SET-TRANSITIVE PERMUTATION GROUPS

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1. Introduction. The concept of an s-ply transitive ($1 \leqslant s \leqslant n$) permutation group on n symbols is of considerable importance in the classical theory of finite permutation groups, which was in the height of its development in the period around the turn of the century. The obvious generalization to a permutation group which is s set-transitive (i.e., a group which, for each pair of s-element unordered subsets S, T of the given n symbols, contains a permutation which carries S into T) seems to have received little attention. A discussion **(8**, p. 257**)** of the symmetry of an arbitrary n-person game leads in a natural way to the notion of a set-transitive permutation group (i.e., a group which is s set-transitive for all s) on the n players of the game. In the preface to **(8)**, credit is given to C. Chevalley for solving the problem of determining all set-transitive groups. Since, to our knowledge, nothing has appeared in the literature on this problem, we believe that a complete and relatively simple solution is of interest.

In §2, the definitions are given, and the alternating and symmetric groups A_n and S_n, along with the trivial cases for $n \leqslant 3$, are considered. The properties of s set-transitive groups which are used for their enumeration are derived in §3, the principal results being contained in Theorems 5 and 6, which state that these groups are transitive and primitive. In this connection, a recent paper by Bays **(1)**, relating a concept of primitivity for ordered pairs to the degree of transitivity of a permutation group, is of interest. In §4, a theorem on the distribution of prime numbers **(2)**, is used to eliminate the possibility of set-transitive groups for $n \geqslant 82$. Various special results in **(5)** and **(7)** are used to obtain Theorem 10, which states that set-transitive groups, other than A_n and S_n, are possible only for $n = 5$, 6, and 9. Finally, in §5, all of the set-transitive groups of degree 5, 6, and 9 are determined. The results of this section are given in Theorem 11.

2. Definitions. We begin with a formal statement of the principal definitions and their immediate consequences. Since every permutation on n symbols sends the complete set of n symbols into itself, we exclude this trivial case from the following definition. Thus the case $n = 1$ is excluded. Further, the identity group I on n symbols can clearly be omitted from consideration, so that in the sequel, by a group $\mathfrak{G}$, we will mean a permutation group $\mathfrak{G} \neq I$ on the set N of $n \geqslant 2$ symbols, $N = [1, 2, \ldots, n]$.

Definition 1. A group $\mathfrak{G}$ is s *set-transitive* ($1 \leqslant s \leqslant n - 1$) if for every pair of subsets S, T of $N = [1, 2, \ldots, n]$, each containing s elements, there exists a permutation in $\mathfrak{G}$ which carries S into T.

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It should be noticed that according to this definition, "1 set-transitive" and "transitive" mean the same thing. We have the following immediate consequences of Definition 1:

(i) If the group $\mathfrak{G}$ contains an s set-transitive subgroup $\mathfrak{H}$, then $\mathfrak{G}$ is s set-transitive.

(ii) If the group $\mathfrak{G}$ is k-ply transitive, then $\mathfrak{G}$ is s set-transitive for all $s \leqslant k$.

(iii) If the group $\mathfrak{G}$ contains permutations which carry the set $S = [1,2, \ldots, s]$ into any other set T containing s elements, then $\mathfrak{G}$ is s set-transitive.

With $N = [1,2,3,4,5,6,7]$, the group $\mathfrak{G} = \{(1234567), (235)(476)\}$, is an example of a group which is 2 set-transitive but not doubly transitive. $\mathfrak{G}$ is not 3 or 4 set-transitive.

DEFINITION 2. A group $\mathfrak{G}$ is *set-transitive* if $\mathfrak{G}$ is s set-transitive for all s $(1 \leqslant s \leqslant n - 1)$.

Since the alternating group A_n $(n \geqslant 3)$ is $(n - 2)$-ply transitive and since A_2 is intransitive, we have

THEOREM 1. *The alternating group A_n is set-transitive except for $n = 2$. The symmetric group S_n is set-transitive.*

Proof. For $n \geqslant 3$, we have by (ii) that A_n is s set-transitive for all $s \leqslant n - 2$. Since, in particular, A_n is transitive, A_n contains a permutation which sends n into any other symbol j. Therefore A_n contains a permutation which sends $S = [1, 2, \ldots, n - 1]$ into any other $n - 1$ element set T. By (iii), A_n is $n - 1$ set-transitive, and hence set-transitive. Since S_n is n-ply transitive for all n, S_n is set-transitive.

