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RESEARCH ARTICLE

Optimal allocation of policy limits in layer reinsurance treaties

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Abstract

Layer reinsurance treaty is a common form obtained in the problem of optimal reinsurance design. In this paper, we study allocations of policy limits in layer reinsurance treaties with dependent risks. We investigate the effects of orderings and heterogeneity among policy limits on the expected utility functions of the terminal wealth from the viewpoint of risk-averse insurers faced with right tail weakly stochastic arrangement increasing losses. Orderings on optimal allocations are presented for normal layer reinsurance contracts under certain conditions. Parallel studies are also conducted for randomized layer reinsurance contracts. As a special case, the worst allocations of policy limits are also identified when the exact dependence structure among the losses is unknown. Numerical examples are presented to shed light on the theoretical findings.

1. Introduction

Consider a policyholder (or insurer) exposed to n random losses during a fixed time period. Through paying some amount of pre-specified premium, a coverage (e.g., deductible or policy limit) will be granted from an insurer (or reinsurer) (cf. [28]). Cheung [11] might be the first to study the problem of optimal allocation of deductibles and policy limits in a set of (re)insurance contracts. Under the setting of independent risks, by maximizing the expected utility of the policyholder's terminal wealth after receiving compensation from insurance company, he showed that a larger policy limit should be allocated to a larger size of certain risk, while a smaller deductible should be accompanied with a greater risk. Hua and Cheung [24] investigated stochastic orders of scalar products of random vectors and applied their results in the study of the optimal allocation of policy limits and deductibles. By contrast, Hua and Cheung [25] investigated worst allocations of policy limits and deductibles from the viewpoint of an insurer by applying the equivalent utility premium principle yielding the maximum fair premiums. Zhuang *et al.* [41] studied the orderings among optimal allocations of policy limits and deductibles with respect to the well-known family of distortion risk measures (with convex/concave distortion functions) by applying bivariate characterizations of stochastic ordering relations. Lu and Meng [31] further studied the same allocation problem for risk-neutral policyholders when the random losses are arrayed according to the likelihood ratio order. Li and You [30] addressed the allocation problem of upper limits and deductibles discussed in Hua and Cheung [24] by modeling occurrence frequencies of the risks with Archimedean copulas. For more recent relevant studies, we refer interested readers to Manesh and Khaledi [32], Manesh *et al.* [33], Li and Li [29], and You and Li [39].

Reinsurance is an effective tool for insurance companies to transfer potential aggregate risk to reinsurers (cf. [2]). The past several decades have witnessed extensive development on optimal reinsurance

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design from different perspectives; see for example Borch [5,6], Arrow [3], Raviv [35], Young [40], Kaluszka [27], Cai *et al.* [9], Cheung [12], Cui *et al.* [18], Cheung *et al.* [13,14], and the references therein. Under the assumption that both the insurer and reinsurer are obligated to pay more when the underlying claim is getting larger,[1] Chi and Tan [16] proved that the layer reinsurance contract is quite robust in the sense that it is always optimal over both value-at-risk (VaR) and tail value-at-risk (TVaR) measures under general premium principles including Wang's and Dutch premium principles as special cases. Recently, under the setting that all insurers use VaR or range value-at-risk (RVaR) measures, Bäuerle and Glauner [4] considered the optimal reinsurance problem from a macroeconomic point of view when there are n insurance companies each bearing a certain risk and one representative reinsurer, and showed the optimality of the layer reinsurance treaty under certain conditions. Another recent work in support of the optimality of the layer reinsurance treaty is conducted by Chi *et al.* [17], who showed that layer reinsurance contract is optimal by examining the effect of background risk on optimal reinsurance design under the criterion of maximizing the probability of reaching a goal.

To the best of the authors' knowledge, few studies are available on the problem of optimal allocations of deductibles and policy limits for a set of heterogeneous and dependent risks compensated by the form of layer reinsurance treaties. For a risk-averse insurer exposed to n right tail weakly stochastic arrangement increasing (RWSAI) random losses, in this paper, we shall probe into the orderings of the optimal allocations of policy limits in layer reinsurance treaties by maximizing the expected utility of the insurer's terminal wealth.

