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On the Oscillation of a Second Order Strictly Sublinear Differential Equation

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Abstract. We establish a flexible oscillation criterion based on an averaging technique that improves upon a result due to C. G. Philos.

1 Introduction

The averaging techniques originate from a paper Wintner [15] published in 1949 where Fite's [5] oscillation criterion had been significantly generalized. Fite's theorem [5, p. 347] is the first result regarding the oscillation of a linear ordinary differential equation without pointwise estimates on the coefficients. Several important developments in this area can be found in the surveys by Philos and Purnaras [14, §1] and Wong [18].

An innovative paper by Kamenev [6] emphasized studies about averaging techniques for an investigation of nonlinear oscillations. Such results have immense implications for the study of celestial and fluid mechanics as well as biology and the social sciences. We mention the description of several categories of applied examples undertaken in recent monographs [1, 2].

In this respect, let us consider here the second order nonlinear ordinary differential equation

$$x'' + a(t)f(x) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where several technical conditions, listed next, are assumed to hold.

- The function $a\colon [t_0, +\infty) \to \mathbb{R}$ is continuous, while the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous over $\mathbb{R}$, is continuously differentiable over $\mathbb{R} - \{0\}$, and has the sign property, that is, $f(x)x > 0$ for any $x \neq 0$.
- The function f has a monotonic behavior given by $f'(x) > 0$ for $x \neq 0$.
- In the terminology due to Wong [16], [18, pp. 419–420] and Naito [10] we suppose that f is *strictly sublinear*. This means that there exists $C > 0$ such that

$$f'(x) \int_{0+}^{x} \frac{dy}{f(y)} \geq C \quad \text{for all } x > 0$$

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and respectively

$$f'(x)\int_{0-}^{x}\frac{dy}{f(y)}\geq C \quad \text{for all } x<0.$$

Following [3, 12], we introduce the quantity $I_f \in (0,1)$ via the formula

$$I_f=\min\left\{\frac{\inf_{x>0} f'(x)F(x)}{1+\inf_{x>0} f'(x)F(x)},\frac{\inf_{x<0} f'(x)F(x)}{1+\inf_{x<0} f'(x)F(x)}\right\},$$

where

$$F(x)=\int_{0+}^{x}\frac{dy}{f(y)} \text{ for } x>0 \quad \text{and} \quad F(x)=\int_{0-}^{x}\frac{dy}{f(y)} \text{ for } x<0.$$

In the sequel we take as maximal value of C the quantity $I_f/(1-I_f)$.

A celebrated particular case and source of many substantial investigations in the field of averaged oscillations is the *Butler conjecture* [4, p. 144], [7, p. 548], which consists of deciding whether or not the Emden–Fowler equation

$$x''+t^{\lambda}\sin t\cdot x^{\gamma}=0, \quad t\geq t_0>0, \lambda\in\mathbb{R}, \gamma\in(0,1), \tag{1.2}$$

is oscillatory. In two seminal papers on this subject, Kwong and Wong [7, 8] made use of a special Belohorec type quantity, namely

$$\frac{1}{T}\int_{t}^{T}\int_{t}^{s}[\varphi(\tau)]^{\gamma}a(\tau)\,d\tau ds, \quad T\geq t\geq t_0, \tag{1.3}$$

(let us denote its limit when $T\to+\infty$ with $A(t)$) and established that equation (1.2) oscillates for $\lambda>1-\gamma$ and does not oscillate for $\lambda<-\gamma$. Actually, Kwong and Wong established further the oscillation of equation (1.2) for $\lambda>-\gamma$ [8, p. 717]. The critical value $\lambda=-\gamma$ also implies oscillation, as demonstrated independently by Onose [11] and Kwong and Wong [8].

