LEBESGUE CONSTANTS FOR REGULAR TAYLOR SUMMABILITY

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(received September 21, 1964)

1. Introduction. The n^{th} Taylor mean of order r of a sequence $\{s_n\}$ is given by

$$\sigma_n^r = \sum_{k=0}^{\infty} a_{nk} s_k \tag{1.1}$$

where

$$\frac{(1-r)^{n+1}\theta^n}{(1-r\theta)^{n+1}} = \sum_{k=0}^{\infty} a_{nk}\theta^k, \qquad |r\theta| < 1. \tag{1.2}$$

Cowling [1] has shown that this method is regular if and only if $0 \leq r < 1$. Since $r = 0$ corresponds to ordinary convergence, it will be assumed here that $0 < r < 1$.

The n^{th} Taylor-Lebesgue constant of order r is given by

$$L_T^r(n) = \frac{2}{\pi}\int_0^{\pi/2} \left|\sum_{k=0}^{\infty} a_{nk} \sin(2k+1)u\right| \frac{du}{\sin u}. \tag{1.3}$$

These constants have already been studied by Ishiguro [2] and Lorch and Newman [6] who showed, independently, that

$$L_T^r(n) = \frac{2}{\pi^2}\log\frac{2n}{r} + \alpha + o(1) \qquad \text{as } n \to \infty \tag{1.4}$$

Canad. Math. Bull. vol. 8, no. 6, 1965

where

$$(1.5) \qquad \alpha = -\frac{2}{\pi^2}\gamma + \frac{2}{\pi}\int_0^1 \frac{\sin t}{t}\,dt - \frac{2}{\pi}\int_1^\infty \left[\frac{2}{\pi} - |\sin t|\right]\frac{dt}{t}$$

where γ is Euler's constant.

Here it will be proved that

$$(1.6) \qquad L_T^r(n-1) = L_B\left(\frac{n}{r}\right) + o(1) \qquad \text{as } n \to \infty .$$

where $L_B(x)$ is the x^{th} Borel-Lebesgue constant. Using the result of Lorch [4],

$$(1.7) \qquad L_B\left(\frac{n}{r}\right) = \left[\frac{2}{\pi^2}\log\frac{n}{r} - \gamma - \log\frac{\pi^2}{2} - \int_0^\pi \psi\left(\frac{t}{\pi}\right)\sin t\,dt\right] + O\left(\frac{1}{\sqrt{n}}\right) \qquad \text{as } n \to \infty$$

where

$$(1.8) \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} ,$$

it follows that

$$(1.9) \qquad L_T^r(n) = L_T^r(n-1) + O\left(\frac{1}{n}\right) \qquad \text{as } n \to \infty$$

$$= L_B\left(\frac{n}{r}\right) + o(1) \qquad \text{as } n \to \infty .$$

2. <u>Proof of 1.6.</u> Using (1.2),

$$(2.1) \qquad L_T^r(n-1) = \frac{2}{\pi}\int_0^{\pi/2} \left(\frac{1-r}{\rho}\right)^n |\sin[(2n-1)u + n\theta]| \frac{du}{\sin u} ,$$

https://doi.org/10.4153/CMB-1965-060-2 Published online by Cambridge University Press

where

$$(2.2) \qquad \rho e^{-i\theta} = 1 - r e^{2iu} .$$

The following lemmas and corollaries are required, the proofs being given in Section 4:

LEMMA 1

$$\left(\frac{1-r}{\rho}\right)^2 \leqq e^{-2Au^2} \qquad \text{for } 0 \leqq u \leqq \frac{\pi}{2},$$

where A is a positive constant.

COROLLARY 1

$$\left(\frac{1-r}{\rho}\right)^n \leqq e^{-Anu^2} \qquad \text{for } 0 \leqq u \leqq \frac{\pi}{2} .$$

LEMMA 2

$$\left| \left(\frac{1-r}{\rho}\right)^2 - e^{-\frac{4ru^2}{(1-r)^2}} \right| \leqq Bu^4 \qquad \text{for } u \geqq 0 ,$$

where B is a positive constant.

COROLLARY 2

$$\left| \left(\frac{1-r}{\rho}\right)^n - e^{-\frac{2nru^2}{(1-r)^2}} \right| \leqq Bnu^4 \qquad \text{for } u \geqq 0 .$$

LEMMA 3 [Due to Miracle (see 4.9 of [7])].

$$\left|\theta - \frac{2ru}{1-r}\right| \leqq Cu^3 \qquad \text{for } 0 \leqq u \leqq \frac{\pi}{2} ,$$

where C is a positive constant.

