Bull. Aust. Math. Soc. **92** (2015), 123–132
doi:10.1017/S0004972715000192

QUOTIENT AND PSEUDO UNIT IN NONUNITAL OPERATOR SYSTEM

LI LIU and JIAN-ZE LI✉

(Received 17 June 2014; accepted 31 December 2014; first published online 2 April 2015)

Abstract

We define the quotient and complete NUOS-quotient map (NUOS stands for nonunital operator system) in the category of nonunital operator systems. We prove that the greatest reduced tensor product $\max^0$ is projective in some sense. Moreover, we define a pseudo unit in a nonunital operator system and give some necessary and sufficient conditions under which a nonunital operator system has an operator system structure.

2010 *Mathematics subject classification*: primary 46L06; secondary 46L07.

Keywords and phrases: nonunital operator system, complete NUOS-quotient map, pseudo unit.

1. Introduction and preliminaries

Tensor products of C^*-algebras have played an important role in the development of the theory of C^*-algebras. Recently, operator system theory has developed further and tensor products of operator systems have been studied deeply (see [1, 3–5, 9–11]). Properties of operator systems related to nuclearity have been shown to have essential connections with operator system exactness, the weak expectation property, the local lifting property and so on (see [6]). In particular, the operator system quotient and complete order quotient map were defined in [2]. The maximal tensor product in an operator system was proved to be projective.

The theory of nonunital operator systems was developed in [8, 12, 13]. A nonunital operator system is an abstract characterisation of a $*$-invariant subspace of $\mathcal{L}(\mathcal{H})$ up to a completely isometric complete order isomorphism. Given the absence of an order unit, we have to study both the matrix order and the matrix norm structures at the same time. In this respect, unitalisation is an important tool turning a nonunital operator system into a operator system with a universal property, similar to the C^*-algebra case.

The minimal and maximal tensor products of nonunital operator systems were studied in [7]. The minimal tensor product was proved to be injective. It was shown that there are very few (Min, Max)-nuclear nonunital operator systems. Moreover, the

The second author was supported by the National Natural Science Foundation of China (11371201).

concept of reduced tensor products was introduced through unitalisation. The greatest reduced tensor product $\max^0$ was studied and (Min, $\max^0$)-nuclearity was proved to have a strong connection with C^*-nuclearity.

In this article, we continue the study of the tensor product theory of nonunital operator systems. Moreover, we give a definition of a pseudo unit in a nonunital operator system and give some necessary and sufficient conditions under which a nonunital operator system is an operator system.

In Section 2, we define the quotient of nonunital operator systems through the regularisation of a matrix ordered operator space. We prove that this approach is actually the same as the approach of unitalisation. Then we give the definition of a complete NUOS-quotient map and prove that the greatest reduced tensor product $\max^0$ is a projective tensor product in some sense.

In Section 3, we give a necessary and sufficient condition under which a nonunital operator system is an operator system. We give a definition of the pseudo unit in a nonunital operator system. Let X be a nonunital operator system with a pseudo unit e_0. We define an index $n_{cb}^+(X; e_0)$ similar to the MOS-index defined in [8] (MOS stands for matrix ordered operator system and is defined below). Then we prove that (X, e_0) is an operator system if and only if $n_{cb}^+(X; e_0) = 1$.

Before we start, let us first recall some related results and set some notation. In this article, all vector spaces are over $\mathbb{C}$. We denote by $\mathfrak{S}_1(E)$ the unit sphere of a normed space E. We denote by 1_S the order unit of an operator system S. Let S and T be operator systems. We denote by $\mathcal{S}(S; T)$ the unital completely positive maps from S to T and define $\mathcal{S}_n^S := \mathcal{S}(S; M_n)$. Let V and W be operator spaces. We denote by $\|\cdot\|_\vee$ and $\|\cdot\|_\wedge$ the injective and projective tensor product matrix norms respectively on $V \otimes W$.

Let S be an operator system and $J \subseteq S$ be a subspace. Then J is called a *kernel* of S if there exist an operator system T and $\varphi \in \mathcal{S}(S, T)$ such that $J = \ker \varphi$ (see [6]). The *quotient operator system* S/J is determined by the family of positive cones

$$M_n(S/J)_+ := \{(u_{i,j}) : \forall \varepsilon > 0, \exists k_{i,j} \in J, \varepsilon I_n \otimes 1_S + (u_{i,j} + k_{i,j})_{i,j} \in M_n(S)_+\}$$

for all $n \in \mathbb{N}$. Suppose that T is an operator system and $\varphi \in \mathcal{S}(S, T)$ is surjective. Then φ is called a *complete order quotient map* if, for any $Q \in M_n(T)_+$ and $\varepsilon > 0$, there exists $P \in M_n(S)$ such that $P + \varepsilon I_n \otimes 1_S \in M_n(S)_+$ and $\varphi_n(P) = Q$ (see [2]).