The case $-\gamma<\lambda\leq 1-\gamma$, leading to oscillation, was studied in [17, p. 478] for the class of equations (1.1) described previously by means of the divergence condition

$$\limsup_{T\to+\infty}\Big[\int_{t_0}^{T}\frac{[\max\{A(s),0\}]^2}{s}\,ds\Big]\cdot\Big[\int_{t_0}^{T}s\Big[\frac{\varphi'(s)}{\varphi(s)}\Big]^2 ds\Big]^{-1}=+\infty.$$

Nonoscillation of equations (1.1) is discussed in [7, 9, 10].

By combining Onose's technique with the Kwong–Wong methods, Philos [12, 13] established several interesting criteria of oscillation for equations (1.1). In fact, the result in [13, Theorem 2], which motivates the present note, reads as follows.

Theorem 1 *Suppose that*

$$\lim_{T\to+\infty}\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{s}\tau^{I_f}a(\tau)\,d\tau ds \textit{ exists in } \mathbb{R} \tag{1.4}$$

and define

$$A(t) = \lim_{T\to+\infty} \frac{1}{T} \int_t^T \int_t^s \tau^{I_f} a(\tau)\, d\tau ds, \quad t \geq t_0.$$

Also, suppose that

$$\liminf_{t\to+\infty} A(t) > -\infty. \tag{1.5}$$

Moreover, assume that

$$\int_{t_0}^{\infty} \frac{[A(t) + r(t)]^2}{t}\, dt = +\infty$$

for every continuous real-valued function $r(t)$ on $[t_0, +\infty)$ with $\lim_{t\to+\infty} r(t) = 0$. Then equation (1.1) *is oscillatory.*

A variant of the result, improving the theorem of Onose, which was designed for equation (1.2), is detailed in [13, Theorem 1]. Here, $I_f = \gamma$. The main difference with respect to Theorem 1 is that, in the latter situation, the hypothesis (1.5) is omitted.

It is therefore natural to ask whether or not condition (1.5) can be removed from the hypotheses of Theorem 1. The main contribution here is to answer this question affirmatively. Also, we shall operate a modification of hypothesis (1.4) making it look more like the Kwong–Wong average (1.3).

2 Main Result

Theorem 2 *Fix $\beta \in (0, I_f]$. Let φ be a positive and real-valued, twice continuously differentiable function defined on $[t_0, +\infty)$ such that*

$$\varphi'(t) > 0, \ \varphi''(t) \leq 0 \quad \textit{for all } t \textit{ large enough}$$

and

$$\lim_{t\to+\infty} t\left[\frac{\varphi'(t)}{\varphi(t)}\right] = w_1 > 0 \quad \textit{and} \quad \lim_{t\to+\infty} t\left[\frac{\varphi''(t)}{\varphi'(t)}\right] = w_2,$$

where $w_1 - w_2 = 1$.

Suppose that

$$\lim_{T\to+\infty} \frac{1}{T} \int_{t_0}^T \int_{t_0}^s [\varphi(\tau)]^\beta a(\tau)\, d\tau ds \textit{ exists in } \mathbb{R}$$

and define

$$A_\beta(t) = \lim_{T\to+\infty} \frac{1}{T} \int_t^T \int_t^s [\varphi(\tau)]^\beta a(\tau)\, d\tau ds, \quad t \geq t_0.$$

Moreover, assume that

$$\int_{t_0}^{\infty} \frac{[A_\beta(t) + r(t)]^2}{t}\, dt = +\infty$$

for every real-valued, continuous function $r(t)$ which exists on $[t_0, +\infty)$ and satisfies $\lim_{t\to+\infty} r(t) = 0$. Then equation (1.1) *is oscillatory.*

The following lemma justifies the relationship between the numbers w_1 and w_2.