It follows from the theorem that, in the determination of all set-transitive groups, we need only consider those groups $\mathfrak{G} \neq I$ which do not contain the alternating group. For $n = 2$, there are none. For $n = 3$, the only such groups are the three cyclic groups $\{(12)\}$, $\{(13)\}$, and $\{(23)\}$ which are not 1 or 2 set-transitive. Having disposed of these trivial cases, we will henceforth assume that $n \geqslant 4$.

3. Properties of s set-transitive groups. In this section we derive properties of s set-transitive groups which are needed for their enumeration. The principal result is that an s set-transitive group is primitive if $s > 1$.

THEOREM 2. *A group $\mathfrak{G}$, which is the conjugate in S_n of an s set-transitive group $\mathfrak{H}$, is s set-transitive.*

Proof. Let σ be a permutation in S_n such that $\mathfrak{G} = \sigma \mathfrak{H} \sigma^{-1}$. Let $S = [1, 2, \ldots, s]$, and $J = [j_1, j_2, \ldots j_s]$ be an arbitrary s-element subset. Define the set $I = [i_1, i_2, \ldots, i_s]$ by $\sigma^{-1} S = I$, and the set $K = [k_1, k_2, \ldots, k_s]$ by $\sigma^{-1} J = K$. Since $\mathfrak{H}$ is s set-transitive, there exists a permutation $\tau \in \mathfrak{H}$ such that $\tau I = K$. Then

$$\sigma\tau\sigma^{-1} \in \mathfrak{G}, \quad \sigma\tau\sigma^{-1} S = \sigma\tau I = \sigma K = J.$$

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THEOREM 3. *If the group $\mathfrak{G}$ is s set-transitive, then $\mathfrak{G}$ is $n - s$ set-transitive.*

Proof. Let $S = [1, 2, \ldots, n - s]$, $C(S) = [n - s + 1, n - s + 2, \ldots, n]$. Let $J = [j_1, j_2, \ldots, j_{n-s}]$ be an arbitrary $(n - s)$-element subset, and $C(J) = [i_1, i_2, \ldots, i_s]$ be the complement of J in N. There exists a permutation $\sigma \in \mathfrak{G}$ such that $\sigma C(S) = C(J)$, so that $\sigma(S) = J$.

THEOREM 4. *If the group $\mathfrak{G}$ is s set-transitive, then the order of $\mathfrak{G}$ is $m\binom{n}{s}$, where m is the order of the subgroup $\mathfrak{G}_1$ of $\mathfrak{G}$ consisting of those permutations which carry the subset $S = [1, 2, \ldots, s]$ into itself.*

Proof. It is clear that the subset $\mathfrak{G}_1$ of $\mathfrak{G}$ consisting of those permutations which carry $S = [1, 2, \ldots, s]$ into itself is a subgroup. Since there are $t = \binom{n}{s}$ distinct s-element subsets of N, denote them by $I_1 = S, I_2, \ldots, I_t$, and denote by $\sigma_1, \sigma_2, \ldots \sigma_t$ a set of permutations in $\mathfrak{G}$ such that $\sigma_k I_1 = I_k$, for $k = 1, 2, \ldots, t$. Then $\sigma_1, \sigma_2, \ldots, \sigma_t$ form a complete set of representatives of $\mathfrak{G}$ modulo $\mathfrak{G}_1$, so that the order of $\mathfrak{G}$ is mt, where m is the order of $\mathfrak{G}_1$, and $t = \binom{n}{s}$.

COROLLARY. *If the group $\mathfrak{G}$ is set-transitive, then the order of $\mathfrak{G}$ is divisible by the least common multiple of the binomial coefficients*

$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}.$$

The 2 set-transitive group $\mathfrak{G} = \{(1234567), (235)(476)\}$ of degree 7 has order 21. Since $\binom{7}{2} = 21$, $\mathfrak{G}$ has minimum order for a 2 set-transitive group of this degree. Since

$$\binom{7}{3} = \binom{7}{4} = 35,$$

$\mathfrak{G}$ cannot be 3 or 4 set-transitive.