Albrecher and Cani [1] advocated that randomized reinsurance treaties might be offered from the reinsurer for the insurer to conduct efficient risk management. Though it might be counter-intuitive to impose external randomness on the determination of the retained loss, the authors showed why and to what extent a randomized reinsurance treaty can be interesting for the insurer (see the introduction and Section 6 of Albrecher and Cani [1] for more relevant detailed discussions). As the second aim of the present work, we shall consider the problem of optimal allocations of policy limits for randomized layer reinsurance treaties designed for a set of heterogeneous and dependent risks. More specifically, we study the orderings of the optimal limits for risk-averse insurers by maximizing the expected utility of the terminal wealth when the vector of randomization indicators is left tail weakly stochastic arrangement increasing (LWSAI) and the random losses are RWSAI. As a special case, we also identify the worst allocations of policy limits when the exact dependence structure among the losses are unknown.

Compared with the probabilistic techniques used in some existing works dealing with allocations of deductibles and policy limits, the novelty of the present paper is summarized as follows:

- Most of existing literature assumes that the losses are independent and stochastically ordered, while our setting considers dependent and stochastically ordered losses characterized by RWSAI or LWSAI. Thus, the probabilistic techniques are very different with the corresponding ones of existing studies.
- The tool of majorization orders and their basic properties play a key role in our proofs, which are rarely seen in the existing studies. The method of classification discussion is also applied for reaching the desired allocation result.
- For the study of optimal allocations on policy limits in randomized layer reinsurance treaties, we model the willingness of compensation indicator events by a set of dependent Bernoulli random variables depicted by SAI. The proof relies on the decomposition of these indicator events, which are also relatively new compared with existing studies.

The remainder of this paper is rolled out as follows: Section 2 recalls some pertinent notations, definitions, and helpful lemmas used in the sequel. In Section 3, we study the orderings among optimal allocations of policy limits in layer reinsurance treaties when the dependence structure among the random losses faced by the insurer is modeled via RWSAI. Section 4 deals with the similar allocation problem for randomized layer reinsurance treaties when the vector of randomization indicators is LWSAI and the random losses are RWSAI. The worst allocation of policy limits is further identified when the

[1] This requirement on the indemnity function is, in general, termed as satisfying the *incentive-compatibility* or *no-sabotage* condition in optimal (re)insurance design (cf. [10]).

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dependence structure among random losses is depicted by the comonotonicity. Section 5 concludes the paper with some remarks.

2. Preliminaries

In this section, we recall some definitions, notations, and helpful lemmas used in this study. For real vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, the Hadamard product is denoted as $\boldsymbol{\lambda} \circ \boldsymbol{x} = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ and $\boldsymbol{x}_{\{i,j\}}$ stands for the subvector of $\boldsymbol{x}$ with its ith and jth entries deleted. Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$. Let $x_{1:n} \le \ldots \le x_{n:n}$ be the increasing sequence of the components of $\boldsymbol{x}$. Denote $\mathcal{I}_+^n = \{\boldsymbol{x} : 0 \le x_1 \le x_2 \le \ldots \le x_n\}$ and $\mathcal{D}_+^n = \{\boldsymbol{x} : x_1 \ge x_2 \ge \ldots \ge x_n \ge 0\}$. All inverse functions are assumed to exist whenever they appear.

Definition 2.1. *A real vector $\boldsymbol{x} \in \mathbb{R}^n$ is said to*

(i) majorize vector $\boldsymbol{y} \in \mathbb{R}^n$ (written as $\boldsymbol{x} \overset{\mathrm{m}}{\ge} \boldsymbol{y}$) if $\sum_{i=j}^n x_{i:n} \ge \sum_{i=j}^n y_{i:n}$, for $j = 2, \ldots, n$, and $\sum_{i=1}^n x_{i:n} = \sum_{i=1}^n y_{i:n}$;
(ii) weakly submajorize vector $\boldsymbol{y} \in \mathbb{R}^n$ (written as $\boldsymbol{x} \ge_{\mathrm{w}} \boldsymbol{y}$) if $\sum_{i=j}^n x_{i:n} \ge \sum_{i=j}^n y_{i:n}$, for all $j = 1, \ldots, n$.