Lemma 1 *Let φ be a positive, real-valued, twice continuously differentiable function defined on $[t_0, +\infty)$ such that*

$$\varphi'(t) > 0,\ \varphi''(t) \le 0 \quad \text{for all } t \text{ large enough}$$

and ($w_{1,2} \in \mathbb{R}$)

$$\lim_{t\to+\infty} t\Big[\frac{\varphi'(t)}{\varphi(t)}\Big] = w_1 > 0 \quad \text{and} \quad \lim_{t\to+\infty} t\Big[\frac{\varphi''(t)}{\varphi'(t)}\Big] = w_2.$$

Then we have

$$w_1 - w_2 = 1. \tag{2.1}$$

Proof We deduce that

$$\begin{aligned} w_1^2 - w_2 w_1 &= \lim_{t\to+\infty}\Big\{\Big[t\frac{\varphi'(t)}{\varphi(t)}\Big]^2 - \Big[t\frac{\varphi''(t)}{\varphi'(t)}\Big]\Big[t\frac{\varphi'(t)}{\varphi(t)}\Big]\Big\} \\ &= \lim_{t\to+\infty} t^2\Big\{\Big[\frac{\varphi'(t)}{\varphi(t)}\Big]^2 - \frac{\varphi''(t)}{\varphi(t)}\Big\} = \lim_{t\to+\infty}\frac{\frac{d}{dt}\Big[\frac{\varphi'(t)}{\varphi(t)}\Big]}{\frac{d}{dt}\big(\frac{1}{t}\big)}. \end{aligned}$$

Now according to the L'Hôpital rule, we have

$$w_1^2 - w_2 w_1 = \lim_{t\to+\infty}\Big[t\frac{\varphi'(t)}{\varphi(t)}\Big] = w_1.$$

Finally, since $w_1 > 0$, we obtain (2.1). ■

A simple example of auxiliary function φ, besides the identity, is given by $\varphi(t) = t(\ln t)^{-1}$, where $t \ge t_0 > \mathrm{e}^2$. Here, $w_1 = 1$, $w_2 = 0$.

Proof of Theorem 2 Suppose for the sake of contradiction that $x(t)$ is a nonoscillatory solution of (1.1). This means we can without loss of generality consider the function x to be positive-valued throughout $[T_0, +\infty)$ for a certain $T_0 > t_0$.

Following [3, 13], we introduce the variables

$$w(t) = [\varphi(t)]^{\beta} F(x(t)) \quad \text{and} \quad W(t) = 1 + \frac{1}{F(x(t)) f'(x(t))},$$

where $t \ge T_0$.

A straightforward computation yields

$$w'(t) = \beta\Big[\frac{\varphi'(t)}{\varphi(t)}\Big]w(t) + [\varphi(t)]^{\beta}\frac{x'(t)}{f(x(t))}$$

$$w''(t) = [\varphi(t)]^{\beta}\frac{x''(t)}{f(x(t))} + \beta\Big\{\frac{\varphi''(t)}{\varphi(t)} - [1-\beta W(t)]\Big[\frac{\varphi'(t)}{\varphi(t)}\Big]^{2}\Big\}w(t)$$
$$- \frac{1}{w(t)}\Big[w'(t) - \beta W(t)\frac{\varphi'(t)}{\varphi(t)}w(t)\Big]^{2}F(x(t))f'(x(t))$$

for $t \geq T_0$.

Further, we replace $x''(t)$ with $-a(t)f(x(t))$ and integrate the expression of $w''(t)$ twice over $[t,T]$. These lead to

$$-\frac{w(T)}{T} + \frac{w(t)}{T} + \Big(1-\frac{t}{T}\Big)w'(t) = \frac{1}{T}\int_t^T\int_t^s[\varphi(\tau)]^{\beta}a(\tau)\,d\tau ds$$
$$+\frac{1}{T}\int_t^T H_1(s)ds + \frac{1}{T}\int_t^T H_2(s)\,ds,$$

where

$$H_1(s) = \beta\int_t^s\Big\{[1-\beta W(\tau)]\Big[\frac{\varphi'(\tau)}{\varphi(\tau)}\Big]^{2} - \frac{\varphi''(\tau)}{\varphi(\tau)}\Big\}w(\tau)\,d\tau,$$
$$H_2(s) = \int_t^s\frac{1}{w(\tau)}\Big[w'(\tau) - \beta W(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)}w(\tau)\Big]^{2}F(x(\tau))f'(x(\tau))\,d\tau$$

for all $s \geq t \geq T_0$.