THEOREM 5. *If the group $\mathfrak{G}$ is s set-transitive for at least one s, then $\mathfrak{G}$ is transitive.*

Proof. Assume that $\mathfrak{G}$ is intransitive and let $L \subset N$ be the smallest transitivity set of $\mathfrak{G}$. Then L has l elements where $l \leqslant \frac{1}{2}n$, and we may assume by Theorem 3 that $l \leqslant s$. Since $l \leqslant s \leqslant n - 1$, there exists an s-element subset S of N such that $L \subseteq S$ and such that $C(S)$, the complement of S in N, contains an element not in L. By removing an element of L from S and replacing it by an element from $C(S)$, we can construct an s-element subset T such that $L \not\subseteq T$. Since $\mathfrak{G}$ is s set-transitive, there exists a permutation $\sigma \in \mathfrak{G}$ such that $\sigma S = T$. Since L is a transitivity set of $\mathfrak{G}$, $\sigma L = L$, and since $L \subseteq S$,

$\sigma L = L \subseteq \sigma S = T$. But this contradicts the choice of T, proving that $\mathfrak{G}$ must be transitive.

THEOREM 6. *If the group $\mathfrak{G}$ is s set-transitive for at least one $s > 1$, then $\mathfrak{G}$ is primitive.*

Proof. By Theorem 3, we may suppose that $s \leqslant \frac{1}{2}n$. $\mathfrak{G}$ is transitive by Theorem 5, and assume that $\mathfrak{G}$ is imprimitive. Then the set $N = [1, 2, \ldots, n]$ can be partitioned into $r \geqslant 2$ subsets N_i, each containing $l \geqslant 2$ elements, such that every permutation in $\mathfrak{G}$ carries each N_i into some N_j.

If $s \leqslant l$, there is an s-element subset S which is a subset of N_1. Since $s > 1$, there is an s-element subset T which contains elements from both N_1 and N_2. But since $\mathfrak{G}$ is s set-transitive, there exists a permutation in $\mathfrak{G}$ which carries S into T and this contradicts the assumption that $\mathfrak{G}$ is imprimitive.

If $l < s$, then since $s \leqslant \frac{1}{2}n$, we have $l < \frac{1}{2}n$ and $r \geqslant 3$. Then there is an s-element subset S which contains elements only from the sets of imprimitivity, $N_1, N_2, \ldots, N_k$ where $1 < k \leqslant \frac{1}{2}(r + 1)$, such that S contains $N_1, N_2, \ldots, N_{k-1}$. Since $r \geqslant 3$, $r - k \geqslant \frac{1}{2}(r - 1) \geqslant 1$, and the $r - k$ sets of imprimitivity, $N_{k+1}, \ldots, N_r$, are disjoint from those from which S was constructed. Therefore there is an s-element subset T, constructed by replacing one of the elements of N_1 by an element from N_{k+1}, which contains elements from $k + 1$ different sets of imprimitivity. Again, the existence of a permutation in $\mathfrak{G}$ which carries S into T contradicts the assumption that $\mathfrak{G}$ is imprimitive.

In the next section, we use a classical result (**3**, p. 199) due to Jordan and Netto, which connects primitivity with the degree of transitivity of $\mathfrak{G}$, to show that set-transitive groups are rare exceptions.

4. Determination of the values of n for which set-transitive groups may exist.

THEOREM 7. *If the group $\mathfrak{G}$ is s set-transitive $(s > 1)$, and if there exists a prime p such that $\frac{1}{2}n < p \leqslant n$ and such that p divides $\binom{n}{s}$, then $\mathfrak{G}$ is $(n-p+1)$-ply transitive.*

Proof. Since the order of $\mathfrak{G}$ is $m\binom{n}{s}$ by Theorem 4, if a prime p divides $\binom{n}{s}$, $\mathfrak{G}$ contains an element of order p. The elements of order p in $\mathfrak{G}$, when written as a product of cycles on disjoint letters, are products of cycles of length p. Since $p > \frac{1}{2}n$, the elements of order p in $\mathfrak{G}$ are cycles of length p. Such a cycle generates a cyclic subgroup $\mathfrak{H}$ of $\mathfrak{G}$ which is of degree p. Thus $\mathfrak{H}$ is primitive and leaves $n - p$ letters unchanged. Since $\mathfrak{G}$ is primitive by Theorem 6, we obtain the conclusion of the theorem by employing the result mentioned above (**3**, p. 199) which states that if a primitive group $\mathfrak{G}$ contains a primitive subgroup of degree m which leaves the remaining $n - m$ letters unchanged, then $\mathfrak{G}$ is $(n - m + 1)$-ply transitive.