It is evident that $\boldsymbol{x} \overset{\mathrm{m}}{\ge} \boldsymbol{y}$ implies $\boldsymbol{x} \ge_{\mathrm{w}} \boldsymbol{y}$, while the reverse statement is not true in general. Majorization order is quite useful in establishing various inequalities arising from actuarial science, applied probability, reliability theory, and so on. The following lemma, implying that the weak submajorization order is preserved under transformation induced by increasing convex functions, is borrowed from Theorem 5.A.2 of Marshall *et al.* [34] and plays a key role in proving the main results.

Lemma 2.2. *Consider two real vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. For all increasing convex function ϕ, it follows that $(x_1, x_2, \ldots, x_n) \ge_{\mathrm{w}} (y_1, y_2, \ldots, y_n)$ implies*

$$(\phi(x_1), \phi(x_2), \ldots, \phi(x_n)) \ge_{\mathrm{w}} (\phi(y_1), \phi(y_2), \ldots, \phi(y_n)).$$

For more discussions on their properties and applications, one can refer to [34].

Definition 2.3. *Let $F[\bar{F}]$ and $G[\bar{G}]$ be the distribution [survival] functions of X and Y, respectively. Then, X is said to be smaller than Y in the*

(i) usual stochastic order (denoted by $X \le_{\mathrm{st}} Y$) if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for any increasing $\phi : \mathbb{R} \mapsto \mathbb{R}$, or equivalently, $\bar{F}(t) \le \bar{G}(t)$ for all $t \in \mathbb{R}$;
(ii) hazard rate order (denoted by $X \le_{\mathrm{hr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ is increasing in $t \in \mathbb{R}$;
(iii) reversed hazard rate order (denoted by $X \le_{\mathrm{rh}} Y$) if $G(t)/F(t)$ is increasing in $t \in \mathbb{R}$;
(iv) stop-loss order (denoted by $X \le_{\mathrm{sl}} Y$) if $\mathbb{E}[(X-t)_+] \le \mathbb{E}[(Y-t)_+]$ for all $t \in \mathbb{R}$.

In the context of applied probability, the stop-loss order is termed as the *increasing convex order* defined in the sense that $X \le_{\mathrm{sl}} Y$ if and only if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for all increasing convex function ϕ. Henceforth, we shall sometimes adopt the notation "$\le_{\mathrm{icx}}$" as an alternative of the stop-loss order "$\le_{\mathrm{sl}}$." It is known that the hazard rate order implies the usual stochastic order, which in turn implies the stop-loss order. According to Theorem 4.A.3. of Shaked and Shanthikumar [36], $X \le_{\mathrm{icx}} Y$ if and only if

$$\int_x^\infty \bar{F}(u)\,\mathrm{d}u \le \int_x^\infty \bar{G}(u)\,\mathrm{d}u, \quad \text{for all } x \in \mathbb{R},$$

or equivalently,

$$\int_\alpha^1 F_X^{-1}(t)\,\mathrm{d}t \le \int_\alpha^1 F_Y^{-1}(t)\,\mathrm{d}t, \quad \text{for all } 0 \le \alpha \le 1.$$

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Let $\{\tau(1), \ldots, \tau(n)\}$ be any permutation of $\{1, 2, \ldots, n\}$ under the permutation operator τ. For any vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we use $\tau(\boldsymbol{x})$ to denote the permuted vector $(x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(n)})$. Let $\boldsymbol{x}_\downarrow$ denote the decreasing rearrangement of $\boldsymbol{x}$, $\boldsymbol{\lambda}_\uparrow$ denote the increasing rearrangement of $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}_\downarrow$ denote the decreasing rearrangement of $\boldsymbol{\lambda}$.

Definition 2.4. *A real-valued function $g(\boldsymbol{x}, \boldsymbol{\lambda})$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be arrangement increasing (AI) if*

(i) g is permutation invariant, that is, $g(\boldsymbol{x}, \boldsymbol{\lambda}) = g(\tau(\boldsymbol{x}), \tau(\boldsymbol{\lambda}))$ for any permutation τ; and
(ii) g exhibits permutation order, that is, $g(\boldsymbol{x}_\downarrow, \boldsymbol{\lambda}_\uparrow) \leq g(\boldsymbol{x}_\downarrow, \tau(\boldsymbol{\lambda})) \leq g(\boldsymbol{x}_\downarrow, \boldsymbol{\lambda}_\downarrow)$ for any permutation $(\tau(1), \ldots, \tau(n))$.

The next lemma is indebted to Cheung [11].

