_j)_i^- := G^{-1}(N_j) \cap C_i^-\\ X(S_j)_i^+ &:= G^{-1}(S_j) \cap C_i^+ \quad X(S_j)_i^- := G^{-1}(S_j) \cap C_i^-. \end{aligned}$$

Now using the notation of (8) we define

$$F_i^+(x) = (f_2(x_i(b_i)), \cdots, f_{n+1}(x_i(b_i))) : \mathbf{R}^n \to \mathbf{R}^n$$
$$F_i^-(x) = (f_2(x_i(a_i)), \cdots, f_{n+1}(x_i(a_i))) : \mathbf{R}^n \to \mathbf{R}^n.$$

Moreover, we call the cube C *good* for F relative to f_1 if the sets $G^{-1}(N_1)$ and $G^{-1}(S_1)$ are both finite, each point of that inverse image of N_1, S_1 lies in the *interior* of some side of C, and the Jacobian determinants $\left|J\left(F_i^+\right)\right|$ and $\left|J\left(F_i^-\right)\right|$ have isolated zeros at every point of $G^{-1}(N_1)$, $G^{-1}(S_1)$.

We are now ready to state the main theorem of this note.

THEOREM 2.1. *Let F, C, G be as above, and suppose that C is good relative to f_1. Then*

$$\begin{aligned} \deg F_C &= \sum_{i=1}^{n+1} (-1)^{i+1} l.d.F_i^+\left(y_i^+\right) + \sum_{i=1}^{n+1} (-1)^{i} l.d.F_i^-\left(y_i^-\right) \\ &= \sum_{i=1}^{n+1} (-1)^{i} l.d.F_i^+\left(z_i^+\right) + \sum_{i=1}^{n+1} (-1)^{i+1} l.d.F_i^-\left(z_i^-\right) \end{aligned}$$

where $y_i^+ \in X(N_1)_i^+$, $y_i^- \in X(N_1)_i^-$, $z_i^+ \in X(S_1)_i^+$ and $z_i^- \in X(S_1)_i^-$.

Before we proceed with the proof of the theorem, note that a similar result can be obtained if the north N_1 or south pole S_1 is replaced with another north ot south pole N_k or S_k. In that case, however, the cube C must be *good* relative to f_k.

PROOF OF THEOREM 2.1: For the sake of clarity we prove this theorem in the case where the north pole N_1 is a regular value of G. That is, the points y_i^+ and y_i^- are noncritical points of G. So, let us take a point $y_i^+ = (y_1, \cdots, y_{i-1}, b_i, y_{i+1}, \cdots, y_{n+1})$ and see what is its contribution to the degree $\deg F_C$. We define the series of maps

$$K_i \xrightarrow{p_i} Int\left(C_i^+\right) \hookrightarrow C \xrightarrow{G} U_i \hookrightarrow S^n \xrightarrow{g_1} \mathbf{R}^n \tag{11}$$

where $\hookrightarrow$ denotes the inclusion map. From (11) we get the composite map

$$A := g_1 \circ G \circ p_i : K_i \subset \mathbf{R}^n \to \mathbf{R}^n.$$

Let dg_1, dG, dp_i be the corresponding linear maps. Let

$$y_0 = \left(y_1, \cdots, y_{i-1}, \widehat{b}_i, y_{i+1}, \cdots, y_{n+1}\right).$$

Then at y_0 the Jacobian determinant of A is nonzero, since y_i^+ is a noncritical point of G. Moreov, we have

$$|dA(y_0)| = |dg_1(N_1)| \cdot \left|dG\left(y_i^+\right)\right| \cdot |dp_i(y_0)|$$

and thus

$$\begin{aligned} \left|dG\left(y_i^+\right)\right| &= |dg_1(N_1)| \cdot |dp_i(y_0)| \cdot |dA(y_0)| \\ &= (-1)^{n-1}(-1)^{n-i} l.d.A(y_0), \text{ from (6), (10)} \\ &= (-1)^{i+1} l.d.A(y_0). \end{aligned}$$

https://doi.org/10.1017/S0004972700030768 Published online by Cambridge University Press

But, an easy computation shows that $l.d.A(y_0) = l.d.F_i^+\left(y_i^+\right)$. A similar argument shows that the contribution to the degree of F at a y_i^- is $(-1)^i l.d.F_i^-\left(y_i^-\right)$. Thus, when we repeat the above process over all preimages of N_1 under G we get the desired expression for $\deg F_C$. ∎

We close this section with the final step of the recursion, that is, the computation of the degree in dimension 2, (see also [**5**]). For this, let $F = (f(x,y), g(x,y))$ be a polynomial vector field in $\mathbf{R}^2$ and let $C = \partial[a_1, b_1] \times [a_2, b_2]$ be a cube (rectangle) good for F relative to g. C has the counterclockwise (positive) orientation that it inherits as the boundary of the (open) rectangle. Let z be a zero of G that lies on a side of C. Then when we move on C according to the orientation, we observe the *sign* of $f \cdot g$ passing through z. If the *sign*$f \cdot g$ changes from $+ \to -$ the contribution to the degree is 1, while if it changes from $- \to +$ the contribution is -1. By summing up all this contributions over all sides of C and dividing by 2 we get the desired degree.