Since the integrals $H_{1,2}(s)$ have limits in $[0,+\infty]$ for $s \to +\infty$ as their integrands are functions with nonnegative values throughout $[t,+\infty)$, we conclude, via the L'Hôpital rule, that the averages of $H_{1,2}$ over $[t,T]$ have the same limits when $T \to +\infty$. This yields that (a) the integrals

$$\int_{t_0}^{\infty}\Big\{[1-\beta W(t)]\Big[\frac{\varphi'(t)}{\varphi(t)}\Big]^{2} - \frac{\varphi''(t)}{\varphi(t)}\Big\}w(t)\,dt \tag{2.2}$$

and

$$\int_{t_0}^{\infty}\frac{1}{w(t)}\Big[w'(t) - \beta W(t)\frac{\varphi'(t)}{\varphi(t)}w(t)\Big]^{2}F(x(t))f'(x(t))\,dt \tag{2.3}$$

are convergent, (b) $\lim_{T\to+\infty}\frac{w(T)}{T}$ exists and is finite (which means, actually, that $\frac{w(t)}{t} \leq k$ for all $t \geq T_0$ and a certain number $k > 0$) and (c) the next integral representation holds true

$$w'(t) = \lim_{T\to+\infty}\frac{w(T)}{T} + A_{\beta}(t) + g(t), \tag{2.4}$$

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where

$$g(t) = \beta \int_t^{\infty} \Big\{ [1 - \beta W(s)] \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 - \frac{\varphi''(s)}{\varphi(s)} \Big\} w(s)\, ds + \int_t^{\infty} \frac{1}{w(s)} \Big[w'(s) - \beta W(s) \frac{\varphi'(s)}{\varphi(s)} w(s) \Big]^2 F(x(s)) f'(x(s))\, ds$$

for all $t \geq T_0$.

Further, we define

$$r(t) = \begin{cases} \lim_{T\to+\infty} \frac{w(T)}{T} + g(t) - w(t) \left[\frac{\varphi'(t)}{\varphi(t)} - \frac{\varphi''(t)}{\varphi'(t)} \right] & t \geq T_0, \\ r(T_0) & t_0 \leq t \leq T_0. \end{cases}$$

It is easy to see that $r(t)$ is a continuous function that exists throughout $[t_0, +\infty)$ and satisfies $\lim_{t\to+\infty} r(t) = 0$.

Allow us to establish an essential fact at this point. We claim that *the following integral is convergent*:

$$\int_{T_0}^{\infty} \Big[1 - \beta W(s) - \frac{\varphi''(s)\varphi(s)}{[\varphi'(s)]^2} \Big]^2 \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 w(s)\, ds.$$

Denoting it by $I(T_0)$, we deduce that

$$I(T_0) \leq \int_{T_0}^{\infty} 2[1 - \beta W(s)] \Big\{ [1 - \beta W(s)] \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 - \frac{\varphi''(s)}{\varphi(s)} \Big\} w(s)\, ds + \int_{T_0}^{\infty} \Big[\frac{\varphi''(s)}{\varphi'(s)} \Big]^2 w(s)\, ds.$$

Also, since we have $0 \leq 1 - \frac{\beta}{I_f} \leq 1 - \beta W(t) \leq 1$ and

$$\Big[\frac{\varphi''(t)}{\varphi'(t)} \Big]^2 = \frac{-\varphi''(t)}{\varphi(t)} \cdot \left\{ -t \Big[\frac{\varphi''(t)}{\varphi'(t)} \Big] \cdot \frac{1}{t \big[\frac{\varphi'(t)}{\varphi(t)} \big]} \right\} \leq c \cdot \frac{-\varphi''(t)}{\varphi(t)}$$

for all $t \geq T_0$, where the number $c > \frac{-w_2}{w_1}$ is taken large enough, we conclude that $I(T_0)$ is finite (recall the convergence of (2.2)). The claim is thus established.