Let X be a matrix ordered $*$-vector space with an operator space structure such that the $*$-operation is completely isometric and $M_n(X)_+ \subseteq M_n(X)$ is closed for all $n \in \mathbb{N}$. Then X is called a *matrix ordered operator space*, or simply MOS.

Let X, Y and Z be MOSs. We denote by $\mathrm{qMor}(X \times Y; Z)$ the set of all completely contractive completely positive bilinear maps from $X \times Y$ to Z, and set $Q_n^{X,Y} := \mathrm{qMor}(X \times Y; M_n)$. We denote by $\mathrm{qMor}(X; Y)$ the set of completely positive completely contractive maps from X to Y, and set $Q_n^X := \mathrm{qMor}(X; M_n)$. The *MOS-index* of X in [8] is defined as follows:

$$n_{cb}^+(X) := \inf_{\substack{\|u\|=1; k\in\mathbb{N} \\ u \in M_k(X)}} \sup_{\substack{n\in\mathbb{N} \\ f \in Q_n^X}} \|f_k(u)\|.$$

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

Theorem 1.1 [8, Lemma 2.4 and Theorem 2.6]. *Let X be a MOS.*

(i) *The evaluation map*

$$\iota_X : X \to \bigoplus_{n\in\mathbb{N}} C(Q_n^X, M_n)$$

is a complete order monomorphism.

(ii) $n_{cb}^+(X) = 1$ *if and only if there exist a Hilbert space $\mathcal{H}$ and a completely isometric complete order monomorphism from X to $\mathcal{L}(\mathcal{H})$.*

Let X be a MOS with $n_{cb}^+(X) = 1$. Then X is called a *(possibly) nonunital operator system*, or simply NUOS. By Theorem 1.1, a nonunital operator system is an abstract characterisation of a $*$-invariant subspace of $\mathcal{L}(\mathcal{H})$ up to a completely isometric complete order isomorphism. The nonunital operator system $\iota_X(X)$ is called the *regularisation* of a MOS X, denoted by X_{reg}. Let X and Y be MOSs. Then $\varphi \in \mathrm{qMor}(X_{\mathrm{reg}}; Y_{\mathrm{reg}})$ whenever $\varphi \in \mathrm{qMor}(X; Y)$.

The *unitalisation* of a nonunital operator system X is an operator system S with a map $i : X \to S$ such that i is a completely isometric complete order monomorphism and, for any operator system T and $\varphi \in \mathrm{qMor}(X; T)$, there exists $\tilde{\varphi} \in \mathcal{S}(S; T)$ satisfying $\tilde{\varphi} \circ i = \varphi$. There exists a unique unitalisation for any nonunital operator system X, denoted by X_1. In fact, a specific form of X_1 is $\iota_X(X) + \mathbb{C} \cdot I$.

Proposition 1.2 [12, Lemma 4.9]. *Let X and Y be nonunital operator systems and $\varphi \in \mathrm{qMor}(X; Y)$. Then $\varphi_1 : X_1 \to Y_1$ defined by*

$$\varphi_1(\iota_X(x) + \lambda \cdot 1_{X_1}) = \iota_Y(\varphi(x)) + \lambda \cdot 1_{Y_1}$$

is unital completely positive.

An injective linear mapping $\varphi : X \to Y$ is called a *MOS-embedding* if φ_1 defined in Proposition 1.2 is a complete order monomorphism. A self-adjoint subspace $X_0 \subseteq X$ is called a *MOS-subspace* if the inclusion mapping is a MOS-embedding. It is the case that X_0 is a MOS-subspace of X if and only if any $\varphi \in \mathrm{qMor}(X_0; \mathcal{L}(\mathcal{H}))$ has an extension in $\mathrm{qMor}(X; \mathcal{L}(\mathcal{H}))$ for any Hilbert space $\mathcal{H}$.

Let X and Y be nonunital operator systems. Suppose that $\kappa_{X,Y} : X \times Y \to X \otimes Y$ is the canonical map and α is a nonunital operator system structure on $X \otimes Y$. Then α is said to be *compatible* if $\kappa_{X,Y} \in \mathrm{qMor}(X \times Y; X \otimes_\alpha Y)$ and $Q_m^X \otimes Q_n^Y \subseteq Q_{mn}^{X\otimes_\alpha Y}$ for any $m, n \in \mathbb{N}$.