COROLLARY 1. *If the group $\mathfrak{G}$ is s set-transitive $(s > 1)$, and if there exists a prime p such that* $\max(s, n - s) < p \leqslant n$, *then $\mathfrak{G}$ is $(n - p + 1)$-ply transitive.*

Proof. If $p > \max(s, n - s)$, then $p > \frac{1}{2}n$ and p divides

$$\binom{n}{s} = \frac{n!}{s!\,(n-s)!},$$

so that the hypotheses of the theorem are satisfied.

In the determination of set-transitive groups, the critical value of s is $s = [\frac{1}{2}n]$, where as usual this symbol denotes the greatest integer in $\frac{1}{2}n$.

COROLLARY 2. *If the group $\mathfrak{G}$ is $[\frac{1}{2}n]$ set-transitive, and if there exists a prime p such that $\frac{1}{2}(n + 1) < p \leqslant n$, then $\mathfrak{G}$ is $(n - p + 1)$-ply transitive.*

Proof. For $p > \frac{1}{2}(n + 1) \geqslant [\frac{1}{2}(n + 1)] = \max([\frac{1}{2}n], n - [\frac{1}{2}n])$, and the hypotheses of Corollary 1 are satisfied for $s = [\frac{1}{2}n]$.

We now make use of various known limits of transitivity to eliminate the possibility of the existence of $[\frac{1}{2}n]$ set-transitive, and therefore set-transitive, groups. The principal criterion is given in the following theorem.

THEOREM 8. *If there exists a prime p such that $\frac{1}{2}(n + 1) < p < \frac{2}{3}n$, then a group $\mathfrak{G}$ on n symbols, which does not contain the alternating group A_n, cannot be $[\frac{1}{2}n]$ set-transitive.*

Proof. Assume that $\mathfrak{G}$ is $[\frac{1}{2}n]$ set-transitive. Then if a prime p exists in the given range, $\mathfrak{G}$ is $(n - p + 1)$-ply transitive by Corollary 2, Theorem 7. But since $\frac{1}{3}n + 1 = n - \frac{2}{3}n + 1 < n - p + 1$, and since $\frac{1}{3}n + 1$ is an upper limit for the degree of transitivity (**3**, p. 152) for a group $\mathfrak{G}$ not containing A_n, we have a contradiction.

There are many refinements of Bertrand's postulate which states that a prime exists in the range between x and $2x$. One such result which is convenient for our purposes is due to Breusch (**2**). He shows that for $x \geqslant 48$, there always exists a prime between x and $9x/8$. For $n \geqslant 82$, $x = \frac{1}{2}(n + 14) \geqslant 48$, and there exists a prime between $\frac{1}{2}(n + 14)$ and $9(n + 14)/16$. Since $\frac{1}{2}(n + 1) < \frac{1}{2}(n + 14)$, and $9(n + 14)/16 < \frac{2}{3}n$ for $n \geqslant 82$, there exists a prime in the range given in the hypothesis of Theorem 8. By examining a table of primes, we find a prime strictly between $\frac{1}{2}(n + 1)$ and $\frac{2}{3}n$ for all $n \geqslant 26$, and we have

THEOREM 9. *A group $\mathfrak{G}$ on $n \geqslant 26$ symbols, which does not contain the alternating group A_n, cannot be $[\frac{1}{2}n]$ set-transitive, and therefore is not set-transitive.*