Now, by eventually enlarging T_0, we ask that the double inequality holds true

$$\frac{3w_1}{2s} \geq \frac{\varphi'(s)}{\varphi(s)} \geq \frac{w_1}{2s}, \quad s \geq T_0.$$

We obtain that

$$\begin{aligned}
\int_{T_0}^{t} \frac{1}{w(s)} \Big[w'(s) - \beta W(s) \frac{\varphi'(s)}{\varphi(s)} w(s) \Big]^2 F(x(s)) f'(x(s))\, ds \\
&\geq \frac{2I_f}{3kw_1(1-I_f)} \int_{T_0}^{t} \frac{\varphi'(s)}{\varphi(s)} \Big[w'(s) - \beta W(s) \frac{\varphi'(s)}{\varphi(s)} w(s) \Big]^2 ds \\
&= \frac{2I_f}{3kw_1(1-I_f)} \int_{T_0}^{t} \frac{\varphi'(s)}{\varphi(s)} \Big\{ [A_\beta(s) + r(s)] \\
&\quad + \Big\{ \Big[1 - \beta W(s) \Big] \frac{\varphi'(s)}{\varphi(s)} - \frac{\varphi''(s)}{\varphi'(s)} \Big\} w(s) \Big\}^2 ds \\
&\geq \frac{2I_f}{3kw_1(1-I_f)} \int_{T_0}^{t} \frac{\varphi'(s)}{\varphi(s)} [A_\beta(s) + r(s)]^2 ds + \frac{4I_f}{3kw_1(1-I_f)} \\
&\quad \times \int_{T_0}^{t} [A_\beta(s) + r(s)] \Big\{ \Big[1 - \beta W(s) \Big] \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 - \frac{\varphi''(s)}{\varphi(s)} \Big\} w(s) ds \\
&\geq \frac{I_f}{3k(1-I_f)} \int_{T_0}^{t} \frac{[A_\beta(s) + r(s)]^2}{s} ds + \frac{4I_f}{3kw_1(1-I_f)} \\
&\quad \times \int_{T_0}^{t} [A_\beta(s) + r(s)] \Big\{ [1 - \beta W(s)] \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 - \frac{\varphi''(s)}{\varphi(s)} \Big\} w(s)\, ds,
\end{aligned}$$

where $t \geq T_0$.

Further, we must show that the integral

$$\int_{T_0}^{t} [A_\beta(s) + r(s)] \Big\{ [1 - \beta W(s)] \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 - \frac{\varphi''(s)}{\varphi(s)} \Big\} w(s)\, ds$$

is bounded from below for all $t \geq T_0$. This is an essential feature of [13, pp. 113–114], established by making use of restriction (1.5).

In order to avoid hypothesis (1.5), we apply the Cauchy–Schwarz inequality in integral form, namely

$$\begin{aligned}
\int_{T_0}^{t} |A_\beta(s) + r(s)| \Big\{ [1 - \beta W(s)] \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 - \frac{\varphi''(s)}{\varphi(s)} \Big\} w(s)\, ds \\
&= \int_{T_0}^{t} |A_\beta(s) + r(s)| \frac{\varphi'(s)}{\varphi(s)} \sqrt{w(s)} \\
&\quad \times \Big[1 - \beta W(s) - \frac{\varphi''(s)\varphi(s)}{[\varphi'(s)]^2} \Big] \frac{\varphi'(s)}{\varphi(s)} \sqrt{w(s)} ds \\
&\leq \Big\{ \int_{T_0}^{t} [A_\beta(s) + r(s)]^2 \Big[\frac{\varphi'(s)}{\varphi(s)} \Big]^2 w(s) ds \Big\}^{1/2} \sqrt{I(T_0)}
\end{aligned}$$