The minimal and maximal tensor products of a nonunital operator system were defined in [7]. We denote them by $X \otimes_{\mathrm{Min}} Y$ and $X \otimes_{\mathrm{Max}} Y$, respectively. In particular, $\max^0$, the greatest induced tensor product of nonunital operator systems, was introduced.

For any $n \in \mathbb{N}$, we set

$$\check{Q}_n^{X,Y} := \{\phi \otimes \psi : \phi \in Q_k^X; \psi \in Q_l^Y;\ \text{for any } k, l \in \mathbb{N} \text{ with } k \cdot l = n\}$$

and consider

$$\check{\rho} : X \otimes Y \to \bigoplus_{n\in\mathbb{N}} \ell^\infty(\check{Q}_n^{X,Y}; M_n)$$

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

to be the map given by evaluations. We denote the operator subsystem $\check{\rho}(X \otimes Y)$ by $X \otimes_{\mathrm{Min}} Y$. The matrix norm on $X \otimes_{\mathrm{Min}} Y$ is $\|\cdot\|_{\vee}$, while the matrix cone is given by

$$\begin{aligned} &M_n(X \otimes_{\mathrm{Min}} Y)_+ \\ &\quad := \{u \in M_n(X \otimes Y) : (\phi \otimes \psi)_n(u) \geq 0;\ \text{for any } k, l \in \mathbb{N}, \phi \in Q_k^X \text{ and } \psi \in Q_l^Y\}. \end{aligned}$$

Then Min is the smallest compatible nonunital operator system structure on $X \otimes Y$.

For any $n \in \mathbb{N}$, let C_n be the $\|\cdot\|_{\wedge}$-closure of

$$D_n := \{\alpha(u \otimes v)\alpha^* : u \in M_k(X)_+; v \in M_l(Y)_+; \alpha \in M_{n,kl}; k, l \in \mathbb{N}\}.$$

Then $X \otimes_{\wedge} Y$ together with $\{C_n\}_{n\in\mathbb{N}}$ is a compatible MOS-structure on $X \otimes Y$. We denote its regularisation by Max. The regularisation Max is the greatest compatible nonunital operator system structure on $X \otimes Y$.

Proposition 1.3 [7, Example 2.4(b)]. *Let X and Y be nonunital operator systems. Then $X \otimes_{\mathrm{Max}} Y$ is induced by the map*

$$\hat{\rho} : X \otimes Y \to \bigoplus_{n\in\mathbb{N}} \ell^{\infty}(Q_n^{X,Y}; M_n).$$

In other words, a bilinear map $\Phi : X \times Y \to \mathcal{L}(\mathcal{H})$ is completely contractive and completely positive if and only if its linearisation $\Phi_L : X \otimes_{\mathrm{Max}} Y \to \mathcal{L}(\mathcal{H})$ is completely contractive and completely positive.

Theorem 1.4 [7, Example 2.4(a) and Theorem 2.6(c)]. *Let $X \subseteq \mathcal{L}(\mathcal{H})$ and $Y \subseteq \mathcal{L}(\mathcal{K})$ be nonunital operator systems, and let S and T be operator systems. Then:*

(i) *$X \otimes_{\mathrm{Min}} Y \subseteq \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$;*

(ii) *if $X_0 \subseteq X$ and $Y_0 \subseteq Y$ are MOS-subspaces, then $X_0 \otimes_{\mathrm{Min}} Y_0 \subseteq X \otimes_{\mathrm{Min}} Y$;*

(iii) *$S \otimes_{\mathrm{Min}} T = S \otimes_{\min} T$ and $S \otimes_{\mathrm{Max}} T = S \otimes_{\max^0} T = S \otimes_{\max} T$.*

2. Quotient and complete NUOS-quotient map

Let X be a nonunital operator system and $K \subseteq X$ be a subspace. Then $K \subseteq X$ is called a *NUOS-kernel* if there exist a nonunital operator system Y and $\varphi \in \mathrm{qMor}(X; Y)$ such that $K = \ker \varphi$. It is clear that $K \subseteq X$ is a NUOS-kernel if and only if $K \subseteq X_1$ is a kernel. Moreover, any kernel of an operator system is also a NUOS-kernel.

