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Congruences on distributive pseudocomplemented lattices

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Let $B_0 \subset B_1 \subset B_2 \subset \ldots \subset B_n \subset \ldots \subset B_\omega$ be all the non-trivial varieties of distributive pseudocomplemented lattices $(L; \vee, \wedge, *, 0, 1)$ considered as algebras of type $(2, 2, 1, 0, 0)$. A subset J of such an algebra L is a congruence-kernel if and only if it is a lattice-ideal and $x^{**} \in J$ for each $x \in J$. The smallest congruence having J as its kernel is $\Theta(J)$, where $a \equiv b \left(\Theta(J)\right)$, $(a, b \in L)$ if and only if $a \wedge c^* = b \wedge c^*$ for some $c \in J$. For given $0 \leq n \leq \omega$, let $\Sigma_n(J)$ be the smallest congruence having J as its kernel and such that the associated quotient algebra is in B_n. Of course, $\Sigma_\omega(J) = \Theta(J)$ and the main result of this paper shows that for $1 \leq n < \omega$,

$$\Sigma_n(J) = \cap\{\Theta(P_1 \cap P_2 \cap \ldots \cap P_n) : J \subseteq P_1, P_2, \ldots, P_n \subset L \text{ are minimal prime ideals}\}.$$

It is also shown that $\Sigma_\omega(J)$ is the smallest congruence on the lattice $(L; \vee, \wedge, 0, 1)$ having J as its kernel for each congruence kernel J if and only if $L \in B_1$. On the other hand, for any congruence kernel J, $\Sigma_0(J)$ is always the largest congruence on the lattice $(L; \vee, \wedge, 0, 1)$ having J as its kernel and so $\Sigma_0(J)$ is the unique congruence with J as its kernel and such that quotient algebra is boolean.

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1. Congruence-kernels

A congruence Θ on a distributive pseudocomplemented lattice $(L; \vee, \wedge, *, 0, 1)$ is a congruence of the lattice $(L; \vee, \wedge, 0, 1)$ which also has the substitution property for the $*$-operation. At times it will be convenient to refer to such a congruence as a $*$-congruence. For such a congruence Θ, $\ker(\Theta) = \{x \in L : x \equiv 0(\Theta)\}$ and $\operatorname{coker}(\Theta) = \{x \in L : x \equiv 1(\Theta)\}$ are called, respectively, the *kernel* and *cokernel* of Θ. A subset J of L is called a *congruence-kernel* (respectively *congruence-cokernel*) if $J = \ker(\Theta)$ (respectively $J = \operatorname{coker}(\Theta)$) for some $*$-congruence Θ on L.

The purpose of this section is to characterize the congruence-kernels of a given algebra $(L; \vee, \wedge, *, 0, 1)$ and also to describe the smallest and largest congruences on L which have a given congruence-kernel as their common kernel. The necessary prerequisite information on the variety of all distributive pseudocomplemented lattices can be obtained from Chapter 3 of Grätzer's recent book [4]. We begin with some preliminary results on distributive lattices which will be useful both in this section and in the later sections.

Let J be an ideal in a distributive lattice L. Then a prime ideal P in L is called a *minimal prime ideal belonging to* J if P is a minimal member of the set of all prime ideals in L which contain J. If L has a smallest member, 0, then a *minimal prime ideal* of L is simply a minimal prime ideal belonging to the zero-ideal $(0]$. The following lemma summarizes some useful information which appears in the literature.

LEMMA 1.1. *The following results hold in a distributive lattice* L.

(i) *A prime ideal* P *containing an ideal* J *is a minimal prime ideal belonging to* J *if and only if, for each* $x \in P$, *there exists* $y \in L \backslash P$ *such that* $x \wedge y \in J$.

(ii) *Suppose that* $0 \in L$ *and that* Q *is a prime ideal in* L. *Then,* $0(Q) = \{x \in L : x \wedge y = 0$ *for some* $y \in L \backslash Q\}$ *is the intersection of all the minimal prime ideals of* L *which are contained in* Q.

(iii) *If* L *is pseudocomplemented then a prime ideal* P *is a minimal prime ideal if and only if* $x^{**} \in P$ *for each* $x \in P$; *and the set of all minimal prime ideals of* L *endowed with the*

hull-kernel topology, is a compact Hausdorff space.

Proof. *(i)* follows from [7, Lemma 3.1] and the distributivity of L.

(ii) is established in [1, Proposition 2.2].

The first part of *(iii)* follows from *(i)*, see also [4, Lemma 5, p. 169], while the second part is a consequence of [9, Proposition 3.2]. □

If J is an ideal in a distributive lattice L then $\Theta_{\mathrm{Lat}}(J)$ and $R(J)$ will be used to respectively denote the smallest congruence and the largest congruence on L possessing J as a congruence class. It is well-known that $x \equiv y \ (\Theta_{\mathrm{Lat}}(J))$, $(x, y \in L)$, if and only if $x \vee a = y \vee a$ for some $a \in J$, while $x \equiv y \ (R(J))$, $(x, y \in L)$, if and only if, for any $b \in L$, $x \wedge b \in J$ if and only if $y \wedge b \in J$. For detailed information on $\Theta_{\mathrm{Lat}}(J)$ and $R(J)$ we refer respectively to [4] and [10]. In [10, Proposition 4.1] it is shown that if $1 \in L$ then $\{x \in L : x \equiv 1 (R(J))\}$ is the filter $D(J)$ where $D(J) = \{a \in L : J{:}a = J\}$ and $J{:}a$ is the ideal $\{b \in L : a \wedge b \in J\}$. When $0 \in L$, R will be used to denote $R((0])$ and D will be used to denote $D((0])$. If L is pseudocomplemented then $x \equiv y \ (R)$, $(x, y \in L)$, if and only if $x^{**} = y^{**}$ and $D = \{x \in L : x^{**} = 1\}$, the filter of dense elements.