For $n < 26$, a table of primes shows that a prime lies in the required range for $n = 8, 11, 12$ and for all n such that $17 \leqslant n \leqslant 24$. Since the cases for $n < 4$ have been previously discussed, we have only the cases

$$n = 4, 5, 6, 7, 9, 10, 13, 14, 15, 16, 25$$

to investigate. For these cases we first employ a better result on the limit of transitivity due to Miller (**7**, vol. III, p. 439) which states that if $n = kp + r$, where p is a prime greater than the positive integer k and where $r > k$, then a group $\mathfrak{G}$ on n symbols, not containing the alternating group A_n, cannot be more than r-ply transitive, unless $k = 1$ and $r = 2$. As an example, with $n = 25$, we obtain from Corollary 2, Theorem 7 with $p = 17$, that if $\mathfrak{G}$ is 12 set-transitive, then $\mathfrak{G}$ is 9-ply transitive. But $25 = 1 \cdot 19 + 6$ with $k = 1$, $p = 19$, and $r = 6$, so that $\mathfrak{G}$ cannot be more than 6-ply transitive. Therefore there are no groups on $n = 25$ symbols, other than A_{25} and S_{25}, which are $[\frac{1}{2}n] = 12$ set-transitive. In this way, the cases $n = 10, 14, 15, 16$, and 25 are eliminated.

Miller has proved (**7**, vol. I, p. 200) that a transitive group of degree 13, which does not contain the alternating group A_{13}, is at most doubly transitive. By Corollary 2, Theorem 7 with $p = 11$, we have that if $\mathfrak{G}$ is $[\frac{1}{2}n] = 6$ set-transitive, then $\mathfrak{G}$ is triply transitive. Similarly, for $n = 7$, a transitive group not containing A_7 is at most doubly transitive (**5**, p. 186; **6**, p. 338; **7**, vol. I, pp. 1-9), while Corollary 2, Theorem 7, with $p = 5$, gives that if $\mathfrak{G}$ is $[\frac{1}{2}n] = 3$ set-transitive, then $\mathfrak{G}$ is triply transitive.

A 2 set-transitive group $\mathfrak{G}$ on $n = 4$ symbols has an order divisible by $\binom{4}{2} = 6$ by Theorem 4. The only such transitive groups are A_4 and S_4 (**6**, p. 338; **7**, vol. I, pp. 1-9). The cases $n = 2$ and 3 were eliminated in §1.

We summarize the above results in the following theorem.

THEOREM 10. *A group $\mathfrak{G}$ on n symbols, which does not contain the alternating group A_n, cannot be $[\frac{1}{2}n]$ set-transitive, and therefore is not set-transitive, with the exceptions of $n = 5, 6$, and* 9.

5. Determination of the set-transitive groups for $n = 5, 6$, and 9. In the following determination of set-transitive groups, we use the table of transitive groups on $n \leqslant 9$ symbols given by Cole in (**6**). Although this list has two omissions for $n = 8$, it has been verified by Miller (**7**, vol. I, pp. 1-9, 12-14) and others for $n = 5, 6$, and 9.

If a group $\mathfrak{G}$ on 5 symbols is 2 set-transitive, then by Theorem 4 the order of $\mathfrak{G}$ is divisible by $\binom{5}{2} = 10$. Thus the only possibilities for a 2 set-transitive group, other than A_5 or S_5, are the transitive groups:

$$G_1 = \{(12345), (1325)\}, \qquad \text{order } 20,$$
$$H_1 = \{(12345), (12)(35)\}, \qquad \text{order } 10,$$

and their conjugate subgroups in S_5. Since G_1 is doubly transitive, it is 2 set-transitive, and therefore 3 set-transitive by Theorem 3. Since G_1 is transitive, it is 4 set-transitive by this same theorem. Thus G_1 and its conjugate subgroups in S_5 (Theorem 2) are set-transitive. If H_1 were 2 set-transitive, then the order of H_1 would be $10m$, where m is the order of the subgroup of H_1 which carries the set [1,2] into itself. Thus $m = 1$, and the identity is the only

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element of H_1 which carries [1,2] into itself. But $(12)(35) \in H_1$ carries [1,2] into itself. Therefore H_1 is not 2 set-transitive.