3. The Algorithm in the Cases $n = 2$ and 3

The algorithms of this section are implemented in the *Axiom* symbolic algebra system, but any type or object oriented language could be used to describe the process. For the computations with the real algebraic numbers we use the method of interval coding, see [**3**]. In this section R denotes the real closure of an ordered field K. The algorithm is generic.

3.1. Case $n = 2$: Let $F(x,y) := (f_1(x,y), f_2(x,y)) : R^2 \to R^2$ be a polynomial mapping. Let $C = [a_1, b_1] \times [a_2, b_2]$ be a rectangle in R^2. The algorithm *degree2* computes the degree of F in C. The boundary ∂C, of box C, has the positive orientation. The algorithm proceeds as follows:

Algorithm 1. degree2

input: *The polynomials $f_1(x,y)$, $f_2(x,y)$, and the rectangle C as above.*

output: *An integer equal to* $\deg F_C$.

Description:

The basic steps are:

1. Call $h_1(x) := f_1(x, a_2)$, $h_2(x) := f_2(x, a_2)$, $g_1(y) := f_1(b_1, y)$ and $g_2(y) := f_2(b_1, y))$.

2. Compute the real roots of $h_1(x)$ ($g_1(y)$) in the intervals $[a_1, b_1]$ ($[a_2, b_2]$), respectively. Let $(r_i)_{1 \leqslant i \leqslant k}$ be these real roots. If $h_1(a_1) = 0$ or $g_1(a_2) = 0$, or $h_1(b_1) = 0$ or $g_1(b_2) = 0$, or there exists an i such that $h_2(r_i) = 0$, or $g_2(r_i) = 0$, then redefine the rectangle C. Else, go to step 3.

3. Compute the sign of $h_1(x)h_2(x)$, and of $g_1(y)g_2(y)$, on the left and on the right of each real root r_i, respectively.

4. If $sign(h_1(x)h_2(x))$ changes from $+ \to -$ the contribution to the degree is 1, while if it changes from $- \to +$ the contribution is -1.

5. Start by the left contribution of the real root r_1. Sum up all the contributions over the sides $y = b_2$ and $x = b_1$; denote this sum by l_1.

6. Call $h_1(x) := f_1(x, b_2)$, $h_2(x) := f_2(x, b_2)$, $g_1(y) := f_1(a_1, y)$ and $g_2(y) := f_2(a_1, y)$.

7. Compute the real roots of $h_1(x)$ ($g_1(y)$) in the intervals $[a_1, b_1]$ ($[a_2, b_2]$), respectively. Let $(s_j)_{1 \leqslant j \leqslant m}$ be these real roots. If $h_1(a_1) = 0$ or $g_1(a_2) = 0$, or $h_1(b_1) = 0$ or $g_1(b_2) = 0$, or there exists j such that $h_2(s_j) = 0$ or $g_2(s_j) = 0$, then redefine the rectangle C. Else, go to step 8.

8. Compute the sign of $h_1(x)h_2(x)$ and of $g_1(y)g_2(y)$, on the left and on the right of each real root s_j.

9. Start by the right contribution of the real root r_m. If $sign(h_1(x)h_2(x))$ changes from $+ \to -$ the contribution is 1, while if it changes from $- \to +$ the contribution is -1.

10. Sum up all the contributions over the sides $y = b_2$ and $x = b_1$and denote this sum by l_2.

11. The degree is equal to $\dfrac{l_1 + l_2}{2}$.

3.2. CASE $n = 3$: Let $F(x_1, x_2, x_3) := (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))$ be a polynomial map of $R^3 \to R^3$, and $C := \prod_{i=1}^{3} [a_i, b_i]$ be a box in R^3. We use the standard implementation of the Gröbner basis of the *Axiom* symbolic system.

ALGORITHM 2. degree3

input: *The polynomials*
$f_1(x_1, x_2, x_3)$, $f_2(x_1, x_2, x_3)$, $f_3(x_1, x_2, x_3)$, *and the box* C, *as above.*

output: *An integer equal to* $\deg F_C$.

DESCRIPTION:

Notation : If $x \in R^3$, $d \in R$ and $i = 1 \ldots 3$, then we define $x^i(d) = \Big(., \underbrace{d}_{i-position}, .\Big)$, and $x^i = \Big(., \underbrace{\widehat{x}_i}_{i-position}, .\Big) \in R^2$, where $\widehat{\ }$ denotes omission, as in Section 2. If $j = 1 \ldots 3$, we denote by $a_{j,i}$ the i-element of the j-interval, for example, $a_{3,2}$ is b_3, and $a_{2,1}$ is a_2. By $[a_{j,1}, a_{j,2}]^j$ we denote the rectancle in R^2: $\ldots \times \underbrace{\widehat{[a_{j,1}, a_{j,2}]}}_{j-position} \times \ldots$, where $\widehat{\ }$ denotes omission, as above.