If $K \subseteq X$ is a NUOS-kernel and $q : X \to X/K$ is the canonical map, the operator space quotient X/K has a natural $*$-operation which is completely isometric. For any $n \in \mathbb{N}$, we define $M_n(X/K)_+$ to be the closure of

$$\begin{aligned} q_n(M_n(X)_+) &:= \{(u_{i,j} + K) \in M_n(X/K) : (u_{i,j}) \in M_n(X)_+\} \\ &= \{(u_{i,j} + K) \in M_n(X/K) : \exists v_{i,j} \in K \text{ with } (u_{i,j} + v_{i,j}) \in M_n(X)_+\} \end{aligned}$$

and denote this MOS by $(X/K)_{\mathrm{MOS}}$. We define the *nonunital operator system quotient* of X by K to be $(X/K)_{\mathrm{NUOS}} = ((X/K)_{\mathrm{MOS}})_{\mathrm{reg}}$.

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

Proposition 2.1. *Let X and Y be nonunital operator systems and $\varphi \in \mathrm{qMor}(X;Y)$.*

(i) *Suppose that $K \subseteq X$ is a NUOS-kernel and $K \subseteq \ker\varphi$. The induced map $\tilde{\varphi} : (X/K)_{\mathrm{NUOS}} \to Y$ given by $\tilde{\varphi}(x+K) = \varphi(x)$ is completely positive and completely contractive.*

(ii) *Suppose that $K \subseteq X$ is a NUOS-kernel, Z is a nonunital operator system and $\psi \in \mathrm{qMor}(X;Z)$. If $K \subseteq \ker\psi$ and Z has the property that whenever Y is a nonunital operator system and $\varphi \in \mathrm{qMor}(X;Y)$ with $K \subseteq \ker\varphi$, there exists a unique $\hat{\varphi} \in \mathrm{qMor}(Z;Y)$ such that $\hat{\varphi} \circ \psi = \varphi$, then there exists a completely isometric complete order isomorphism $\gamma : Z \to (X/K)_{\mathrm{NUOS}}$ such that $\gamma \circ \psi = q$.*

Proof. (i) For any $u = (x_{i,j} + K) \in q_n(M_n(X)_+)$, there exists $(y_{i,j}) \in M_n(K)$ such that $(x_{i,j} + y_{i,j}) \in M_n(X)_+$. We have

$$\tilde{\varphi}_n(u) = (\tilde{\varphi}(x_{i,j} + K)) = (\varphi(x_{i,j} + y_{i,j})) \geq 0,$$

since $\varphi \in \mathrm{qMor}(X;Y)$. Therefore, for any $u \in M_n((X/K)_{\mathrm{MOS}})_+$, we have $\tilde{\varphi}_n(u) \geq 0$. Note that $\tilde{\varphi}$ is completely contractive. We have $\tilde{\varphi} \in \mathrm{qMor}((X/K)_{\mathrm{MOS}};Y)$, which implies that $\tilde{\varphi} \in \mathrm{qMor}((X/K)_{\mathrm{NUOS}};Y)$. To prove (ii), we only need to note that $q : X \to (X/K)_{\mathrm{NUOS}}$ is completely contractive and completely positive. □

Proposition 2.2.

(i) *Let S be an operator system and let $J \subseteq S$ be a kernel. Then*

$$\mathrm{id} : (S/J)_{\mathrm{NUOS}} \to S/J$$

is a completely contractive complete order isomorphism.

(ii) *Let X be a nonunital operator system and let $K \subseteq X$ be a NUOS-kernel. Then the inclusion map*

$$\tau : (X/K)_{\mathrm{NUOS}} \to X_1/K$$

is a completely isometric complete order monomorphism.

(iii) *Let X be a nonunital operator system and let $K \subseteq X$ be a NUOS-kernel. Then*

$$((X/K)_{\mathrm{NUOS}})_1 = X_1/K.$$

Proof. (i) The canonical map $q : S \to S/J$ is unital completely positive and thus $q \in \mathrm{qMor}(S;S/J)$, which implies that $\mathrm{id} \in \mathrm{qMor}((S/J)_{\mathrm{NUOS}};S/J)$ by Proposition 2.1(i). If $u \in M_n(S/J)_+$ and $\varepsilon_k \to 0$ as $k \to +\infty$, then $u + \varepsilon_k I_n \in q_n(M_n(S)_+)$ and thus $u \in M_n((S/J)_{\mathrm{NUOS}})_+$. Therefore, $\mathrm{id} : (S/J)_{\mathrm{NUOS}} \to S/J$ is also a complete order isomorphism.