COROLLARY 3

$$\left|\,|\sin[(2n-1)u + n\theta]| - |\sin[(2\frac{n}{r}+1)\frac{ru}{1-r}]|\,\right| \leqq Cnu^3 + \frac{ru}{1-r}$$

$$\text{for } 0 \leqq u \leqq \frac{\pi}{2} .$$

The following standard results are also required:

(2.3) $\frac{2}{\pi} u \leqq \sin u \leqq u$ for $0 \leqq u \leqq \frac{\pi}{2}$

(2.4) $0 \leqq u - \sin u \leqq u^3$ for $u \geqq 0$.

The integral in (2.1) can be reduced, in three steps, to a simpler, approximate integral, the error committed at each step being $o(1)$ as $n \to \infty$. To facilitate computation, it is convenient to reduce the range of integration by replacing $\pi/2$ by δ where $\delta = n^{-\varepsilon}$. Further analysis will show that ε must be chosen to lie between 1/3 and 1/2. Hence, to be specific, let

(2.5) $$\delta = n^{-3/8} .$$

Then the error committed by replacing $\pi/2$ by δ is

$$\frac{2}{\pi} \int_{\delta}^{\pi/2} \left(\frac{1-r}{\rho}\right)^n |\sin[(2n-1)u + n\theta]| \frac{du}{\sin u}$$

$$\leqq \frac{2}{\pi} \int_{\delta}^{\pi/2} e^{-Anu^2} \frac{du}{(2/\pi)u} \quad \text{by Corollary 1 and (2.3)}$$

$$\leqq \frac{\pi}{2} \frac{e^{-An\delta^2}}{\delta}$$

$$= \frac{\pi}{2} n^{3/8} e^{-An^{1/4}} \quad \text{by (2.5)}$$

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$= o(1)$, exponentially, as $n \to \infty$.

The steps in the reduction of the integral are then

(a) the replacing of $\left(\frac{1-r}{\rho}\right)^n$ by $e^{\frac{2nru^2}{(1-r)^2}}$,

(b) the replacing of $\sin[(2n-1)u + n\theta]$ by $\sin[(2\frac{n}{r}+1)\frac{ru}{1-r}]$,

and

(c) the replacing of $\sin u$ by u.

The error committed in (a) is

$$\left| \frac{2}{\pi} \int_0^\delta \left[\left(\frac{1-r}{\rho}\right)^n - e^{\frac{2nru^2}{(1-r)^2}} \right] \right| \sin[(2n-1)u + n\theta] \left| \frac{du}{\sin u} \right|$$

$$\leqq \frac{2}{\pi} \int_0^\delta Bnu^4 \frac{du}{2/\pi\, u} \qquad \text{by Corollary 2 and (2.3)}$$

$$= \frac{Bn^{-1/2}}{4} \qquad \text{by (2.5)}$$

$$= o(1) \qquad \text{as } n \to \infty.$$

The error committed in (b) is

$$\left| \frac{2}{\pi} \int_0^\delta e^{-\frac{2nru^2}{(1-r)^2}} \left[\left|\sin[(2n-1)u + n\theta]\right| - \left|\sin[(2\frac{n}{r}+1)\frac{ru}{1-r}]\right| \right] \frac{du}{\sin u} \right|$$

$$\leqq \frac{2}{\pi} \int_0^\delta \left(Cnu^3 + \frac{ru}{1-r}\right) \frac{du}{2/\pi\, u} \qquad \text{by Corollary 3 and (2.3)}$$

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$$= \frac{Cn^{-1/8}}{3} + \frac{rn^{-3/8}}{1-r} \qquad \text{by (2.5)}$$

$$= o(1) \qquad \text{as } n \to \infty .$$

The error committed in (c) is

$$\left| \frac{2}{\pi} \int_0^{\delta} e^{-\frac{2nru^2}{(1-r)^2}} \left| \sin\left[\left(2\frac{n}{r}+1\right)\frac{ru}{1-r}\right]\right| \left|\frac{1}{\sin u} - \frac{1}{u}\right| du \right|$$