Lemma 2.5. *(i) The function $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(\boldsymbol{x}, \boldsymbol{\lambda}) = -\sum_{i=1}^n (x_i - \lambda_i)_+$ is an AI function in $(\boldsymbol{x}, \boldsymbol{\lambda})$, that is, $-f(\boldsymbol{x}, \boldsymbol{\lambda})$ is arrangement decreasing (AD) function in $(\boldsymbol{x}, \boldsymbol{\lambda})$.*
(ii) The function $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $g(\boldsymbol{x}, \boldsymbol{d}) = \sum_{i=1}^n (x_i \wedge d_i)$ is an AI function in $(\boldsymbol{x}, \boldsymbol{d})$.

For any (i, j) with $1 \leq i < j \leq n$, let $\tau_{ij}(\boldsymbol{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$ and

$$\mathcal{G}_s^{i,j}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) \geq g(\tau_{ij}(\boldsymbol{x})) \text{ for any } x_i \leq x_j\},$$
$$\mathcal{G}_l^{i,j}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is decreasing in } x_i \leq x_j\},$$
$$\mathcal{G}_r^{i,j}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is increasing in } x_j \geq x_i\}.$$

Definition 2.6. *A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is said to be*

(i) stochastic arrangement increasing (SAI) if $\mathbb{E}[g(\boldsymbol{X})] \geq \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$ for any $g \in \mathcal{G}_s^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$;
(ii) left tail weakly stochastic arrangement increasing (LWSAI) if $\mathbb{E}[g(\boldsymbol{X})] \geq \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$ for any $g \in \mathcal{G}_l^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$;
(iii) right tail weakly stochastic arrangement increasing (RWSAI) if $\mathbb{E}[g(\boldsymbol{X})] \geq \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$ for any $g \in \mathcal{G}_r^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$.

The following lemma gives bivariate characterization of RWSAI random vectors, which plays a key role in proving our main results.

Lemma 2.7 [38]**.** *(X_1, X_2) is RWSAI if and only if $\mathbb{E}[g_2(X_1, X_2)] \geq \mathbb{E}[g_1(X_1, X_2)]$ for all g_1 and g_2 such that*

(i) $g_2(x_1, x_2) - g_1(x_1, x_2)$ is increasing in $x_2 \geq x_1$ for any x_1, and
(ii) $g_2(x_1, x_2) + g_2(x_2, x_1) \geq g_1(x_1, x_2) + g_1(x_2, x_1)$ for any $x_2 \geq x_1$.

For ease of reference, we denote $\boldsymbol{0} = (0, \ldots, 0)$, $\boldsymbol{1} = (1, \ldots, 1)$, $\boldsymbol{I} = (I_1, \ldots, I_n)$ comprised of n Bernoulli random variables, $p(\boldsymbol{\lambda}) = \mathbb{P}(\boldsymbol{I} = \boldsymbol{\lambda})$,

$$\Lambda_k = \{\boldsymbol{\lambda} \,|\, \lambda_i = 0 \text{ or } 1,\ i = 1, 2, \ldots, n,\ \lambda_1 + \cdots + \lambda_n = k\}, \quad k = 0, \ldots, n,$$

and let, for $1 \leq i \neq j \leq n$ and $k = 1, \ldots, n-1$,

$$\Lambda_k^{i,j}(0,1) = \{\boldsymbol{\lambda} \in \Lambda_k \,|\, \lambda_i = 0, \lambda_j = 1\}, \quad \Lambda_k^{i,j}(0,0) = \{\boldsymbol{\lambda} \in \Lambda_k \,|\, \lambda_i = \lambda_j = 0\},$$
$$\Lambda_k^{i,j}(1,0) = \{\boldsymbol{\lambda} \in \Lambda_k \,|\, \lambda_i = 1, \lambda_j = 0\}, \quad \Lambda_k^{i,j}(1,1) = \{\boldsymbol{\lambda} \in \Lambda_k \,|\, \lambda_i = \lambda_j = 1\}.$$

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It is easy to see that $\Lambda_1^{i,j}(1,1) = \Lambda_{n-1}^{i,j}(0,0) = \emptyset$ and

$$\Lambda_k = \Lambda_k^{i,j}(0,1) \cup \Lambda_k^{i,j}(0,0) \cup \Lambda_k^{i,j}(1,0) \cup \Lambda_k^{i,j}(1,1).$$

