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FREE ABELIAN TOPOLOGICAL GROUPS ON COUNTABLE CW-COMPLEXES

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Let n be a positive integer, B^n the closed unit ball in Euclidean n-space, and X any countable CW-complex of dimension at most n. It is shown that the free Abelian topological group on B^n, $F(B^n)$, has $F(X)$ as a closed subgroup. It is also shown that for every differentiable manifold Y of dimension at most n, $F(Y)$ is a closed subgroup of $F(B^n)$.

INTRODUCTION

In recent years there has been an investigation of which free Abelian topological groups can be embedded as subgroups of the free Abelian topological group, $F(B^n)$, on the closed ball, B^n, for n a positive integer. In [**5**] it was shown that if S^n denotes the n-sphere, then $F(S^n) \leqslant F(B^n)$; that is, $F(S^n)$ is a (topological) subgroup of $F(B^n)$. This was extended in [**6**] to show that if $F(X)$ is a closed subgroup of $F(B^n)$ and $X \sqcup_f B^n$ is an adjunction of B^n and X along the boundary of B^n, then $F(X \sqcup_f B^n) \leqslant F(B^n)$. In this paper we obtain the 'full story' by proving that if Y is a relative countable CW-complex over X of dimension at most n, then $F(Y) \leqslant F(B^n)$. In particular, the free Abelian topological group on any countable CW-complex of dimension at most n can be embedded in $F(B^n)$. From this we deduce that if Y is any m-dimensional manifold, for $m \leqslant n$, then $F(Y) \leqslant F(B^n)$. (Note that Y can be an n-dimensional differentiable manifold not embeddable in B^n.) This includes, as a special case, the known result that $F(\mathbb{R}^n) < F(B^n)$. So our results include those of [**3, 4, 5, 6**].

PRELIMINARIES

We first record the necessary definitions and background results.

A Hausdorff topological space X is said to be a *k_ω-space* with k_ω-decomposition $X = \bigcup_n X_n$ if X_n is compact, $X_n \subseteq X_{n+1}$ for $n = 1, 2, 3, \ldots$ and X has the weak topology with respect to the spaces X_n.

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DEFINITION. If X is a topological space with distinguished point e, the Abelian topological group $F(X)$ is said to be the (Graev) *free Abelian topological group on X* if

(a) the underlying group of $F(X)$ is the free Abelian group with free basis $X \setminus \{e\}$ and identity e, and

(b) the topology of $F(X)$ is the finest topology on the underlying group which makes it into a topological group and induces the given topology on X.

If X is any completely regular space, then $F(X)$ exists, is unique, and is independent of the choice of e in X. Further, $F(X)$ is algebraically the free Abelian group on $X \setminus \{e\}$. If X is also Hausdorff, then $F(X)$ is Hausdorff and has X as a closed subspace [7]. For k_ω-spaces, one can say rather more:

THEOREM A [7]. *Let $X = \cup X_n$ be any k_ω-space with distinguished point e. Then $F(X)$ is a k_ω-space and $F(X)$ has k_ω-decomposition $F(X) = \bigcup_n gp_n(X_n)$, where $gp_n(X_n)$ is the set of words of length not exceeding n in the subgroup generated by X_n.*

REMARK. It is known [2] that every k_ω-topological group is a complete topological group.

DEFINITION. Let X be a k_ω-space, and let $Y = \cup Y_n$ be a closed k_ω-subspace of $F(X)$. Then Y is said to be *regularly situated* with respect to X if for each natural number n there is an integer m such that $\mathrm{gp}(Y) \cap \mathrm{gp}_n(X_n) \subseteq \mathrm{gp}_m(Y_m)$.

THEOREM B [7]. *If X is a k_ω-space and Y is a closed subset of $F(X)$ such that $Y \setminus \{e\}$ is a free algebraic basis for $gp(Y)$, and Y is regularly situated with respect to X, then $gp(Y)$ is $F(Y)$*

RELATIVE CW-COMPLEXES

Our starting point will be the main result of [6]:

THEOREM C. *If for some positive integer n, $F(B^n)$ has $F(X)$ as a closed topological subgroup, then $F(B^n)$ also has $F(X \sqcup_f B^n)$ as a closed topological subgroup, where $f : S^{n-1} \to X$ is any continuous map, and S^{n-1} is regarded as the boundary of B^n.*

This theorem can be applied repeatedly to show that we can adjoin to X a finite number of cells of dimension up to n. (See [6].) In particular this yields the fact that $F(B^n)$ contains $F(Y)$, for Y any finite cell complex of dimension at most n. We would like to remove the 'finiteness' condition. Of course, as $F(Y)$ is to be closed in $F(B^n)$, Y must be a k_ω-space and so is not an uncountable CW-complex. Thus the best possible result would be to embed $F(Y)$, for Y any countable CW-complex. Indeed, this follows from our main theorem.