We prove (ii) and (iii). Note that the inclusion map from $(X/K)_{\mathrm{NUOS}}$ to $(X_1/K)_{\mathrm{NUOS}}$ is completely contractive and completely positive. By part (i), the inclusion map from $(X/K)_{\mathrm{NUOS}}$ into X_1/K is a completely contractive complete order monomorphism. Therefore, the canonical map from $((X/K)_{\mathrm{NUOS}})_1$ onto X_1/K preserving the order unit is unital completely positive. Let q be the canonical map from X to X/K. Note that $q \in \mathrm{qMor}(X;(X/K)_{\mathrm{NUOS}})$. We have $q_1 \in \mathcal{S}(X_1;((X/K)_{\mathrm{NUOS}})_1)$. Since $K \subseteq X_1$ is a kernel and $K = \ker q_1$, the induced map $\tilde{q}_1 : X_1/K \to ((X/K)_{\mathrm{NUOS}})_1$ is unital completely positive. Since $\tilde{q}_1|_{X/K} = \mathrm{id}_{X/K}$, we have the desired conclusions. □

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

By Proposition 2.2, given a nonunital operator system X and a NUOS-kernel $K \subseteq X$, we can also define the nonunital operator system quotient $(X/K)_{\mathrm{NUOS}}$ with the structure induced by the inclusion $X/K \subseteq X_1/K$. In fact, these two approaches are the same.

Remark 2.3. The concept of quotient in the nonunital operator system defined above is an extension of the operator system quotient defined in [6]. In fact, if S is an operator system and $K \subseteq S$ is a kernel, then $(S/K)_{\mathrm{NUOS}}$ is exactly the operator system quotient S/K. To prove this, we only need to note that $(S/K)_{\mathrm{NUOS}} \subseteq S_1/K$ and $S/K \subseteq S_1/K$ by Proposition 2.2(iii) and [12, Lemma 4.9(c)]. If S is an operator system and $N \subseteq S$ is a subspace, then N is a NUOS-kernel if and only if N is a kernel in the operator system sense. This is clear based on [1, Lemma 5.1.6]. On the other hand, this concept of quotient in the nonunital operator system case applies to all C^*-algebras, while only a unital C^*-algebra has the operator system quotient.

Definition 2.4. Let X and Y be nonunital operator systems and let $\varphi \in \mathrm{qMor}(X; Y)$ be surjective. Then φ is called a *complete NUOS-quotient map* if:

(i) the induced map $\tilde{\varphi} : (X/ker\varphi)_{\mathrm{NUOS}} \to Y$ is completely isometric;
(ii) for any $v \in M_n(Y)_+$ and $\varepsilon > 0$, we can find $u \in M_n(X)$ such that $\varphi_n(u) = v$ and there exists $u_0 \in M_n(X)_+$ such that $\|u - u_0\|_{(X/ker\,\varphi)_{\mathrm{MOS}}} < \varepsilon$.

Proposition 2.5. *Let X and Y be nonunital operator systems and let $\varphi \in \mathrm{qMor}(X; Y)$ be surjective.*

(i) *φ is a complete NUOS-quotient map if and only if $\tilde{\varphi} : (X/ker\,\varphi)_{\mathrm{NUOS}} \to Y$ is a completely isometric complete order isomorphism.*
(ii) *The map $\varphi \in \mathrm{qMor}(X; Y)$ is a complete NUOS-quotient map if and only if the induced map $\varphi_1 : X_1 \to Y_1$ is a complete order quotient map.*

Proof. Part (i) is clear. For (ii), if $\varphi_1 : X_1 \to Y_1$ is a complete order quotient map, then we have $X_1/\ker\varphi = Y_1$ by [2, Theorem 3.2], which implies that $(X/\ker\varphi)_{\mathrm{NUOS}} = Y$ by Proposition 2.2(iii). Conversely, if $\varphi \in \mathrm{qMor}(X; Y)$ is a complete NUOS-quotient map, then $(X/\ker\varphi)_{\mathrm{NUOS}} = Y$. By Proposition 2.2(iii), we have $X_1/\ker\varphi = Y_1$. □

Lemma 2.6. *Let X be a nonunital operator system and let S and T be operator systems. Suppose that $X \subseteq S$ is a MOS-subspace.*

(i) *If $K \subseteq X$ is a NUOS-kernel and $K \subseteq S$ is a kernel, then $(X/K)_{\mathrm{NUOS}} \subseteq S/K$.*
(ii) *$X \otimes_{max^0} T \subseteq X_1 \otimes_{\max} T$ is a MOS-subspace.*

Proof. (i) By Proposition 2.2(ii), it is clear that $(X/K)_{\mathrm{NUOS}} \subseteq X_1/K$ and $X_1/K \subseteq S_1/K$. Note that there exists a unital completely positive map from S_1 to S by [12, Lemma 4.9(c)]. Then the inclusion map from $(S/K)_{\mathrm{NUOS}}$ to $(S_1/K)_{\mathrm{NUOS}}$ is a complete order monomorphism. By Proposition 2.2(i), the inclusion map from S/K to S_1/K is also a complete order monomorphism. Therefore, we have $(X/K)_{\mathrm{NUOS}} \subseteq S/K$.