LEMMA 1.2. *Let $h(J)$ denote the set of all minimal prime ideals belonging to a given ideal J in a distributive lattice L. Then*

(i) $R(J) = \cap\{R(P) : P \in h(J)\}$, *and*

(ii) $\Theta_{\mathrm{Lat}}(J) = \cap\{\Theta_{\mathrm{Lat}}(P) : P \in h(J)\}$.

Proof. *(i)* Suppose that $x, y \in L$ and $x \equiv y \ (R(J))$. If $x \in P \in h(J)$ then, according to Lemma 1.1 *(i)*, $x \wedge z \in J$ for some $z \in L \backslash P$. Then $y \wedge z \in J$ and so $y \in P$. By symmetry, we conclude that $x \in P$ if and only if $y \in P$, that is, $x \equiv y \ (R(P))$. Thus, $R(J) \subseteq \cap\{R(P) : P \in h(J)\}$. But $J = \cap\{P : P \in h(J)\}$ and so J is a congruence class of L modulo $\cap\{R(P) : P \in h(J)\}$. Hence $\cap\{R(P) : P \in h(J)\} \subseteq R(J)$. Part *(i)* is now proved.

(ii) For any $P \in h(J)$, it is clear that $\Theta_{\mathrm{Lat}}(J) \subseteq \Theta_{\mathrm{Lat}}(P)$.

Suppose that $x, y \in L$ and $x \nmid y \ (\Theta_{\mathrm{Lat}}(J))$ and consider $F = \{a \in L : x \vee a = y \vee a\}$. Then F is non-empty since $x \vee y \in F$, and F is a filter. By the hypothesis on x and y, it follows that J and F are disjoint. Using the Birkhoff-Stone Theorem and the dual form of Zorn's Lemma, it follows that there exists $Q \in h(J)$ such that Q and F are disjoint. Hence, $x \nmid y \ (\Theta(Q))$. The equality asserted in *(ii)* is now readily established. □

If F is a filter in an arbitrary lattice L then the relation $\Psi(F)$ defined by: $x \equiv y \ (\Psi(F))$ $(x, y \in L)$ if and only if $x \wedge f = y \wedge f$ for some $f \in F$, is always an equivalence relation. When L is a distributive lattice, $\Psi(F)$ is the smallest congruence having F as a congruence class. Special cases of the following lemma are well-known, for example, [5, Lemma 7, p. 36] and [6, Theorem 2]; we include a simple direct proof.

LEMMA 1.3. *For any two filters F_1 and F_2 in a lattice L, the equivalence relations $\Psi(F_1)$ and $\Psi(F_2)$ are permutable.*

Proof. Suppose that $x, y \in L$ and that $x \equiv y \ (\Psi(F_1) \circ \Psi(F_2))$. Then, $x \wedge f_1 = z \wedge f_1$ and $z \wedge f_2 = y \wedge f_2$ for some $z \in L$, $f_1 \in F_1$ and $f_2 \in F_2$. Thus, $x \wedge f_1 \wedge f_2 = y \wedge f_1 \wedge f_2$. Put $w = (x \wedge f_2) \vee (y \wedge f_1)$. Then

$$x = x \vee (x \wedge f_1 \wedge f_2) = x \vee (y \wedge f_1 \wedge f_2) \equiv (x \wedge f_2) \vee (y \wedge f_1) = w \ (\Psi(F_2)),$$

and $w = (x \wedge f_2) \vee (y \wedge f_1) \equiv (x \wedge f_1 \wedge f_2) \vee y = y \ (\Psi(F_1))$. That is, $x \equiv y \ (\Psi(F_2) \circ \Psi(F_1))$. The lemma now follows from symmetry. □

The next result describes congruence-cokernels in a distributive pseudocomplemented lattice; it will be useful throughout the entire paper.

PROPOSITION 1.4. *Let $(L; \vee, \wedge, *, 0, 1)$ be a distributive pseudocomplemented lattice. A subset F of L is a congruence-cokernel if and only if F is a filter in L. Moreover, the smallest $*$-congruence having a given filter F as its cokernel is $\Psi(F)$.*

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Proof. It is sufficient to prove that, for a given filter F, $\Psi(F)$ has the substitution property for the *-operation. Thus suppose that $x, y \in L$ and $x \equiv y\left(\Psi(F)\right)$, that is, $x \wedge f = y \wedge f$ for some $f \in F$. Then $x^{**} \wedge f^{**} = (x \wedge f)^{**} = (y \wedge f)^{**} = y^{**} \wedge f^{**}$ so that $x^* \underline{\vee} f^* = y^* \underline{\vee} f^*$ in the skeleton $\left(S(L); \underline{\vee}, \wedge, *, 0, 1\right)$ of L. Then $x^* \wedge f^{**} = y^* \wedge f^{**}$, since $S(L)$ is a boolean algebra, and hence $x^* \wedge f = y^* \wedge f$. □

If J is an ideal in a distributive psuedocomplemented lattice L, let $J_* = \{x \in L : x \geq a^* \text{ for some } a \in J\}$. It is easy to see that J_* is a filter. We now come to our first characterization of congruence-kernels.

THEOREM 1.5. *Let J be a non-empty subset of a distributive pseudo-complemented lattice $(L; \vee, \wedge, *, 0, 1)$. Then the following conditions are equivalent:*

(i) J is a congruence-kernel,

(ii) J is an ideal of the lattice L and $x^{**} \in J$ for each $x \in J$,

(iii) J is an ideal of the lattice L and each minimal prime ideal belonging to J is a minimal prime ideal,

(iv) J is an intersection of minimal prime ideals of the lattice L.

Moreover, if J satisfies any of the equivalent conditions (i)-(iv) then the smallest congruence on $(L; \vee, \wedge, *, 0, 1)$ having J as its kernel is $\Theta(J)$, where $x \equiv y \left(\Theta(J)\right)$ $(x, y \in L)$ if and only if $x \wedge a^* = y \wedge a^*$ for some $a \in J$. Thus, $\Theta(J) = \Psi(J_*)$.