Lemma 2.8 [8]. *A multivariate Bernoulli random vector $\boldsymbol{I}$ is LWSAI if and only if $p(\tau_{ij}(\boldsymbol{\lambda})) \le p(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0,1)$, $1 \le i < j \le n$ and $k = 1, \ldots, n-1$, where $p(\boldsymbol{\lambda}) = \mathbb{P}(\boldsymbol{I} = \boldsymbol{\lambda})$ and $p(\tau_{ij}(\boldsymbol{\lambda})) = \mathbb{P}(\boldsymbol{I} = \tau_{ij}(\boldsymbol{\lambda}))$.*

Formally, for a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ with univariate marginal distributions $F_1, \ldots, F_n$ and survival functions $\bar{F}_1, \ldots, \bar{F}_n$, the well-known Sklar's theorem [37] states that there exist $C : [0,1]^n \mapsto [0,1]$ and $\bar{C} : [0,1]^n \mapsto [0,1]$ such that its distribution function F and survival function $\bar{F}$ can be represented as, for all $\boldsymbol{x} = (x_1, \ldots, x_n)$,

$$F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_n(x_n)) \quad \text{and} \quad \bar{F}(\boldsymbol{x}) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)).$$

Then, C and $\bar{C}$ are called the *copula* and *survival copula* of $\boldsymbol{X}$, respectively. For a decreasing and continuous function $\phi : \mathbb{R}_+ \cup \{+\infty\} \mapsto [0,1]$ with $\phi(0) = 1$, $\phi(+\infty) = 0$ and the pseudo-inverse $\psi := \phi^{-1}$, the function

$$C_\phi(u_1, \cdots, u_n) = \phi(\psi(u_1) + \cdots + \psi(u_n)), \quad \text{for all } u_i \in [0,1],\ i = 1, 2, \ldots, n,$$

is called an *Archimedean copula* with generator ψ if $(-1)^k \phi^{(k)}(x) \ge 0$ for $k = 0, \ldots, n-2$ and $(-1)^{n-2}\phi^{(n-2)}(x)$ is decreasing and convex.

The following lemma provides sufficient conditions based on stochastic orders and Archimedean copulas to characterize LWSAI and RWSAI random vectors.

Lemma 2.9 [7,8]. *Assume the random vector $(X_1, \ldots, X_n)$ has a positive joint density function and $X_1 \le_{\mathrm{hr[rh]}} \cdots \le_{\mathrm{hr[rh]}} X_n$. If the joint survival function of $(X_1, \ldots, X_n)$ is linked by an Archimedean copula $C(u_1, \ldots, u_n) = \phi(\sum_{k=1}^n \psi(u_k))$ and $x\psi'(x)$ is increasing in $x \in [0,1]$, then $(X_1, \ldots, X_n)$ is RWSAI [LWSAI].*

The notions of RWSAI and LWSAI can be roughly understood in the sense that a set of random variables are generally positively dependent and ordered in some stochastic sense, which agrees with many practical scenarios in actuarial science. For instance, all business lines in an insurance portfolio are normally positively dependent under external common shock induced by systemic risks such as some severe natural disasters including the earthquake and flood. From the viewpoint of mathematical technicality, it has standard steps on functional characterizations to verify whether or not a random vector is RWSAI or LWSAI according to Theorem 3.2 or 3.3 of You and Li [38]. Besides, according to Lemma 2.9, we can parametrize RWSAI/LWSAI losses by applying Archimedean copulas and some traditional stochastic orders, which produces a wide class of multivariate distributions containing the commonly used multivariate t, multivariate F, and multivariate Pareto distributions as special cases; see Hollander *et al.* [23]. Therefore, the notions of RWSAI and LWSAI have nice mathematical tractability and can be expediently adapted for modeling various practical insurance scenarios.

A subset $I \subseteq \mathbb{R}^n$ is said to be *comonotonic* if, for any $(x_1, \ldots, x_n) \in I$ and $(y_1, \ldots, y_n) \in I$, either $x_i \le y_i$ for $i = 1, \ldots, n$ or $x_i \ge y_i$ for $i = 1, \ldots, n$. A random vector $\boldsymbol{X}$ is said to be *comonotonic* if there is a comonotonic subset I such that $\mathbb{P}(\boldsymbol{X} \in I) = 1$. The following lemma states that the summation of the comonotonic random variables is maximized via the convex order.