By Theorem 4, if a group $\mathfrak{G}$ on 6 symbols is 3 set-transitive, then the order of $\mathfrak{G}$ is $m\binom{6}{3} = 20m$, where m is the order of the subgroup of $\mathfrak{G}$ which sends the set [1,2,3] into itself. The only possibilities for a 3 set-transitive group, other than A_6 or S_6, are the transitive groups:

$$G_2 = \{(12345),(12)(35),(13465),(1325)\}, \qquad \text{order } 120,$$
$$H_2 = \{(12345),(12)(35),(13465)\}, \qquad \text{order } 60,$$

and their conjugate subgroups in S_6. Since G_2 is triply transitive, it is both 3 and 2 set-transitive, and by Theorem 3, G_2 is set-transitive, as are its conjugates in S_6. If H_2 were 3 set-transitive, then the order of the subgroup of H_2 which carries the set [1,2,3] into itself would be 3. However, the permutations

$$(1),(13)(45),(123)(465),(23)(56)$$

in H_2 carry [1,2,3] into itself. Therefore H_2 is not 3 set-transitive.

Again using Theorem 4, a 4 set-transitive group on 9 symbols has order $m\binom{9}{4} = 126m$. The only possibilities, other than A_9 and S_9, are the transitive groups:

$$G_3 = \{(1254673),\ (15)(29)(47)(68),\ (124)(765)\}, \qquad \text{order } 1512,$$
$$H_3 = \{(1254673),\ (15)(29)(47)(68)\}, \qquad \text{order } 504,$$

and their conjugate subgroups in S_9. Since H_3 is triply transitive, H_3 is 1, 2, 3, 6, 7 and 8 set-transitive by Theorem 3. By this same theorem if H_3 is 4 set-transitive, it is 5 set-transitive and consequently set-transitive. That a permutation can be found in H_3 which sends the set [1,2,3,4] into each of the 126 four element subsets of $N = [1,2,3,4,5,6,7,8,9]$ has been checked directly by the authors. Therefore H_3 is set-transitive, and since H_3 is a subgroup of G_3, G_3 is also set-transitive. We summarize these results in the following theorem which, as was indicated at the beginning of this section, depends in part on the correctness of the list of transitive groups (**6**, p. 338) for $n = 5$, 6, and 9.

THEOREM 11. *The only groups on n symbols, other than the symmetric and alternating groups S_n and A_n, which are $[\frac{1}{2}n]$ set-transitive, are the groups G_1, G_2, H_3, and G_3, and their conjugates, on* 5, 6, 9, *and* 9 *symbols, respectively. These four groups are set-transitive.*

In the verification that the group H_3 is 4 set-transitive, an element of the form $\tau\sigma^n$, where τ is an element of order two or three in H_3 and $\sigma = (1254673) \in H_3$, can be found which carries the set [1,2,3,4] into each four element subset $[a,b,c,d]$. For example, with $\tau = (64)(72)(51)(39)$ and $n = 4$, $\tau\sigma^n$ carries [1,2,3,4] into [2,3,5,9]. In the same way, with $\tau = (756)(412)(839)$ and $n = 0,1,4,5,6$, $\tau\sigma^n$ carries [1,2,3,4] into [1,2,4,9], [2,5,6,9], [1,6,7,9], [2,3,7,9], and [1,3,5,9], respectively.

It may be of some interest to give an additional description of the four set-transitive groups G_1, G_2, H_3, and G_3. As an abstract group, G_1 is metacyclic with defining relations $R^5 = S^4 = 1$, $S^{-1}RS = R^2$ (**7**, vol. III, p. 241). G_2 is isomorphic to the symmetric group S_5, and has the abstract defining relations $R^5 = S^4 = (RS^2)^3 = (R^3S)^2 = 1$ (**7**, vol. III, p. 241). The group H_3 is the simple group $LF(2,8)$ consisting of all the linear fractional transformations

$$x' = \frac{ax + b}{cx + d}$$

where a, b, c, d are elements of $GF(2^3)$ such that $ad - bc \neq 0$. As an abstract group, $H_3 = \{A, B\}$ has defining relations

$$A^7 = B^2 = (AB)^3 = (A^3BA^5BA^3B)^2 = 1$$

(**4**, p. 174). Finally, G_3 is isomorphic to the group of automorphisms of H_3, and $G_3 = \{A, B, C\}$, where C satisfies the relations

$$C^3 = 1,\ CAC^{-1} = A^2,\ CBC^{-1} = ABA^4BA^4BA.$$

Thus G_3 has order 1512 and contains H_3 as a normal subgroup of index 3. The generators A, B, and C are given by $A = (1254673)$, $B = (15)(29)(47)(68)$, and $C = (124)(765)$ in terms of the given generators of H_3 and G_3 as permutation groups on nine symbols.

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