Proof. *(i) ⇒ (ii).* Suppose that $J = \ker(\Theta)$ for some *-congruence Θ. Then J is certainly a lattice-ideal and, for $x \in J$, $x \equiv 0\ (\Theta)$, so that $x^{**} \equiv 0^{**} = 0\ (\Theta)$, and hence *(ii)* holds for J.

(ii) ⇒ (i). From *(ii)* and Proposition 1.4, $\Psi(J_*)$ is a *-congruence. Because $x^{**} \in J$ for $x \in J$, $\ker\left(\Psi(J_*)\right) = J$. If Φ is a *-congruence with $\ker(\Phi) = J$ then $x \wedge a^* = y \wedge a^*$ for $a \in J$ implies $x \equiv y\ (\Phi)$ since $a^* \equiv 0^* = 1\ (\Phi)$. Thus, *(ii) ⇒ (i)*, and we have also described the smallest *-congruence having J as its kernel.

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The implication *(iv)* $\Rightarrow$ *(ii)* follows from Lemma 1.1 *(iii)* so that it remains to establish that *(ii)* $\Rightarrow$ *(iii)* and *(iii)* $\Rightarrow$ *(iv)*.

(ii) $\Rightarrow$ *(iii)*. Let J satisfy *(ii)* and suppose that P is a minimal prime ideal belonging to J. Let $x \in P$. By Lemma 1.1 *(i)*, $x \wedge z \in J$ for some $z \in L \backslash P$. Then $x^{**} \wedge z^{**} = (x \wedge z)^{**} \in J$ and so $x^{**} \in P$, since P is prime and $z \leq z^{**}$. Then, by Lemma 1.1 *(iii)*, P is a minimal prime ideal.

(iii) $\Rightarrow$ *(iv)*. Each ideal in a distributive lattice is the intersection of all the minimal prime ideals belonging to it. Hence *(iv)* follows from *(iii)*. □

THEOREM 1.6. *Let J be a given congruence-kernel in a distributive pseudocomplemented lattice $(L; \vee, \wedge, {}^*, 0, 1)$. The following conditions relating to an equivalence relation Φ on L are equivalent:*

(i) *Φ is the largest *-congruence such that $J = \ker\Phi$,*

(ii) $\Phi = R(J)$,

(iii) *for any $x, y \in L$, $x \equiv y \ (\Phi)$ if and only if $x^{**} \wedge a^* = y^{**} \wedge a^*$ for some $a \in J$,*

(iv) $\Phi = \Theta(J) \vee R$ *in the lattice of (*-)congruences,*

(v) *for any $x, y \in L$, $x \equiv y \ (\Phi)$ if and only if $x \wedge (b \vee b^*) \wedge a^* = y \wedge (b \vee b^*) \wedge a^*$ for some $b \in L$ and $a \in J$,*

(vi) *Φ is the smallest *-congruence with $J = \ker\Phi$ and such that L/Φ is a boolean algebra,*

(vii) *Φ is the unique *-congruence with $J = \ker\Phi$ and such that L/Φ is a boolean algebra.*

Proof. *(i)* $\Leftrightarrow$ *(ii)*. By Theorem 1.5, each minimal prime ideal belonging to J is a minimal prime ideal. Hence, by Lemma 1.1, $R(P)$ is a *-congruence for each $P \in h(J)$ and so Lemma 1.2 implies that $R(J)$ is a *-congruence. It is now evident that *(i)* and *(ii)* are equivalent.

(iii) $\Leftrightarrow$ *(v)*. Because $(b \vee b^*)^{**} = 1$ for each $b \in L$, *(iii)* follows from *(v)*. On the other hand suppose that $x^{**} \wedge a^* = y^{**} \wedge a^*$ for some $a \in J$. Since $x \vee x^*$ and $y \vee y^*$ are dense,

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$(x \vee x^*) \wedge (y \vee y^*) = b \vee b^*$ for some $b \in L$ and then

$$x \wedge (b \vee b^*) \wedge a^* = \left(x^{**} \wedge (x \vee x^*)\right) \wedge (b \vee b^*) \wedge a^*$$
$$= x^{**} \wedge (b \vee b^*) \wedge a^* = y^{**} \wedge (b \vee b^*) \wedge a^* = y \wedge (b \vee b^*) \wedge a^* .$$

Thus, *(iii)* and *(v)* are equivalent.

(iv) $\Longleftrightarrow$ *(v)*. Because L is pseudocomplemented, $R = \Psi(D)$, (*cf.* [10, Proposition 5.4]). Thus, from Theorem 1.5,

$$\Theta(J) \vee R = \Psi(J_*) \vee \Psi(D) = \Psi(J_* \vee D) ,$$

where

$$J_* \vee D = \{x \in L : x = b \wedge d, b \in J_*, d \in D\}$$

is the join of J_* and D in the lattice of filters on L. Hence *(iv)* and *(v)* are equivalent.

(ii) $\Longleftrightarrow$ *(iii)*. Because of the equivalence of *(iii)*, *(iv)*, and *(v)* it is clear that the relation Φ of *(iii)* is a *-congruence with $\ker\Phi = J$ and hence $\Phi \subseteq R(J)$. It remains to establish the reverse inequality.

Let P be a minimal prime ideal and suppose that $x \equiv y \ (R(P))$ for given $x, y \in L$. Then $x, y \in P$ or $x, y \in L\backslash P$. In the first case $x \vee y \in P$ and $x^{**} \wedge (x \vee y)^* = x^{**} \wedge x^* \wedge y^* = 0 = y^{**} \wedge (x \vee y)^*$, while in the second case $x \wedge y \in L\backslash P$ so that $(x \wedge y)^* \in P$ and $x^{**} \wedge \left((x \wedge y)^*\right)^* = x^{**} \wedge (x \wedge y)^{**} = x^{**} \wedge y^{**} = y^{**} \wedge \left((x \wedge y)^*\right)^*$. Thus, for any minimal prime ideal P, $x \equiv y \ (R(P))$ $(x, y \in L)$ if and only if $x^{**} \wedge p^* = y^{**} \wedge p^*$ for some $p \in P$.

