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# The Number of Non-Zero Digits of $n!$

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*Abstract.* Let $b$ be an integer with $b > 1$. In this note, we prove that the number of non-zero digits in the base $b$ representation of $n!$ grows at least as fast as a constant, depending on $b$, times $\log n$.

In his beautiful book [3], R. K. Guy writes (see page 262) that Erdős noticed that $n! = 2^a + 2^b$ for some non-negative integers $a$ and $b$ implies that $n \le 4$. An obvious problem which arises in this context is investigating what happens with the number of non-zero binary digits of $n!$ for large values of $n$. More general, for any positive integer $b > 1$ one can ask what happens with the number of non-zero digits of $n!$ in base $b$. A natural guess is that this number tends to infinity with $n$. In this note, we give a lower bound for such number in terms of $n$ and $b$. To fix notations, for any positive integers $m$ and $b$ with $b > 1$ let $l_b(m)$ be the number of non-zero digits of $m$ in base $b$.

Our result is

***Theorem*** *The following inequality holds*

$$\big(l_b(n!) + 1\big)\log b + \log\big(l_b(n!)\big) \ge \log(n+1). \tag{1}$$

Lower bounds of a similar type as the right hand side of (1) for $l_b(|u_n|)$ where $(u_n)_{n\ge 0}$ is a non-degenerate linear recurence sequence satisfying certain mild technical assumptions were obtained by C. L. Stewart in [5] (see also [4] for a slightly more general result).

**Proof** Write $l = l_b(n!)$ and

$$n! = c_1 b^{a_1} + \cdots + c_l b^{a_l}, \tag{2}$$

where $a_1 > \cdots > a_l \ge 0$ and $c_i \in \{0, \ldots, b-1\}$ for $i = 1, \ldots, l$. Since certainly $l \ge 1$, it follows that it suffices to assume that $n + 1 \ge b$ (otherwise inequality (1) is automatically satisfied).

Let $m$ be the largest positive integer such that $b^m - 1 \le n$. Notice that $m \ge 1$ because $n \ge b - 1$. For every $i = 1, \ldots, l$ let $\alpha_i \in \{0, \ldots, m-1\}$ be such that $a_i \equiv \alpha_i (\operatorname{mod} m)$. By reindexing the $a_i$'s, we may assume that $\alpha_1 \ge \cdots \ge \alpha_l \ge 0$. Since $b^{mk} \equiv 1\big(\operatorname{mod}(b^m - 1)\big)$ for all $k \ge 0$, and since $b^m - 1$ divides $n!$, it follows, by reducing equation (2) modulo $b^m - 1$ that

$$c_1 b^{\alpha_1} + \cdots + c_l b^{\alpha_l} \equiv 0\big(\operatorname{mod}(b^m - 1)\big). \tag{3}$$

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Write (3) as

$$(4) \qquad c_1 b^{\alpha_1} + \cdots + c_l b^{\alpha_l} = d_0(b^m - 1),$$

where $d_0$ is some positive integer. Notice that

$$(5) \qquad c_1 b^{\alpha_1} + \cdots + c_l b^{\alpha_l} \le l(b-1)b^{m-1} < (l+1)(b^m - 1).$$

Inequality (5) shows that $d_0 \le l$. Now rewrite equation (4) as

$$(6) \qquad c_1 b^{\alpha_1} + \cdots + c_l b^{\alpha_l} + d_0 = d_0 b^m.$$

We shall now show that

$$(7) \qquad lb^l \ge b^m.$$

Assume that inequality (7) does not hold.

We look at the base $b$ representation of the number appearing in the left side of formula (6). If

$$b^m \le d_0,$$

then inequality (7) follows at once because $d_0 \le l < lb^l$. Hence, $d_0 < b^m$ and now equation (6) implies that

$$b^{\alpha_l} \le d_0.$$

Hence,

$$(8) \qquad d_1 := c_l b^{\alpha_l} + d_0 \le d_0(c_l + 1) \le lb.$$

Rewrite formula (6) as

$$(9) \qquad c_1 b^{\alpha_1} + \cdots + c_{l-1} b^{\alpha_{l-1}} + d_1 = d_0 b^m.$$

If $b^m \le d_1$, then inequality (7) follows again from inequality (8). Hence, $b^m > d_1$ and now equation (9) implies

$$b^{\alpha_{l-1}} \le d_1.$$

Thus,

$$(10) \qquad d_2 := c_{l-1} b^{\alpha_{l-1}} + d_1 \le d_1(c_{l-1} + 1) \le lb^2.$$