(ii) Note that

$$X \otimes_{\max^0} T \subseteq X_1 \otimes_{\max} T_1, \quad X_1 \otimes_{\max} T \subseteq X_1 \otimes_{\max} T_1.$$

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

We have $X \otimes_{\max^0} T \subseteq X_1 \otimes_{\max} T$. For any $\Phi \in \mathrm{qMor}(X \otimes_{\max^0} T; \mathcal{L}(\mathcal{H}))$, we have $\Phi \in \mathrm{qMor}(X \otimes_{\mathrm{Max}} T; \mathcal{L}(\mathcal{H}))$ and thus $\bar{\Phi} : X \times T \to \mathcal{L}(\mathcal{H})$ defined by $\bar{\Phi}(x, z) = \Phi(x \otimes z)$ is completely positive and completely contractive. We can define

$$\bar{\Phi}_1 : X_1 \times T \to \mathcal{L}(\mathcal{H})$$

by $\bar{\Phi}_1((x, \lambda), z) = \bar{\Phi}(x, z) + \lambda I_H$. Then $\bar{\Phi}_1$ is unital completely positive, as is its linearisation $(\bar{\Phi}_1)_L : X_1 \otimes_{\max} T \to \mathcal{L}(\mathcal{H})$, and this is the required extension of Φ. □

Proposition 2.7. *Let X and Y be nonunital operator systems and let T be an operator system. Suppose that $\varphi : X \to Y$ is a complete NUOS-quotient map. Then*

$$\varphi \otimes_{\max^0} \mathrm{id}_T : X \otimes_{\max^0} T \to Y \otimes_{\max^0} T$$

is also a complete NUOS-quotient map.

Proof. If $\varphi : X \to Y$ is a complete NUOS-quotient map, we have that $\varphi_1 : X_1 \to Y_1$ is a complete order quotient map by Proposition 2.5(ii). By [2, Theorem 3.4], the map

$$\varphi_1 \otimes_{\max} \mathrm{id}_T : X_1 \otimes_{\max} T \to Y_1 \otimes_{\max} T$$

is a complete order quotient map. By [2, Theorem 3.2],

$$(X_1 \otimes_{\max} T)/(\ker \varphi \otimes T) = Y_1 \otimes_{\max} T.$$

Therefore, by Lemma 2.6(ii),

$$((X \otimes_{\max^0} T)/(\ker \varphi \otimes T))_{\mathrm{NUOS}} = Y \otimes_{\max^0} T.$$

By Proposition 2.5(i), we see that $\varphi \otimes_{\max^0} \mathrm{id}_T$ is a complete NUOS-quotient map. □

Remark 2.8. The $\max^0$ tensor product of nonunital operator systems has an intrinsic connection with the operator system maximal tensor product. By Proposition 2.7, if $\varphi : X \to Y$ is a complete NUOS-quotient map,

$$((X \otimes_{\max^0} T)/(\ker \varphi \otimes T))_{\mathrm{NUOS}} = (X/\ker \varphi)_{\mathrm{NUOS}} \otimes_{\max^0} T.$$

3. Pseudo unit in a nonunital operator system

In this section, we first give some necessary and sufficient conditions under which a nonunital operator system is an operator system.

Proposition 3.1. *Let X be a nonunital operator system. Then the following are equivalent.*

(i) *X is an operator system.*
(ii) *There exists $\iota \in \mathrm{qMor}(X_1; X)$ such that $\iota|_X = \mathrm{id}_X$.*
(iii) *For any nonunital operator system Y and $\varphi \in \mathrm{qMor}(Y; X)$, there exists an extension $\bar{\varphi} \in \mathrm{qMor}(Y_1; X)$.*
(iv) *For any nonunital operator system Y and $\varphi \in \mathrm{qMor}(X; Y)$, there exists an extension $\tilde{\varphi} \in \mathrm{qMor}(X_1; Y)$.*