Lemma 2.10. *Assume that $(X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ and $(X_1^c, \ldots, X_n^c) \in \mathcal{R}(F_1, \ldots, F_n)$, where $\mathcal{R}(F_1, \ldots, F_n)$ denotes the collection of random vectors whose marginal distribution functions are $F_1, \ldots, F_n$. If $(X_1^c, \ldots, X_n^c)$ is comonotonic, then $\sum_{i=1}^n X_i \le_{\mathrm{cx}} \sum_{i=1}^n X_i^c$.*

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For more study on comonotonicity, interested readers can refer to Dhaene and Goovaerts [20], Kaas *et al.* [26], and Dhaene *et al.* [21,22].

3. Allocations of policy limits without randomization

Let $X_1, X_2, \ldots, X_n$ be n potential losses faced by the insurer. By paying a fix amount of premium to the reinsurance company, we assume that the insurer will be compensated by means of the layer reinsurance contract, that is, the amount of $\min\{(X_i - d_i)_+, l_i\}$ can be obtained from the reinsurer if the random loss X_i is claimed. After buying reinsurance contracts, the insurer is then granted a right to freely allocate a fixed amount of policy limit $l > 0$ and/or deductible $d > 0$ to each potential risk. If $(l_1, l_2, \ldots, l_n)$ and $(d_1, d_2, \ldots, d_n)$ are the allocated policy limits and deductibles, then we have $l_1 + \cdots + l_n = l$, $d_1 + \cdots + d_n = d$, $l_i \geq 0$, and $d_i \geq 0$, for $i = 1, \ldots, n$. We use $\mathcal{S}_n(l)$ and $\mathcal{S}_n(d)$ to denote the classes of all admissible allocations of policy limits and deductibles, respectively.

This section addresses the problem of allocating policy limits and deductibles by maximizing the expected utility of the terminal wealth for a risk-averse insurer. According to the above statements, we want to consider the following optimization problem

$$\max_{\boldsymbol{d}\in\mathcal{S}_n(d),\boldsymbol{l}\in\mathcal{S}_n(l)} \mathbb{E}\left[u\left(\omega - \left(\sum_{i=1}^n X_i - \sum_{i=1}^n((X_i - d_i)_+ \wedge l_i)\right)\right)\right], \tag{1}$$

where the utility function u is increasing and concave, $\omega > 0$ is the initial wealth[2] of the insurer. Let $\tilde{u}(x) = -u(\omega - x)$. Then, the problem boils down to solving

$$\min_{\boldsymbol{d}\in\mathcal{S}_n(d),\boldsymbol{l}\in\mathcal{S}_n(l)} \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^n (X_i \wedge d_i) + \sum_{i=1}^n (X_i - d_i - l_i)_+\right)\right], \tag{2}$$

where $\tilde{u}$ is increasing and convex.

To begin with, let us consider one numerical example illustrating that the optimal allocations of policy limits cannot be presented when the deductibles are fixed and different from each other. The basic idea for the refutation of the ordering result is to replace the integrals of the quantile function of aggregate risks from a bivariate vector by their numerical integrals.

Example 3.1. Consider the Clayton copula described by the generator $\psi(x) = 1/x^\theta - 1$ such that $x\psi'(x)$ is increasing in $x \in (0, 1]$, for $\theta > 0$. Suppose that (X_1, X_2) is a random vector linked by an Archimedean copula with generator function ψ having parameter $\theta = 2$, where X_1 and X_2 have the exponential distribution with scale parameters $\lambda_1 = 0.9$ and $\lambda_2 = 0.2$, respectively. Then, the joint distribution function of $\boldsymbol{X}$ is given by

$$F_{\boldsymbol{X}}(x_1, x_2) = [(1 - e^{-0.9x_1})^{-\theta} + (1 - e^{-0.2x_2})^{-\theta}].$$

According to Theorem 5.7 of Cai and Wei [7], we know (X_1, X_2) is RWSAI.