Now suppose $x, y \in L$ and $x \equiv y \ (R(J))$. Let $h(J) = \{P_\lambda : \lambda \in \Lambda\}$ be the set of all minimal prime ideals containing J. By Lemma 1.2 and Theorem 1.5, $x \equiv y \ (R(P_\lambda))$ for each $\lambda \in \Lambda$. It follows from the preceding paragraph that, for each $\lambda \in \Lambda$, there exists $p_\lambda \in P_\lambda$ such that $x^{**} \wedge p_\lambda^* = y^{**} \wedge p_\lambda^*$. Hence, for each $P_\lambda \in h(J)$, $P_\lambda \in g(p_\lambda^*) = \{Q : Q$ is minimal prime ideal and $p_\lambda^* \notin Q\}$. That is, $\{g(p_\lambda^*) : \lambda \in \Lambda\}$ is an open cover of the closed subset $h(J)$ of the space of minimal prime ideals. Because of Lemma 1.1 *(iii)*, $h(J)$ is compact and so there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$ such that

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$$h(J) \subseteq g\left(p^*_{\lambda_1}\right) \cup g\left(p^*_{\lambda_2}\right) \cup \ldots \cup g\left(p^*_{\lambda_n}\right) = g\left(p^*_{\lambda_1} \vee p^*_{\lambda_2} \vee \ldots \vee p^*_{\lambda_n}\right) .$$

Hence

$$p^{**}_{\lambda_1} \wedge \ldots \wedge p^{**}_{\lambda_n} = \left(p^*_{\lambda_1} \vee \ldots \vee p^*_{\lambda_n}\right)^* \in \cap\left\{Q : Q \in g\left(p^*_{\lambda_1} \vee \ldots \vee p^*_{\lambda_n}\right)\right\}$$

$$\subseteq \cap\{Q : Q \in h(J)\} = J ,$$

by Theorem 1.5. Now $x^{**} \wedge p^*_{\lambda_i} = y^{**} \wedge p^*_{\lambda_i}$ for $i = 1, \ldots, n$ and so

$$x^{**} \wedge \left(p^*_{\lambda_1} \underline{\vee} \ldots \underline{\vee} p^*_{\lambda_n}\right) = y^{**} \wedge \left(p^*_{\lambda_1} \underline{\vee} \ldots \underline{\vee} p^*_{\lambda_n}\right)$$

in $S(L)$. Hence

$$x^{**} \wedge \left(p^{**}_{\lambda_1} \wedge \ldots \wedge p^{**}_{\lambda_n}\right)^* = y^{**} \wedge \left(p^{**}_{\lambda_1} \wedge \ldots \wedge p^{**}_{\lambda_n}\right)^*$$

and $p^{**}_{\lambda_1} \wedge \ldots \wedge p^{**}_{\lambda_n} \in J$. Thus $x \equiv y\ (\Phi)$, where Φ is the congruence of *(iii)*. And *(ii)* and *(iii)* are equivalent.

Thus conditions *(i)* to *(v)* are equivalent.

(v) $\Longleftrightarrow$ *(vi)*. If Φ is the congruence of *(v)* then $\ker(\Phi) = J$ and $b \vee b^* \in \operatorname{coker}(\Phi)$ for each $b \in L$ and so L/Φ is boolean. It is clear that Φ is the smallest *-congruence with these properties and so *(v)* and *(vi)* are equivalent.

(vi) $\Longleftrightarrow$ *(vii)*. Since *(i)* and *(vi)* are equivalent, *(vii)* is equivalent to *(vi)*.

The theorem has now been established. □

The preceding theorem shows that the congruence $R(J)$ has a very simple description when J is a congruence-kernel. In fact many of the properties of the congruence $R(J)$ which are implied by that theorem are characteristic of congruence-kernels.

THEOREM 1.7. *The following conditions on an ideal J of a distributive pseudocomplemented lattice L are equivalent:*

(i) J is a congruence-kernel,

*(ii) $R(J)$ is a *-congruence,*

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(iii) *for any* $x, y \in L$, $x \equiv y \ (R(J))$ *if and only if* $x^{**} \wedge a^{*} = y^{**} \wedge a^{*}$ *for some* $a \in J$,

(iv) *for any* $x, y \in L$, $x \equiv y \ (R(J))$ *if and only if* $x \wedge (b \vee b^{*}) \wedge a^{*} = y \wedge (b \vee b^{*}) \wedge a^{*}$ *for some* $b \in L$ *and* $a \in J$,

(v) $R(J) = \Psi(D(J))$,

(vi) $D(J) = D \vee J_{*}$ *in the lattice of filters,*

(vii) $D \subseteq D(J)$,

(viii) *for any filter* F, *the congruences* $R(J)$ *and* $\Psi(F)$ *are permutable,*

(ix) *the congruences* $R(J)$ *and* R *are permutable,*

(x) $R \subseteq R(J)$.

Proof. The equivalence of conditions *(i)*, *(ii)*, *(iii)* and *(iv)* is an easy consequence of Theorems 1.6 and 1.5.

(iv) $\Rightarrow$ *(v)*. Condition *(iv)* implies that $R(J) = \Psi(\operatorname{coker} R(J))$, and, as has been previously mentioned, $\operatorname{coker}(R(J)) = D(J)$.

(v) $\Rightarrow$ *(ii)*. This follows from Proposition 1.4.

(v) $\Rightarrow$ *(vi)* and *(vi)* $\Rightarrow$ *(vii)* are obvious.

(vii) $\Rightarrow$ *(i)*. Suppose that $D \subseteq D(J)$ and let $x \in J$. Then $x \vee x^{*} \in D(J)$ and so $x^{**} \wedge (x \vee x^{*}) = x$ implies $x^{**} \in J : x = J$. Thus *(i)* holds.

(v) $\Rightarrow$ *(viii)* follows from Lemma 1.3.

(viii) $\Rightarrow$ *(ix)* since $R = \Psi(D)$ holds in any psuedocomplemented lattice.