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

Proof. By [12, Lemma 4.9(c)], (i) implies (ii). Conversely, suppose that there exists $\iota \in \mathrm{qMor}(X_1; X)$ such that $\iota|_X = \mathrm{id}_X$. Let $e := \iota(1_{X_1}) \in X_+$. For any $n \in \mathbb{N}$ and $v \in M_n(X)_{sa}$, there exists $r > 0$ such that $r \cdot (I_n \otimes 1_{X_1}) \geq v$. Note that $\iota \in \mathrm{qMor}(X_1; X)$ and $\iota|_X = \mathrm{id}_X$. Then $r \cdot e_n \geq \iota(v) = v$ and thus e is a matrix order unit. Now we prove that e is an Archimedean matrix order unit. Since $e \geq 0$ and $\|e\| \leq \|1_{X_1}\| = 1$, we have $e \leq 1_{X_1}$. Suppose that $n \in \mathbb{N}$ and $v \in M_n(X)_{sa}$ satisfy $v + r \cdot e_n \geq 0$ for any $r > 0$. Then $v + r \cdot (I_n \otimes 1_{X_1}) \geq 0$ for any $r > 0$ and thus $v \geq 0$. Note that $\iota \in \mathrm{qMor}(X_1; X)$ and $e_n \leq I_n \otimes 1_{X_1}$ for any $n \in \mathbb{N}$. For any $v \in M_n(X)$, we get $-r \cdot e_n \leq v \leq r \cdot e_n$ if and only if $-r \cdot I_n \otimes 1_{X_1} \leq v \leq r \cdot I_n \otimes 1_{X_1}$. Therefore, X is an operator system.

Now we prove that (ii) is equivalent to (iii) and (iv). Suppose that (ii) holds. Let Y be a nonunital operator system and $\varphi \in \mathrm{qMor}(Y; X)$. We can define $\bar{\varphi} : Y_1 \to X$ to be the composition of $\varphi_1 \in \mathrm{qMor}(Y_1; X_1)$ and $\iota \in \mathrm{qMor}(X_1; X)$ and get (iii). Let Y be a nonunital operator system and $\varphi \in \mathrm{qMor}(X; Y)$. We can define $\tilde{\varphi} : X_1 \to Y$ to be the composition of $\varphi \in \mathrm{qMor}(X; Y)$ and $\iota \in \mathrm{qMor}(X_1; X)$ and get (iv). It is clear that (iii) or (iv) implies (ii). □

Recall that a nonunital operator system X is called *MOS-injective* if, for any nonunital operator system Y with a MOS-subspace $Y_0 \subseteq Y$, any $\varphi \in \mathrm{qMor}(Y_0; X)$ has an extension $\tilde{\varphi} \in \mathrm{qMor}(Y; X)$ (see [8]). By Proposition 3.1, any MOS-injective nonunital operator system is an operator system.

Definition 3.2. Let X be a nonunital operator system and $e_0 \in X_+$ with $\mathrm{span}\{x \in X : 0 \leq x \leq e_0\} = X$. There exist a Hilbert space $\mathcal{H}$ and $\eta \in \mathcal{H}$ such that $X \subseteq \mathcal{L}(\mathcal{H})$ and $e_0(\eta) = \eta$. If any completely positive map φ from X to an operator system T sending e_0 to 1_T is completely contractive, then e_0 is called a *pseudo unit*.

We can prove the following lemma by similar arguments to [1, Lemma 5.1.6]

Lemma 3.3. *Let X be a nonunital operator system and $e_0 \in X$ be a pseudo unit. Suppose that $\mathcal{K}$ is a Hilbert space and $\Phi \in \mathrm{qMor}(X; \mathcal{L}(\mathcal{K}))$. Then there exists $\Psi \in \mathrm{qMor}(X; \mathcal{L}(\mathcal{K}))$ such that $\Psi(e_0) = I_{\mathcal{K}}$ and $\Phi(x) = \Phi(e_0)^{1/2}\Psi(x)\Phi(e_0)^{1/2}$ for any $x \in X$.*

Let X be a nonunital operator system and $e_0 \in X$ be a pseudo unit. Write

$$Q_k^{X;e_0} = \{\varphi : \varphi \in Q_k^X; \varphi(e_0) = I_k\}$$

for any $k \in \mathbb{N}$. By Lemma 3.3 and $Q_k^X \neq \emptyset$, we see that $Q_k^{X;e_0}$ is not empty. Let

$$\pi : X \to \bigoplus_{k \in \mathbb{N}} C(Q_k^{X;e_0}; M_k)$$

be the map given by evaluations. We define

$$n_{cb}^+(X; e_0) = \inf_{\substack{u \in M_k(X) \\ \|u\|=1; k \in \mathbb{N}}} \sup_{\substack{f \in Q_n^{X;e_0} \\ n \in \mathbb{N}}} \|f_k(u)\|.$$

Proposition 3.4. *Let X be a nonunital operator system and $e_0 \in X$ be a pseudo unit. Then π is a complete order monomorphism.*