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$$\leq \frac{3}{2} w_1 \sqrt{I(T_0)} \cdot \Big\{ \int_{T_0}^{t} \frac{[A_\beta(s) + r(s)]^2}{s} \cdot \frac{w(s)}{s} ds \Big\}^{1/2}$$

$$\leq \frac{3}{2} w_1 \sqrt{k \cdot I(T_0)} \Big\{ \int_{T_0}^{t} \frac{[A_\beta(s) + r(s)]^2}{s} ds \Big\}^{1/2}$$

$$\leq \frac{3}{4} w_1 \sqrt{k \cdot I(T_0)} \cdot \Big\{ 1 + \int_{T_0}^{t} \frac{[A_\beta(s) + r(s)]^2}{s} ds \Big\}.$$

Finally, we have obtained that

$$\int_{T_0}^{t} \frac{1}{w(s)} \Big[w'(s) - \beta W(s) \frac{\varphi'(s)}{\varphi(s)} w(s) \Big]^2 F(x(s)) f'(x(s))\, ds$$
$$\geq Q \cdot \int_{T_0}^{t} \frac{[A_\beta(s) + r(s)]^2}{s}\, ds - P, \quad t \geq T_0,$$

where

$$Q = \frac{I_f}{3k(1 - I_f)} \big[1 - 3\sqrt{k \cdot I(T_0)} \big] \quad \text{and} \quad P = \frac{I_f}{1 - I_f} \sqrt{\frac{I(T_0)}{k}}.$$

Since $I(T_0)$ is finite, it is obvious that we can consider T_0 large enough so that $Q > 0$. Then, by passing to the limit as $t \to +\infty$ in the latter inequality, we are in contradiction with the convergence of (2.3).

The proof is complete. ■

Example 1 Assume that φ is a function which satisfies the hypotheses of Lemma 1. Then for any $\lambda \in [0, 1]$ and $\beta \in (0, I_f]$ the strictly sublinear ordinary differential equation

$$x'' + \frac{t^\lambda}{[\varphi(t)]^\beta} \sin t \cdot f(x) = 0, \quad t \geq t_0 > 0, \tag{2.5}$$

satisfies the hypotheses of Theorem 2 (thus being oscillatory) whereas, for certain values of λ and β in the above-mentioned range, some of the hypotheses of Theorem 1 are not fulfilled.

To prove this assertion, we first take $\delta \in [0, 1]$ and perform the next computation

$$\int_{t}^{s} \frac{\tau^\delta}{[\varphi(\tau)]^\beta} \sin \tau \, d\tau = U(s) - U(t), \quad s \geq t \geq t_0,$$

where

$$U(s) = \frac{1}{[\varphi(s)]^\beta} (\delta s^{\delta - 1} \sin s - s^\delta \cos s) - \frac{\beta}{[\varphi(s)]^{1+\beta}} s^\delta \varphi'(s) \sin s - \int_{t}^{s} V(\tau)\, d\tau$$

and

$$V(\tau) = \frac{\delta(\delta-1)}{[\varphi(\tau)]^{\beta}}\tau^{\delta-2}\sin\tau - 2\frac{\delta\beta}{[\varphi(\tau)]^{1+\beta}}\tau^{\delta-1}\varphi'(\tau)\sin\tau - \frac{\beta}{[\varphi(\tau)]^{1+\beta}}\tau^{\delta}\varphi''(\tau)\sin\tau + \frac{\beta(1+\beta)}{[\varphi(\tau)]^{2+\beta}}\tau^{\delta}[\varphi'(\tau)]^2\sin\tau.$$

Further, we notice that $V \in L^1((t_0,+\infty),\mathbb{R})$. The claim is established by showing that each of the terms from the sum V is integrable, *e.g.*,

$$\int_t^s \Big| \frac{\tau^{\delta-1}}{[\varphi(\tau)]^{1+\beta}}\varphi'(\tau)\sin\tau \Big| \, d\tau \le \int_t^s \frac{\tau^{\delta}}{[\varphi(\tau)]^{\beta}}\Big[\tau\frac{\varphi'(\tau)}{\varphi(\tau)}\Big]\frac{d\tau}{\tau^2} \le \sup_{s\ge t_0}\Big[s\frac{\varphi'(s)}{\varphi(s)}\Big] \cdot \int_t^{+\infty} \frac{d\tau}{\tau^{2-[\delta-\beta(w_1-\varepsilon)]}}$$