(ix) $\Rightarrow$ *(i)*. Let $x \in J$. Then $0 \equiv x \ (R(J))$ and $x \equiv x^{**} \ (R)$ so that $0 \equiv x^{**} \ (R(J) \circ R)$. By *(ix)*, $0 \equiv z \ (R)$ and $z \equiv x^{**} \ (R(J))$. Hence $z = 0$ and so $x^{**} \in J = \ker R(J)$.

(ii) $\Rightarrow$ *(x)* is clear and *(x)* $\Rightarrow$ *(i)* since $x \equiv x^{**} \ (R)$ for any $x \in L$. The theorem is proved. □

Some other consequence of Theorem 1.5 will be given in the next sections.

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2. The congruences $\Sigma_n(J)$

Because of Theorem 1.5, the congruence-kernels of a distributive pseudocomplemented lattice are precisely its α-ideals, in the sense of [2]. The following lemma is thus an easy consequence of [2, Propositions 3.2, 3.3]; it is easy to provide a direct proof and so details are omitted.

LEMMA 2.1. *The set $K(L)$ of all congruence-kernels of a distributive pseudocomplemented lattice $(L; \vee, \wedge, *, 0, 1)$ is a complete distributive lattice, where the meet is set-theoretic intersection and the join of a family $\{J_\lambda : \lambda \in \Lambda\}$ of congruence-kernels is*

$$\bigvee\{J_\lambda : \lambda \in \Lambda\} = \left\{x \in L : x \leq \left(a_{\lambda_1} \vee \ldots \vee a_{\lambda_n}\right)^{**} \right.$$

$$\left. \text{for some } a_{\lambda_1} \in J_{\lambda_1}, \ldots, a_{\lambda_n} \in J_{\lambda_n} \text{ and } \lambda_1, \ldots, \lambda_n \in \Lambda\right\}.$$

Moreover, the maps $J \to \{b \in S(L) : b = x^{**} \text{ for some } x \in J\}$ and $I \to \{x \in L : x^{**} \in I\}$ are mutually inverse lattice-isomorphisms between $K(L)$ and the lattice of ideals of the skeleton $(S(L); \underline{\vee}, \wedge, *, 0, 1)$. □

LEMMA 2.2. *Let Θ be a $*$-congruence on a distributive pseudocomplemented lattice $(L; \vee, \wedge, *, 0, 1)$ and let $\theta : L \to L/\Theta$ be the canonical epimorphism. Then the set-function induced by θ and the inverse image map $\theta^{\leftarrow}$ are mutually inverse lattice isomorphisms between the interval $[\ker\Theta, L] = \{J \in K(L) : \ker\Theta \subseteq J\}$ of the lattice $K(L)$ of congruence-kernels of L and the lattice $K(L/\Theta)$ of congruence-kernels of L/Θ. Hence the map $P \to \theta(P)$ is a bijection of the set of minimal prime ideals of L which contain $\ker\Theta$ onto the set of minimal prime ideals of L/Θ.*

Proof. Let $J \in K(L)$ be such that $\ker\Theta \subseteq J$ and suppose that $x \in \theta^{\leftarrow}\theta(J)$. Hence $\theta(x) = \theta(a)$ for some $a \in J$, and so $\theta(x \wedge a^*) = \theta(x) \wedge \theta(a)^* = 0$ implies that $x \wedge a^* \in \ker\Theta \subseteq J$. If P is a minimal prime ideal containing J it follows from Lemma 1.1 that $x \in P$. Because of Theorem 1.5, $x \in J$ and so $\theta^{\leftarrow}\theta(J) = J$.

It is now clear that the first statement of the lemma holds and from this we immediately obtain the second statement, because Theorem 1.5 and

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Lemma 2.1 imply that the minimal prime ideals of L are precisely the meet-irreducible elements of $K(L)$. □

REMARK. Lemma 2.2 gives an estimate of the way in which the lattice of congruence-kernels fails to determine the lattice of *-congruences. Of course, the map $\Theta \to \ker(\Theta)$ is a bijection (lattice-isomorphism) of the lattice of *-congruences onto the lattice of congruence-kernels of L if and only if $(L; \vee, \wedge, *, 0, 1)$ is precisely the skeleton $(S(L); \underline{\vee}, \wedge, *, 0, 1)$ that is, if and only if L is a boolean algebra. This is because $\ker(R) = (0]$ if and only if $x = x^{**}$ for each $x \in L$.

Let $\{B_n : 0 \leq n \leq \omega\}$ be all the non-trivial varieties of distributive pseudocomplemented lattices (*cf.* [4, Chapter 3] and [8]). Thus B_0 is the class of boolean algebras, B_1 is the class of Stone lattices and B_ω is the class of all distributive pseudocomplemented lattices. We will need the following characterization of B_n $(1 \leq n < \omega)$ due to Lee [8, Theorems 4, 5].

LEMMA 2.3. *Let $(L; \vee, \wedge, *, 0, 1)$ be a distributive pseudocomplemented lattice and let $1 \leq n < \omega$ be a positive integer. Then the following conditions are equivalent:*

(i) $L \in B_n$,

(ii) *any* $x_1, x_2, \ldots, x_n \in L$ *satisfy the identity*

$$\left(\bigwedge_{i=1}^{n} x_i\right)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots \wedge x_n)^* = 1 ,$$

(iii) *each prime ideal in L contains at most n distinct minimal prime ideals.* □

LEMMA 2.4. *Let $(L; \vee, \wedge, *, 0, 1)$ be a distributive pseudocomplemented lattice. Then*

(i) *for any n (not necessarily distinct) minimal prime ideals $P_1, P_2, \ldots, P_n$ in L, $L/\Theta(P_1 \cap P_2 \cap \ldots \cap P_n) \in B_n$,*

(ii) *for any $1 \leq n < \omega$, $L \in B_n$ implies that*

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$\cap\{\theta(P_1 \cap P_2 \cap \ldots \cap P_n) : P_1, P_2, \ldots, P_n$ *are* n *minimal prime ideals*$\} = \omega_L$

(the smallest congruence on L *).*

Proof. *(i)* $L/\theta(P_1 \cap P_2 \cap \ldots \cap P_n)$ has, at most, n distinct minimal prime ideals. This follows from Lemma 2.2 since $P_1, P_2, \ldots, P_n$ are the only minimal prime ideals of L containing $P_1 \cap P_2 \cap \ldots \cap P_n$. Thus part *(i)* follows from Lemma 2.3.