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

PROOF. Note that ι_X in Theorem 1.1 is injective. We see that π is injective by Lemma 3.3. It is clear that π is completely positive. Suppose that $u \in M_n(X)$ and $\pi_n(u) \geq 0$. To prove that $u \in M_n(X)_+$, we only need to show that $\Phi_n(u) \geq 0$ for any $\Phi \in Q_n^X$ by Theorem 1.1(i). By Lemma 3.3,

$$\Phi_n(u) = (\Phi(u_{i,j})) = (\Phi(e_0)^{1/2}\Psi(u_{i,j})\Phi(e_0)^{1/2})$$

for some $\Psi \in Q_n^{X;e_0}$ and thus $\Phi_n(u) \geq 0$. □

PROPOSITION 3.5. *Let X be a nonunital operator system and $e_0 \in X$ be a pseudo unit. Then $(X; e_0)$ is an operator system if and only if $n_{cb}^+(X; e_0) = 1$.*

PROOF. If $n_{cb}^+(X; e_0) = 1$, the evaluation map π is a completely isometric complete order monomorphism and $\pi(e_0)$ is the identity map. Therefore, $(X; e_0)$ is an operator system. Conversely, if (X, e_0) is an operator system, it is clear that $Q_n^{X;e_0} = \mathcal{S}_n^X$ for any $n \in \mathbb{N}$ and thus $n_{cb}^+(X; e_0) = 1$. □

The following result follows from Propositions 3.1 and 3.5.

THEOREM 3.6. *Let X be a nonunital operator system and $e_0 \in X$ be a pseudo unit. Then the following are equivalent:*

(i) $(X; e_0)$ *is an operator system;*
(ii) $n_{cb}^+(X; e_0) = 1$*;*
(iii) *there exists* $\iota \in \mathrm{qMor}(X_1; X)$ *such that* $\iota|_X = \mathrm{id}_X$ *and* $\iota(1_{X_1}) = e_0$.

Acknowledgements

The authors would like to show their sincere gratitude to Professor Chi-Keung Ng for his guidance and suggestions. The authors would also like to thank the referee for valuable comments, which led to Remarks 2.3 and 2.8.

References

[1] E. G. Effros and Z.-J. Ruan, *Operator Spaces*, London Mathematical Society Monographs, 23 (Clarendon Press, Oxford, 2000).
[2] K. H. Han, 'On maximal tensor products and quotient maps of operator systems', *J. Math. Anal. Appl.* **384** (2011), 375–386.
[3] K. H. Han and V. I. Paulsen, 'An approximation theorem for nuclear operator systems', *J. Funct. Anal.* **261** (2011), 999–1009.
[4] X. J. Huang and C. K. Ng, 'An abstract characterisation of unital operator spaces', *J. Operator Theory* **67** (2012), 289–298.
[5] A. S. Kavruk, V. I. Paulsen, I. G. Todorov and A. M. Tomforde, 'Tensor products of operator systems', *J. Funct. Anal.* **261** (2011), 267–299.
[6] A. S. Kavruk, V. I. Paulsen, I. G. Todorov and A. M. Tomforde, 'Quotients, exactness and nuclearity in the operator system category', *Adv. Math.* **235** (2013), 321–360.
[7] J. Z. Li and C. K. Ng, 'Tensor products for nonunital operator systems', *J. Math. Anal. Appl.* **396**(2) (2012), 601–605.
[8] C. K. Ng, 'Operator subspaces of $\mathcal{L}(H)$ with induced matrix orderings', *Indiana Univ. Math. J.* **60** (2011), 577–610.

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press

[9] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics, 78 (Cambridge University Press, Cambridge, 2002).

[10] V. I. Paulsen, I. G. Todorov and A. M. Tomforde, 'Operator system structures on ordered spaces', *Proc. Lond. Math. Soc.* (3) **102** (2011), 25–49.

[11] V. I. Paulsen and A. M. Tomforde, 'Vector spaces with an order unit', *Indiana Univ. Math. J.* **58** (2009), 1319–1359.

[12] W. Werner, 'Subspaces of $\mathcal{L}(\mathcal{H})$ that are $*$-invariant', *J. Funct. Anal.* **193** (2002), 207–223.

[13] W. Werner, 'Multipliers on matrix ordered operator spaces and some K-groups', *J. Funct. Anal.* **206** (2004), 356–378.

LI LIU, Grenoble University, Laboratoire Jean Kuntzmann,
BP 53, 38 041 Grenoble Cedex 9, France
e-mail: liliu.math@gmail.com

JIAN-ZE LI, Department of Mathematics, Tianjin University,
Tianjin 300072, PR China
e-mail: lijianze@tju.edu.cn

https://doi.org/10.1017/S0004972715000192 Published online by Cambridge University Press