$$\leqq \frac{2}{\pi} \int_0^{\delta} \frac{u - \sin u}{u \sin u}\, du$$

$$\leqq \int_0^{\delta} u\, du \qquad \text{by (2.3) and (2.4)}$$

$$= \frac{n^{-3/4}}{2} \qquad \text{by (2.5)}$$

$$= o(1) \qquad \text{as } n \to \infty .$$

(2.1) then becomes

$$\text{(2.8)} \quad L_T^r(n-1) = \frac{2}{\pi} \int_0^{\delta} e^{-\frac{2nru^2}{(1-r)^2}} \left|\sin\left[\left(2\frac{n}{r}+1\right)\frac{ru}{1-r}\right]\right| \frac{du}{u} + o(1)$$

$$\text{as } n \to \infty .$$

The substitution $t = \frac{ru}{1-r}$ yields

$$\text{(2.9)} \quad L_T^r(n-1) = \frac{2}{\pi} \int_0^{\frac{r\delta}{1-r}} e^{-2\frac{n}{r}t^2} \left|\sin\left[\left(2\frac{n}{r}+1\right)t\right]\right| \frac{dt}{t} + o(1)$$

$$\text{as } n \to \infty .$$

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$\frac{r\delta}{1-r}$ can now be replaced by $\frac{\pi}{2}$. For sufficiently large n, $\frac{r\delta}{1-r} < \frac{\pi}{2}$ and the error committed is

$$\frac{2}{\pi} \int_{\frac{r\delta}{1-r}}^{\pi/2} e^{-2\frac{n}{r}t^2} \left|\sin[(2\frac{n}{r}+1)t]\right| \frac{dt}{t}$$

$$\leq \frac{e^{-2\frac{n}{r}\left(\frac{r\delta}{1-r}\right)^2}}{\frac{r\delta}{1-r}}$$

$$= \frac{(1-r)n^{3/8} e^{-\frac{2rn^{1/4}}{(1-r)^2}}}{r} \qquad \text{by (2.5)}$$

$$= o(1), \qquad \text{exponentially, as } n \to \infty .$$

Hence

$$(2.10) \qquad L_T^r(n-1) = \frac{2}{\pi} \int_0^{\pi/2} e^{-2\frac{n}{r}t^2} \left|\sin[(2\frac{n}{r}+1)t]\right| \frac{dt}{t} + o(1)$$

as $n \to \infty$

$= L_{B3}(\frac{n}{r}) + o(1)$ as $n \to \infty$ in Lorch's notation

(Theorem 3.3 of [4])

$= L_B(\frac{n}{r}) + o(1)$ as $n \to \infty$ by Lorch's theorem

(Theorem 3.3 of [4]).

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3. Remark. The above methods can also be applied to the Euler-Lebesgue constants of order r already studied by Lorch [5] and Livingston [3]. It can be shown that

$$(3.1) \qquad L_E^r(n) = L_B\left(\frac{rn}{1-r}\right) + o(1) \qquad \text{as } n \to \infty$$

where $L_E^r(n)$ denotes the n^{th} Euler-Lebesgue constant of order r.

4. Proofs of Lemmas and Corollaries. In the proofs, the following results are required, along with (2.3) and (2.4):

$$(4.1) \qquad 0 \leqq e^{-x} - (1-x) \leqq x^2 \qquad \text{for } x \geqq 0$$

$$(4.2) \qquad \big||x| - |y|\big| \leqq |x - y| \qquad \text{for all } x \text{ and } y$$

$$(4.3) \qquad |\sin x - \sin y| \leqq |x - y| \qquad \text{for all } x \text{ and } y .$$

From (2.2),

$$(4.4) \qquad \begin{aligned} \rho^2 &= 1 - 2r\cos 2u + r^2 \\ &= (1-r)^2 + 4r\sin^2 u \\ &= (1+r)^2 - 4r\cos^2 u ; \end{aligned}$$

hence

$$(4.5) \qquad 0 < 1 - r \leqq \rho \leqq 1 + r < 2 .$$

Proof of Lemma 1.

$$\left(\frac{1-r}{\rho}\right)^2 = 1 - \frac{4r\sin^2 u}{\rho^2} \qquad \text{by (4.4)}$$

$$\leqq 1 - \frac{4r(2/\pi\, u)^2}{2^2} \qquad \text{by (2.3) and (4.5)}$$

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$$\leqq e^{-\frac{4ru^2}{\pi^2}} \qquad \text{by (4.1)}.$$

Corollary 1 follows immediately.