(ii) The dual of Lemma 1.2 *(ii)* implies that

$$\cap\{\Psi(F) : F \text{ is a prime filter in } L\} = \omega_L .$$

Recall that ideal P is prime if and only if $L\backslash P$ is a prime filter. By Lemma 1.4, $\Psi(L\backslash P)$ is a $*$-congruence and it is clear that $\ker(\Psi(L\backslash P)) = O(P)$ and so $\theta(O(P)) \subseteq \Psi(L\backslash P)$. Combining this information we obtain: $\cap\{\theta(O(P)) : P \text{ is a prime ideal}\} = \omega_L$. Then, part *(ii)* is an easy consequence of Lemma 2.3 *(iii)* and Lemma 1.1 *(ii)*. □

Let J be a congruence-kernel in a distributive pseudocomplemented lattice L. For any $0 \leq n \leq \omega$, let $\Sigma_n(J)$ denote the smallest of all $*$-congruences Θ on L such that $\ker(\Theta) = J$ and $L/\Theta \in B_n$, so that $L/\Sigma_n(J)$ is the maximal homomorphic image of L which lies in B_n. Even though $\Sigma_\omega(J) = \theta(J)$ and $\Sigma_0(J) = R(J)$ (Theorem 1.6), it will at times be convenient to use that notation $\Sigma_\omega(J)$ and $\Sigma_0(J)$. We will also use Σ_n in place of $\Sigma_n((0])$. The existence and description of $\Sigma_n(J)$ is the aim of the next theorem. Granting the existence it is clear that

$$\theta(J) = \Sigma_\omega(J) \subseteq \ldots \subseteq \Sigma_n(J) \subseteq \Sigma_{n-1}(J) \subseteq \ldots \subseteq \Sigma_1(J) \subseteq \Sigma_0(J) = R(J) .$$

For $1 \leq n < \omega$, let D_n be the filter of L generated by all elements of the form

$$\left(\bigwedge_{i=1}^{n} x_i\right)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots \wedge x_n)^*$$

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where $x_1, \ldots, x_n \in L$. Thus

$$D_n = \Big\{ x \in L : x \geq \bigwedge_{j=1}^{m} \Big(\Big(\bigwedge_{i=1}^{n} x_{ij} \Big)^* \vee \bigvee_{i=1}^{n} (x_{1j} \wedge \ldots \wedge x_{i-1j} \wedge x_{ij}^* \wedge x_{i+1j} \wedge \ldots \wedge x_{nj})^* \Big)$$

$$\text{for some } x_{ij}, (i = 1, \ldots, n, j = 1, \ldots, m), \text{ in } L \Big\} .$$

For any $x_1, \ldots, x_n \in L$, it is easy to see that

$$(x_1 \wedge \ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_{i-1} \wedge x_i^* \wedge \ldots \wedge x_n)^* \geq (x_1 \wedge \ldots \wedge x_n)^* \vee (x_1 \wedge \ldots \wedge x_n)^{**}$$

and so $D_n \supseteq D_1 \supseteq D$ for any $1 \leq n \leq \omega$.

THEOREM 2.5. *Let $1 \leq n < \omega$ be a given positive integer and let J be a congruence-kernel in a distributive pseudocomplemented lattice $(L; \vee, \wedge, *, 0, 1)$. The following conditions relating to a $*$-congruence Φ are equivalent:*

(i) $\Phi = \Sigma_n(J)$,

(ii) *for any $x, y \in L$, $x \equiv y(\Phi)$ if and only if $x \wedge b \wedge a^* = y \wedge b \wedge a^*$, where*

$$b = \bigwedge_{j=1}^{m} \Big(\Big(\bigwedge_{i=1}^{n} b_{ij} \Big)^* \vee \bigvee_{i=1}^{n} (b_{1j} \wedge \ldots \wedge b_{i-1j} \wedge b_{ij}^* \wedge b_{i+1j} \wedge \ldots \wedge b_{nj})^* \Big)$$

for some b_{ij} $(i = 1, \ldots, n, j = 1, \ldots, m)$ in L , and $a \in J$,

(iii) $\Phi = \Psi(D_n \vee J_*)$,

(iv) $\Phi = \Sigma_n \vee \Theta(J)$ *in the lattice of ($*$-)congruences,*

(v)

$$\Phi = \bigcap \{ \Theta(P_1 \cap P_2 \cap \ldots \cap P_n) : P_1, P_2, \ldots, P_n \text{ are (not necessarily distinct) minimal prime ideals in } L \text{ which contain } J \} .$$

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Proof. It is clear that *(ii)* and *(iii)* are equivalent. The congruence Φ given in both is, indeed, a *-congruence (Proposition 1.4) and has J as its kernel since $D_n \subseteq D$. As $D_n \subseteq \operatorname{coker}(\Phi)$, $L/\Phi \in B_n$ in view of Lemma 2.3 *(ii)*. It follows that $\Phi = \Sigma_n(J)$ and so *(i)*, *(ii)* and *(iii)* are equivalent.

The equivalence of *(iii)* and *(iv)* follows from Theorem 1.5 and the equivalence of *(i)* and *(iii)* for the case $J = (0]$.

Now let Φ be the congruence given in *(v)*. The theorem will be established if we can show that $\Phi = \Sigma_n(J)$.

For any n minimal prime ideals $P_1, \ldots, P_n$, $L/\Theta(P_1 \cap \ldots \cap P_n) \in B_n$, by Lemma 2.4 *(i)*, and since B_n is a variety, $L/\Phi \in B_n$, as L/Φ is a subdirect product of algebras in B_n. But $\ker\Phi = J$ (Theorem 1.5) and so $\Sigma_n(J) \subseteq \Phi$ by the definition of $\Sigma_n(J)$. Hence it remains to show that $\Phi \subseteq \Sigma_n(J)$.