1. For $j = 1 \ldots 3$ repeat Step$_j$.
2. **Step$_j$:** For $i = 1 \ldots 2$, repeat

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(a) Call $h_2(x^j) := f_2(x^j(a_{j,i}))$, $h_3(x^j) := f_3(x^j(a_{j,i}))$. Compute the Gröbner basis of $h_2(x^j)$, $h_3(x^j)$. Let G be this base. The polynomials of G form a triangular system where the last polynomial is univariate. We find and isolate the real solutions of this system.

(b) **Step 1:** Find the sign of $f_1(x_1, x_2, x_3)$ in the real root of the system of step 2(a). Compute the degree (see Algorithm 1), of $h_2(x^i)$ and $h_3(x^i)$ in the corresponding isolating rectangle using degree2.

(c) **Step 2:** $i := i + 1$.

3. Compute the degree of $F(x_1, x_2, x_3)$ using the formulae of Theorem 2.1.

The binary time complexity of our algorithm is unknown. The complexity is obviously dependent on the amount of the algebraic numbers generated by the real solutions and the Gröbner basis algorithm.

EXAMPLE 1: Consider the system:

$$F(x,y,z) \quad \left\{ \begin{array}{lcl} p_1(x,y,z) & := & x^3 + y^2 - z \\ p_2(x,y,z) & := & y^3 + z^2 + x \\ p_3(x,y,z) & := & z^3 + x^2 - y \end{array} \right\}$$

This system has the following five real solutions (aproximately):

$$\left\{ \begin{array}{ll} 1. & [z = 0.9635, y = 0.60002, x = -1.1289] \\ 2. & [z = -8.715, y = -0.04635 x = -1.261] \\ 3. & [z = -1, y = 0, x = -1] \\ 4. & [z = 0, y = 1, x = -1] \\ 5. & [z = 0, y = 0, x = 0] \end{array} \right.$$

We compute the degree on the boxes $[-3/2, -1/2] \times [-1/2, 1/2] \times [-9, 0]$ and $[-2, 2]^3$. Consider the first box. For $y = 1/2$, the algorithm `groebner(p2(x,1/2,z), p3(x,1/2,z))` of *Axiom* gives:

$$\begin{array}{lcl} g_1(z,x) & := & z - \frac{64}{31}x^3 - \frac{56}{31}x^2 + \frac{16}{31}x - \frac{5}{31} \\ g_2(x) & := & x^4 + x^3 - \frac{5}{8}x^2 + \frac{3}{64}x + \frac{129}{512} \end{array}$$

If ξ is the first real root of $g_2(x)$, and ζ the real root of $g_1(z, \xi)$, then the sign of $p_1(\xi, 1/2, \zeta)$ is negative. The degree of $F(x, z) := (p_2(x, 1/2, z), p_3(x, 1/2, z))$ on the rectangle $[-3/2, -1/2] \times [-9, 0]$ is 1. The point $(\xi, 1/2, \zeta) \in X(S)_2^+$. Thus from Theorem 2.1, the contribution to $\deg F_C$ is $(-1)^3 * 1 = -1$. The second real root of $g_2(x)$, $\xi' \notin [-3/2, -1/2]$. For $x = -1/2$, the algorithm $\texttt{groebner}(\texttt{p2}(-1/2, \texttt{y}, \texttt{z}), \texttt{p3}(-1/2, \texttt{y}, \texttt{z}))$ of *Axiom* gives:

$$\begin{array}{lcl} g_1(z,y) & := & z - \frac{64}{31}y^8 - \frac{16}{31}y^7 + \ldots - \frac{68}{31}y + \frac{15}{31} \\ g_2(y) & := & y^9 - \frac{3}{2}y^6 + \frac{3}{4}y^3 + y^2 - \frac{1}{2}y - \frac{1}{16} \end{array}$$

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Let ξ be the second real root of the univariate polynomial $g_2(y)$, and ζ the real root of $g_1(z,\xi) = 0$. In this case, the sign of $p_1(-1/2,\xi,\zeta)$ is positive, and the degree of $D(y,z) := (p_2(-1/2,y,z), p_3(-1/2,y,z))$ is also 1. Since our point is in $X(N)_1^-$, it similarly follows that the contribution to the $\deg F_C$ is $(-1)^1 * 1 = -1$, as in the case where $y = 1/2$.
All the other cases are discarded. Thus $\deg F_C = -1$

Consider now the box $\partial[-2,2]^3$. Let $x = -2$; if (ξ,ζ) is the solution of the system of the Gröbner basis, of the polynomials $p_2(-2,y,z)$ and $p_3(-2,y,z)$, the sign of $p_1(-2,\xi,\zeta)$ is negative, and the corresponding degree in R^2 is -1. The point $(-2,\xi,\zeta) \in X(S)_1^-$. It follows that the contribution to $\deg F_C$ is $(-1)^{1+1} * (-1) = -1$. For $x = 2$, the sign of p_1 in the corresponding point of R^3 is positive, and the degree in R^2 is -1. Thus, the degree $\deg F_C$ in this case is still -1.

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https://doi.org/10.1017/S0004972700030768 Published online by Cambridge University Press