Proof of Lemma 2.

$$\left(\frac{1-r}{\rho}\right)^2 - e^{-\frac{4ru^2}{(1-r)^2}}$$

$$= 1 - \frac{4r\sin^2 u}{\rho^2} - e^{-\frac{4ru^2}{(1-r)^2}} \qquad \text{by (4.4)}$$

$$= \left(\frac{4ru^2}{\rho^2} - \frac{4r\sin^2 u}{\rho^2}\right) + \left(\frac{4ru^2}{(1-r)^2} - \frac{4ru^2}{\rho^2}\right)$$

$$- \left[e^{-\frac{4ru^2}{(1-r)^2}} - \left(1 - \frac{4ru^2}{(1-r)^2}\right)\right].$$

Hence

$$\left|\left(\frac{1-r}{\rho}\right)^2 - e^{-\frac{4ru^2}{(1-r)^2}}\right|$$

$$\leqq \frac{4r}{\rho^2}(u^2 - \sin^2 u) + 4ru^2\left(\frac{1}{(1-r)^2} - \frac{1}{\rho^2}\right)$$

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$$+ \left[e^{-\frac{4ru^2}{(1-r)^2}} - \left(1 - \frac{4ru^2}{(1-r)^2}\right) \right]$$

$$= f(u) + g(u) + h(u), \quad \text{say}.$$

$$f(u) = \frac{4r}{\rho^2} (u + \sin u)(u - \sin u)$$

$$\leqq \frac{4r}{(1-r)^2} (2u)\, u^3 \qquad \text{by (4.5), (2.3) and (2.4)}$$

$$= \frac{8ru^4}{(1-r)^2} .$$

$$g(u) = 4ru^2 \left[\frac{\rho^2 - (1-r)^2}{(1-r)^2 \rho^2} \right]$$

$$\leqq \frac{4ru^2}{(1-r)^4} (4r \sin^2 u) \qquad \text{by (4.5) and (4.4)}$$

$$\leqq \frac{16r^2u^4}{(1-r)^4} \qquad \text{by (2.3)} .$$

$$h(u) \leqq \left(\frac{4ru^2}{(1-r)^2} \right)^2 \qquad \text{by (4.1)}$$

$$= \frac{16r^2u^4}{(1-r)^4} .$$

Hence $\left| \left(\frac{1-r}{\rho}\right) - e^{-\frac{4ru^2}{(1-r)^2}} \right| \leqq \left(\frac{8r}{(1-r)^2} + \frac{32r^2}{(1-r)^4} \right) u^4 .$

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Proof of Corollary 2.

Let $a = \left(\frac{1-r}{\rho}\right)^2$, $b = e^{-\frac{4ru^2}{(1-r)^2}}$ and $f(x) = x^{n/2}$, so that $f'(x) = \frac{n}{2} x^{n/2-1}$.

By the mean value theorem, $f(a) - f(b) = f'(\xi)(a-b)$, where ξ lies between a and b, inclusive, and hence $0 < \xi < 1$ by (4.5).

Therefore, $f(a) - f(b) = \frac{n}{2} \xi^{n/2-1}(a-b)$

and $|f(a) - f(b)| \leqq \frac{n}{2} |a - b|$.

That is,

$$\left| \left(\frac{1-r}{\rho}\right)^n - e^{-\frac{2nru^2}{(1-r)^2}} \right| \leqq \frac{n}{2} \left| \left(\frac{1-r}{\rho}\right)^2 - e^{-\frac{4ru^2}{(1-r)^2}} \right|$$

$$\leqq Bnu^4 \qquad \text{by Lemma 2.}$$

Proof of Corollary 3.

$$\left| |\sin[(2n-1)u + n\theta]| - \left|\sin\left[\left(2\frac{n}{r}+1\right)\frac{ru}{1-r}\right]\right| \right|$$

$$\leqq \left|\sin[(2n-1)u + n\theta] - \sin\left[\left(2\frac{n}{r}+1\right)\frac{ru}{1-r}\right]\right| \qquad \text{by (4.2)}$$

$$\leqq \left|(2n-1)u + n\theta - \left(2\frac{n}{r}+1\right)\frac{ru}{1-r}\right| \qquad \text{by (4.3)}$$

$$\leqq n\left|\theta - \frac{2ru}{1-r}\right| + \frac{ru}{1-r}$$

$$\leqq Cnu^3 + \frac{ru}{1-r} \qquad \text{by Lemma 3.}$$

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