Let $\sigma : L \to L/\Sigma_n(J)$ denote the canonical epimorphism and, for congruence $\Theta \supseteq \Sigma_n(J)$, let Θ' denote the congruence on $L/\Sigma_n(J)$ induced by Θ. For $\sigma(x)$, $\sigma(y)$ in $L/\Sigma_n(J)$, $(x, y \in L)$, $x \equiv y\ (\Theta')$ if and only if $x \equiv y\ (\Theta)$; for details of this construction see [3, §11, Chapter 3].

Since $\Sigma_n(J) \subseteq \Phi$, Φ' is a well-defined congruence on $L/\Sigma_n(J)$. Using Theorem 1.5 it follows that

$$\Phi' \subseteq \cap\{\Theta(\sigma(P_1 \cap P_2 \cap \ldots \cap P_n) : P_1, P_2, \ldots, P_n \text{ are minimal primes in } L \text{ containing } J\}.$$

As $\ker(\Sigma_n(J)) = J$, Lemma 2.2 implies that

$$\Phi' \subseteq \cap\{\Theta(Q_1 \cap Q_2 \cap \ldots \cap Q_n) : Q_1, Q_2, \ldots, Q_n \text{ are minimal prime ideals in } L/\Sigma_n(J)\}.$$

Hence, by Lemma 2.4 *(ii)*, $\Phi' = \omega_{L/\Sigma_n(J)}$ and so $\Phi = \Sigma_n(J)$. □

Some easy consequences of the above theorem are:

COROLLARY 2.6. *For any* $0 \leq n < \omega$, $D_{n+1} \subseteq D_n \subseteq D$. □

COROLLARY 2.7. *For any* $0 \leq n \leq \omega$, $\Sigma_0(J) = \Sigma_n(J) \vee \Sigma_0$. □

Proof. Combine Theorems 1.6 and 2.5. □

COROLLARY 2.8. *For any* $0 \leq n$, $m \leq \omega$ *and congruence-kernels* I *and* J, *the congruences* $\Sigma_n(I)$ *and* $\Sigma_m(J)$ *are permutable.*

Proof. Combine Lemma 1.3, Theorem 1.5, Theorem 1.6 and Theorem 2.5. □

COROLLARY 2.9. *For any* $1 \leq n < \omega$ *and congruence-kernel* J, $L/\Theta(J) \in B_n$ *if and only if*

$$\Theta(J) = \bigcap\{\Theta(P_1 \cap P_2 \cap \ldots \cap P_n) : P_1, P_2, \ldots, P_n \text{ are minimal prime ideals containing } J\}.$$

We end this section with the following result.

THEOREM 2.10. *For any fixed* $0 \leq n \leq \omega$, *the map* $J \mapsto \Sigma_n(J)$ *is an injection of the lattice of congruence-kernels into the lattice of *-congruences which preserves arbitrary joins. When* $n = 0$, *the map preserves finite meets.*

Proof. It easily follows from Theorem 1.5 and Lemma 2.1 that $\Theta(\bigvee\{J_\lambda : \lambda \in \Lambda\}) = \bigvee\{\Theta(J_\lambda) : \lambda \in \Lambda\}$ for any set $\{J_\lambda : \lambda \in \Lambda\}$ of congruence-kernels. As a consequence, Theorem 2.5 *(iv)* and Theorem 1.6 *(iv)* imply that $J \to \Sigma_n(J)$ preserves arbitrary joins for any $0 \leq n \leq \omega$. The last statement of the theorem follows from Theorem 1.6 *(iii)*. □

In the next section we will consider conditions on L which ensure that the map $J \mapsto \Sigma_n(J)$ preserves finite meets even when $1 \leq n \leq \omega$.

3. Stone lattices

LEMMA 3.1. *Let* J *be a congruence-kernel in the distributive pseudocomplemented lattice* L. *Then* $\Theta(J) = \Theta_{\mathrm{Lat}}(J)$ *if and only if* $J = \{x \in J : J \vee (x^*] = L\}$.

Proof. Suppose that $\Theta(J) = \Theta_{\mathrm{Lat}}(J)$. Thus, $x \in J$ implies

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$x^* \equiv 1 \ (\Theta_{\mathrm{Lat}}(J))$ and so $a \vee x^* = a \vee 1 = 1$ for some $a \in J$. Consequently, $\{x \in J : J \vee (x^*] = L\} = J$.

Conversely, suppose that $J = \{x \in J : J \vee (x^*] = L\}$. Suppose that $a, b \in L$ are such that $a \equiv b \ (\Theta(J))$. Then, by Theorem 1.5, $a \wedge x^* = b \wedge x^*$ for some $x \in J$. Using the hypothesis imposed on J and working in the lattice of ideals of L, we obtain

$$(a] \vee J = \big((a] \vee J\big) \cap L = \big((a] \vee J\big) \cap \big((x^*] \vee J\big) = \big((a] \cap (x^*]\big) \vee J$$
$$= \big((b] \cap (x^*]\big) \vee J = (b] \vee J .$$

Hence $a \equiv b \ (\Theta_{\mathrm{Lat}}(J))$ and so $\Theta(J) = \Theta_{\mathrm{Lat}}(J)$. □

It should be noted that any ideal J of the distributive pseudo-complemented lattice L is necessarily a congruence-kernel if $J = \{x \in J : J \vee (x^*] = L\}$, and we will refer to such ideals as *special congruence-kernels*. It is easy to see that L is a Stone lattice, that is, $L \in B_1$, if and only if each congruence-kernel is a special congruence-kernel. Using this observation we now come to our final result.