for all $t \ge T_1 \ge t_0$, where T_1 is chosen large enough in order to have

$$\varphi(T_1)(t/T_1)^{w_1-\varepsilon} \le \varphi(t) \le \varphi(T_1)(t/T_1)^{w_1+\varepsilon}$$

throughout $[T_1,+\infty)$ for a certain $\varepsilon \in (0,w_1)$. Since $w_2 \le 0$, we have $w_1 \in (0,1]$ which means that $\delta - \beta(w_1-\varepsilon) \in (0,1)$.

Finally, we obtain that

$$\lim_{T\to+\infty}\frac{1}{T}\int_t^T\int_t^s \frac{\tau^{\delta}}{[\varphi(\tau)]^{\beta}}\sin\tau\, d\tau ds = -U(t) - \int_t^{+\infty} V(\tau)\,d\tau = \frac{t^{\delta}}{[\varphi(t)]^{\beta}}\cos t + o(1) \quad \text{when } t \to +\infty.$$

In order to see that (1.5) does not always hold, assume that $\lambda \in (0, 1-I_f]$, $\beta \in (0, \frac{I_f}{w_1+\varepsilon}]$ and take $\delta = \lambda + I_f$. These assumptions imply that $\delta - \beta(w_1+\varepsilon) > 0$. Thus, we obtain $\frac{t^{\delta}}{[\varphi(t)]^{\beta}} \ge (T_1^{w_1+\varepsilon}/\varphi(T_1))^{\beta} \cdot t^{\delta-\beta(w_1+\varepsilon)}$ for all $t \ge T_1$ and so $\liminf_{t\to+\infty} A(t) = -\infty$.

To prove that the functional coefficient $a(t) = \frac{t^{\lambda}}{[\varphi(t)]^{\beta}}\sin t$ of equation (2.5) satisfies the hypotheses of Theorem 2, we follow verbatim the computations from [12, p. 120] and [13, pp. 109-110] and conclude that $A_{\beta}(t) = t^{\lambda}\cos t + o(1)$ when $t \to +\infty$.

Further, we have that

$$\int_{t_0}^{\infty}\frac{[A_{\beta}(t)+r(t)]^2}{t}\,dt \ge \sum_{n=N}^{+\infty}\int_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}}\frac{[t^{\lambda}\cos t + o(1)]^2}{t}\,dt \ge \sum_{n=N}^{+\infty}\int_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}}\frac{\big(\cos t - \frac{1}{2\sqrt{2}}\big)^2}{t}\,dt \ge \frac{1}{8}\sum_{n=N}^{+\infty}\int_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}}\frac{dt}{t} \ge \frac{1}{2}\int_{2N\pi-\frac{\pi}{4}}^{+\infty}\frac{dt}{t} = +\infty$$

for a certain integer $N \geq \max\{\frac{1}{2\pi}\left[t_0 + \frac{\pi}{4}\right], 1\}$ large enough.

In the particular case of φ being the identity and $\beta = I_f$, we regain that equation (1.2), where $\lambda \in [-I_f, 1 - I_f]$, is oscillatory.

We shall close the present note with a comment regarding the function A_β. It has been established in [10, Lemma 2.1] that A_β is continuously differentiable, $A'_\beta(t) = -[\varphi(t)]^\beta a(t)$ throughout $[t_0, +\infty)$ and also

$$\lim_{t\to+\infty} \frac{1}{t} \int_{t_0}^{t} A_\beta(s)\, ds = 0. \tag{2.6}$$

Though at first glance a restriction, condition (2.6) occurs naturally in our computations. To see this, let us assume again that equation (1.1) has an eventually positive solution and thus, as in the preceding proof, we arrive at (2.4). By integrating this formula over $[T_0, t]$, then dividing by t, and passing to the limit as $t \to +\infty$, we obtain (2.6).

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