THEOREM 1. *Let n be a positive integer and Y the space obtained from a space X by attaching a countable number of cells of dimension at most n and giving Y the weak topology* [1]. *If $F(X)$ is a closed topological subgroup of $F(B^n)$, then $F(Y)$ is also a closed topological subgroup of $F(B^n)$.*

PROOF: Let $\{E_i\}_{i=0}^{\infty}$ be a countably infinite family of pairwise disjoint closed balls of dimension n in B^n. For each i, put $G_i = i(E_i - x_i)$, where $x_i \in E_i$. Observe that each G_i is a subspace of $F(B^n)$ homeomorphic to B^n, and $G_i \cap G_j = \{e\}$, for $i \neq j$. Also, by Theorem A, $G = \bigcup_{i=0}^{\infty} G_i$ is a k_ω-space with k_ω-decomposition $G = \bigcup_m \left(\bigcup_{i=0}^{m} G_i\right)$. Further, it is easily seen that the k_ω-space G is regularly situated with respect to B^n. So $\mathrm{gp}(G) = F(G)$ and is a k_ω-space with decomposition $F(G) = \bigcup_m \mathrm{gp}_m\left(\bigcup_{i=0}^{m} G_i\right)$.

Let the space Y be obtained by attaching B_j via maps f_j, $j = 1, 2, \dots$, where each B_j is a cell of dimension $\leqslant n$.

Without loss of generality we assume that $F(X)$ is a closed subgroup of $F(G_0)$ which is itself a closed subgroup of $F(B^n)$. Let $X_1 = X \sqcup_{f_1} B_1$, $X_2 = X_1 \sqcup_{f_2} B_2$, $\dots$, $X_m = X_{m-1} \sqcup_{f_m} B_m$, $\dots$. By Theorem C, (or its Corollary 5 [6] if the dimension of B_1, $\dim(B_1)$, is less than n), there is a topological group isomorphism h_1 of $F(X_1)$ onto its image in $F(G_0 \cup G_1)$. Now assume that h_{m-1} is a topological group isomorphism of $F(X_{m-1})$ into $F\left(\bigcup_{i=0}^{m-1} G_i\right)$. Define $h_m : X_m \to F\left(\bigcup_{i=1}^{m} G_i\right)$ as follows:

$$h_m(x) = \begin{cases} h_{m-1}(x), & x \in X_{m-1} \\ p_m(x) + s_m r_m(x), & x \in B_m \end{cases}$$

where, as in the proof of the Theorem in [6],

(i) $p_m : B_m \to F\left(\bigcup_{i=1}^{m-1} G_i\right)$ is induced by $f_m : \partial B_m \to F\left(\bigcup_{i=1}^{m-1} G_i\right)$, as $F\left(\bigcup_{i=1}^{m-1} G_i\right)$ is contractible relative to the identity, e;

(ii) $r_m : B_m \to S^{\dim(B_m)}$ is a continuous function which maps ∂B_m to $x_0 \in S^{\dim(B_m)}$, maps no other point to x_0, and is one-to-one on $B^m \backslash S^{\dim(B_m)}$;

(iii) $s_m : S^{\dim(B_m)} \to F(G_m)$ is an embedding which extends to a topological isomorphism of $F(S^{\dim(B_m)})$ into $F(G_m)$.

Again, by Theorem C or its Corollary 5 [6], h_m extends to a topological isomorphism of $F(X_m)$ into $F(G_m)$ with $h_m(F(X_m))$ closed in $F(G_m)$.

Let $h : Y \to F(G)$ be defined by $h(y) = h_m(y)$, where $y \in X_m$, for some m. Obviously h is well-defined and one-to-one. As Y has the weak topology with respect to the X_m and each h_m is continuous, h is continuous. We now show that h is a closed mapping. Let A be a closed subset of Y. To show $h(A)$ is closed it suffices to prove that each $h(A) \cap \mathrm{gp}_m\left(\bigcup_{i=0}^{m} G_i\right)$ is closed. But

$$h(A) \cap \mathrm{gp}_m\left(\bigcup_{i=0}^{m} G_i\right) = h(A) \cap h(X_m) = h_m(A \cap X_m)$$

which is closed in $F(G_m)$ and hence also in $F(G)$, as $F(G_m)$ is a k_ω-group and therefore complete. So h is a homeomorphism of Y onto its image.