THEOREM 3.2. *The following conditions on a distributive pseudo-complemented lattice* $(L; \vee, \wedge, {}^*, 0, 1)$ *are equivalent:*

(i) L *is a Stone lattice,*

(ii) *each congruence-kernel in* L *is a special congruence-kernel,*

(iii) $\Theta(J) = \Theta_{\mathrm{Lat}}(J)$ *for each congruence-kernel* J,

(iv) $\Theta_{\mathrm{Lat}}(J)$ *is a* **-congruence for each congruence-kernel* J,

(v) $\Theta(P) = \Theta_{\mathrm{Lat}}(P)$ *for each minimal prime ideal* P,

(vi) $\Theta_{\mathrm{Lat}}(P)$ *is a* **-congruence for each minimal prime ideal* P,

(vii) *each minimal prime ideal is a special congruence-kernel,*

(viii) $\cap\{\Theta(P) : P$ *is a minimal prime ideal*$\} = \omega_L$,

(ix) *for any congruence-kernel* J *and any filter* F, $\Theta_{\mathrm{Lat}}(J)$ *and* $\Psi(F)$ *are permutable congruences,*

(x) *for any congruence-kernels* I *and* J, $\Theta(I \cap J) = \Theta(I) \cap \Theta(J)$,

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(xi) $L \in B_n$ *for some* $1 < n < \omega$ *and, given any congruence-kernels* I *and* J, $\Sigma_n(I) \cap \Sigma_n(J) = \Sigma_n(I \cap J)$,

(xii) *for any* $0 \leq n \leq \omega$, *the map* $J \to \Sigma_n(J)$ *is a lattice monomorphism of the lattice of congruence-kernels into the lattice of* $*$*-congruences.*

Proof. The equivalence of conditions *(i)*, *(ii)*, *(iii)* and *(iv)* follows from Lemma 3.1 as does the equivalence of conditions *(v)*, *(vi)* and *(vii)*.

Condition *(iii)* implies *(v)* by specializiation and *(v)* $\Rightarrow$ *(viii)* follows from Lemma 1.2 *(ii)*.

Condition *(viii)* implies *(i)* by Theorem 2.5, and so the first eight conditions are equivalent.

Condition *(ix)* follows from *(iv)*, by virtue of Lemma 1.3 and Theorem 1.5.

Suppose that *(ix)* holds. Let $x \in L$ and set $J = (x^{**}]$ and $F = [x)$. Then J is a congruence-kernel, $0 \equiv x \left(\Theta_{\mathrm{Lat}}(J)\right)$, and $x \equiv 1 \left(\Psi(F)\right)$ so that $0 \equiv 1 \left(\Theta_{\mathrm{Lat}}(J) \circ \Psi(F)\right)$. By *(ix)*, there exists $a \in L$ such that $0 \equiv a \left(\Psi(F)\right)$ and $a \equiv 1 \left(\Theta_{\mathrm{Lat}}(J)\right)$. Hence $0 = a \wedge x$, that is, $a \leq x^*$, and $a \vee x^{**} = 1$. Thus *(ix)* $\Rightarrow$ *(i)* holds and the first nine conditions are equivalent.

(iii) $\Rightarrow$ *(x)* is clear and *(x)* $\Rightarrow$ *(vii)* follows from Theorem 2.5 *(iv)*, Theorem 1.6 *(iv)* and the distributivity of the lattice of ($*$-)congruences. Of course, *(xii)* $\Rightarrow$ *(x)* and *(x)* $\Rightarrow$ *(i)*, because setting $I = (x^*]$ and $J = (x^{**}]$ for given $x \in L$ implies thatt

$$x^* \vee x^{**} \equiv 1 \left(\Theta(I) \cap \Theta(J)\right) = \Theta(I \cap J) = \Theta\left((0]\right) = \omega_L .$$

Since *(i)* and *(xii)* are equivalent, *(xi)* follows from *(i)* and the theorem will have been proved if we show that *(xi)* implies *(i)*.

Suppose that *(xi)* holds. Put $I = (x^*]$ and $J = (x^{**}]$ for given $x \in L$. Then for $1 < n < \omega$ (fixed),

$$\Sigma_n(I) \cap \Sigma_n(J) = \Sigma_n(I \cap J) = \Sigma_n\left((0]\right) = \omega_L ,$$

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as $L \in B_n$ (Theorem 2.5). By Theorem 2.10,

$$\Sigma_n(I) \vee \Sigma_n(J) = \Sigma_n(I \underline{\vee} J) = \Sigma_n(L) ,$$

as $x^* \underline{\vee} x^{**} = 1$, and so $\Sigma_n(I) \vee \Sigma_n(J) = 1_L$, the largest *-congruence on L. Corollary 2.8 says that $\Sigma_n(I)$ and $\Sigma_n(J)$ are permutable. Thus, from the general theory of universal algebra ([3, §19, p. 120]), the map $a \mapsto ([a]\Sigma_n(I), [a]\Sigma_n(J))$ is an isomorphism of L onto the direct product $L/\Sigma_n(I) \times L/\Sigma_n(J)$. Under this map $x^* \vee x^{**}$ and 1 both correspond to $(1, 1)$ and so $x^* \vee x^{**} = 1$. That is, $L \in B_1$ and so *(xi)* implies *(i)*. □

COROLLARY 3.3. *Let J be a congruence-kernel in a Stone lattice L. Then, $x \equiv y \,(R(J))$, $(x, y \in L)$, if and only if $(x \vee a)^{**} = (y \vee a)^{**}$ for some $a \in J$.*

Proof. By the previous theorem, R permutes with $\Theta_{\mathrm{Lat}}(J) = \Theta(J)$ and so $R(J) = R \circ \Theta_{\mathrm{Lat}}(J)$ (Theorem 1.6 *(iv)*). Thus $x \equiv y \,(R(J))$ implies that $x^{**} = z^{**}$ and $z \vee a = y \vee a$ for some $a \in J$. In any Stone lattice, $(s \vee t)^{**} = s^{**} \vee t^{**}$ is an identity, and so we obtain $(x \vee a)^{**} = (y \vee a)^{**}$ for some $a \in J$. The converse follows from the same identity and Theorem 1.6 *(iii)*. □

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