It should be now clear how the argument works. For any $i = 1, \ldots, l$, let

$$(11) \qquad d_i := c_{l-i+1} b^{\alpha_{l-i+1}} + \cdots + c_l b^{\alpha_l} + d_0.$$

If one assumes that inequality (7) does not hold, then one can use induction on $i$ and the equation

$$c_1 b^{\alpha_1} + \cdots + c_{l-i} b^{\alpha_{l-i}} + d_i = d_0 b^m$$

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to show that

$$d_i \leq lb^i \tag{12}$$

for all $i = 1, \ldots, l$. When $i = l$, we get

$$d_l = d_0 b^m \geq b^m,$$

which together with inequality (12) for $i = l$ implies inequality (7).

We now show that inequality (7) implies inequality (1). Indeed, since $m$ was chosen to be the largest positive integer such that $b^m - 1 \leq n$, it follows that $b^m \geq (n+1)/b$. Hence,

$$lb^l \geq \frac{n+1}{b},$$

or

$$(l+1)\log b + \log l \geq \log(n+1),$$

which is precisely inequality (1).

The Theorem is therefore proved.

***Remark 1*** By analyzing the proof of the Theorem, one sees easily that inequality (1) remains true if one replaces $n!$ by the least common multiple $[1, 2, \ldots, n]$ of all positive integers $1, 2, \ldots, n$.

***Remark 2*** Inequality (6) is probably very weak. Coming back to Erdős's observation, our inequality (1) shows that $l_2(n!) \leq 2$, implies $n \leq 15$, when in fact the largest solution of $l_2(n!) \leq 2$ is $n = 4$. Even worse, our inequality (1) shows that $l_2(n!) \leq 6$ implies $n \leq 767$, when in fact the largest solution of $l_2(n!) \leq 6$ is $n = 9$.

***Remark 3*** Coming back again to Erdős's remark, we notice that if

$$n! = p^a + p^b \tag{13}$$

where $p$ is prime and $a \geq b$ but $(a, b) \neq (0, 0)$ or $(1, 0)$, then $n \leq 4$. We discard the cases $(a, b) = (0, 0)$ or $(1, 0)$ basically because the first one gives the trivial solution $n = 2$ while the second one is equivalent to finding all $n$'s for which $n! - 1$ is a prime, which is another unsolved problem but of a different nature.

Assume that $p$ is odd. Suppose first that $b = 0$. If $a$ is even, then $p^a + 1 \equiv 2 \pmod 8$, hence $n \leq 3$. If $a > 1$ is odd, then the fact that the equation (13) has no solution follows from a result of Erdős and Obláth (see [2]). Assume now that $b > 0$. In this case, $p \leq n$. Since

$$n! = p^b(p^{a-b} + 1),$$

it follows easily that

$$\mathrm{ord}_2(n!) = \mathrm{ord}_2(p^{a-b} + 1) \leq \log_2(p+1) \leq \log_2(n+1).$$

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On the other hand, (see [1])

$$\mathrm{ord}_2(n!) \geq n - \log_2(n+1).$$

Hence,

$$n - \log_2(n+1) \leq \log_2(n+1),$$

which forces $n \leq 5$. One can now check that equation (13) has no solution for $n = 5$.

One may use our Theorem to give an immediate generalization of the result mentioned in Remark 3. Namely

***Corollary*** *Let C and L be positive constants. Then the equation*

$$n! = c_1 p^{a_1} + \cdots + c_l p^{a_l} \tag{14}$$

*where $p$ is prime, $n \geq p$, $l \leq L$ and $c_i$ are non-negative integers such that $c_i \leq C$ has only finitely many solutions.*

**Proof** Since $n \geq p$, it follows that $p - 1$ divides $n!$. Reducing equation (14) modulo $p - 1$, we get

$$\sum_{i=1}^{l} c_i \equiv 0\bigl(\mathrm{mod}(p-1)\bigr).$$

Hence, $p \leq 1 + LC$. Notice now that since $c_i \leq C$, it follows that $n!$ has at most $L\bigl(\lfloor \log_p(C) \rfloor + 1\bigr) \leq L \log_2(2C)$ non-zero digits when written in base $p$. The Theorem now implies that

$$n \leq L \log_2(2C)(1 + LC)^{L \log_2(2C)+1}.$$

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