It remains to show that $h(Y)$ is regularly situated with respect to B^n. As $h(Y)$ is closed in $F(G)$ it is closed in $F(B^n)$ and so has k_ω-decomposition

$$h(Y) = \bigcup_m [h(Y) \cap \mathrm{gp}_m(B^n)].$$

If ℓ is any positive integer,

$$\mathrm{gp}(h(Y)) \cap \mathrm{gp}_\ell(B^n) \subseteq \mathrm{gp}_\ell(h(Y) \cap \mathrm{gp}_\ell(B^n)),$$

and so $h(Y)$ is regularly situated with respect to B^n. Hence $\mathrm{gp}(h(Y))$ is topologically isomorphic to $F(Y)$ and, being a k_ω-group, is a closed subgroup of $F(B^n)$. □

Recall that a *relative countable CW-complex* (Y, X) is the space Y obtained from a space X as follows: the X_0-skeleton consists of X and a countable number of discrete points; the X_1-skeleton consists of X_1 with a countable number of 1-cells attached to it; and so on. Then $Y = \bigcup_{i=0}^{\infty} X_i$ and has the weak topology. A *relative countable CW-complex of dimension* n is one in which the highest dimension of the attached cells is n.

Apply Theorem 1 successively a finite number of times. At each stage attach a countable number of cells of the same dimension. At each application we attach cells of higher dimension than those previously attached. We obtain:

THEOREM 2. *Let (Y, X) be a relative countable CW-complex of dimension n. If $F(X)$ is a closed subgroup of $F(B^n)$, then $F(Y)$ is also a closed subgroup of $F(B^n)$.*

As a special case of Theorem 2, with X a singleton, we obtain:

THEOREM 3. *If X is a countable CW-complex of dimension n, then $F(X)$ is a closed subgroup of $F(B^n)$.*

REMARK. Actually Theorem 1 proves much more than we used in Theorems 2 and 3; namely that the cells do not have to be attached to X but can be attached to X

and cells previously attached; moreover, $\dim(B_1), \dim(B_2), \ldots$ does not have to be a constant sequence, nor a decreasing sequence nor an increasing sequence. So we call a space Y obtained from X by attaching a countable number of cells and with the weak topology a *pseudo relative countable CW-complex*. So Theorem 1 says that if $F(B^n)$ contains $F(X)$ as a closed subgroup, then $F(B^n)$ also contains $F(Y)$ for Y any pseudo relative countable CW-complex of dimension not greater than n. In particular we have:

THEOREM 4. *If Y is a pseudo countable CW-complex of dimension $\leqslant n$, then $F(B^n)$ contains $F(Y)$ as a closed subgroup.*

REMARK. Of course, up to homotopy, every pseudo countable CW-complex is a countable CW-complex, but this has no apparent significance. For example the free Abelian topological group on the Hilbert cube is contractible [6] and so has the homotopy type of a singleton, but cannot be embedded in any $F(B^n)$ as a closed subgroup. This is so since every compact subspace of $F(B^n)$ lies in $\mathrm{gp}_m(B^n)$ for some m, and so has finite dimension.

Theorem 1 has another obvious generalization, which is proved in the same way as Theorem 1 except that we use Corollary 5 of [6] instead of the Theorem in [6].

THEOREM 5. *Let n be a positive integer and Y the space obtained from a space X by attaching a countable number of closed subsets of cells of dimension at most n and giving Y the weak topology. If $F(X)$ is a closed subgroup of $F(B^n)$ then $F(Y)$ is also a closed subgroup.*

Now we note Cairns and Whitney proved that any differentiable manifold of dimension n has a countable triangulation with simplexes of dimension at most n; that is, it is homeomorphic to a countable CW-complex of dimension n. (See [9], pp.124–135.) Thus we obtain:

THEOREM 6. *Let Y be a differentiable manifold of dimension $\leqslant n$. Then $F(Y)$ is a closed subgroup of $F(B^n)$.*

Theorem 6 should be contrasted with the fact that there exist n-dimensional manifolds not embeddable in B^n. For example, Theorem 6 implies that the free Abelian topological group on a torus or on S^2 can be embedded in $F(B^2)$ while neither the torus nor S^2 can be embedded in B^2.

Finally we remark that the method used in Theorem 1 also carries over to the non-Abelian case. But the free topological group on B^1 contains the free topological group on B^n for each positive integer n [8]. Thus we obtain:

THEOREM 7. *Let (Y, X) be a relative countable CW-complex. If the free topological group on B^1 has the free topological group on X as a closed subgroup, then it also has the free topological group on Y as a closed subgroup.*

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