Consider the following allocation polices of policy limits $\boldsymbol{l}_1 = (1, 2) \in \mathcal{I}_+^2$, $\boldsymbol{l}_2 = (2, 1) \in \mathcal{D}_+^2$, $\boldsymbol{l}_3 = (2, 8) \in \mathcal{I}_+^2$, $\boldsymbol{l}_4 = (8, 2) \in \mathcal{D}_+^2$ for fixed deductibles $\boldsymbol{d} = (1, 15) \in \mathcal{I}_+^2$. Denote

$$U_{\boldsymbol{l}_r} = X_1 \wedge d_1 + (X_1 - (d_1 + l_{r1}))_+ + X_2 \wedge d_2 + (X_2 - (d_2 + l_{r2}))_+,$$

where $\boldsymbol{l}_r = (l_{r1}, l_{r2})$, for $r = 1, 2, 3, 4$. Let

$$\Delta_{\boldsymbol{l}_r}(\alpha) := \int_\alpha^1 F_{U_{\boldsymbol{l}_r}}^{-1}(t)\,\mathrm{d}t, \quad \text{for } r = 1, 2, 3, 4 \text{ and } \alpha \in [0, 1].$$

[2] Here, "the initial wealth ω" should be understood as the initial wealth minus the mixed reinsurance premium.

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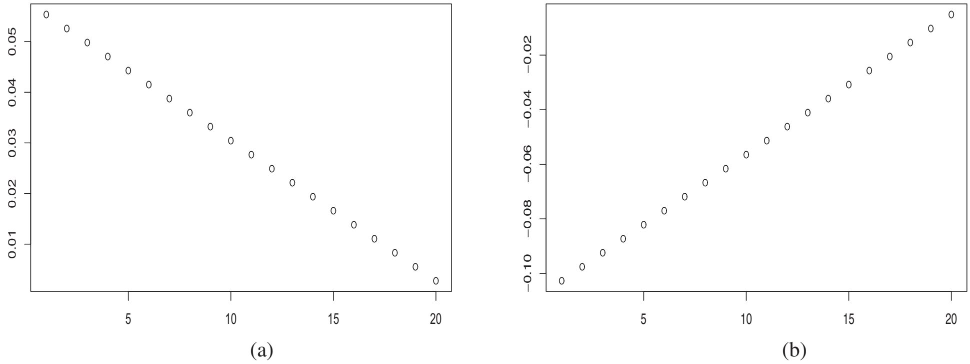

Figure 1. *$\hat{\Delta}_{\boldsymbol{l}_r}(\alpha_j)$ for $j = 1, 2, \ldots, 20$ and $r = 1, 2, 3, 4$. (a) $\hat{\Delta}_{\boldsymbol{l}_1} - \hat{\Delta}_{\boldsymbol{l}_2}$ and (b) $\hat{\Delta}_{\boldsymbol{l}_3} - \hat{\Delta}_{\boldsymbol{l}_4}$.*

Given a sequence of independent and identically distributed (i.i.d.) observations $(X_{i,1}, X_{i,2})$'s sampled from the distribution $F_{\boldsymbol{X}}$, one can obtain the empirical quantile $\hat{F}^{-1}_{U_{\boldsymbol{l}_r}}(t)$ of $F^{-1}_{U_{\boldsymbol{l}_r}}(t)$, for $t \in [0, 1]$. Hence, we can use

$$\hat{\Delta}_{\boldsymbol{l}_r}(\alpha) = \int_{\alpha_j}^{1} \hat{F}^{-1}_{U_{\boldsymbol{l}_r}}(t)\,\mathrm{d}t$$

to approximate $\Delta_{\boldsymbol{l}_r}(\alpha)$. The empirical values of $\Delta_{\boldsymbol{l}_r}(\alpha_j)$, for $r = 1, 2, 3, 4$, are generated under $(\alpha_1, \alpha_2, \ldots, \alpha_{20}) = (0, 0.05, 0.1, \ldots, 0.95)$. As seen from Figure 1(a) and (b), $\hat{\Delta}_{\boldsymbol{l}_1}$ is larger than $\hat{\Delta}_{\boldsymbol{l}_2}$, while $\hat{\Delta}_{\boldsymbol{l}_3}$ is smaller than $\hat{\Delta}_{\boldsymbol{l}_4}$. Due to the strong law of large numbers, it follows that $\Delta_{\boldsymbol{l}_1}(\alpha) > \Delta_{\boldsymbol{l}_4}(\alpha)$ while $\Delta_{\boldsymbol{l}_3}(\alpha) < \Delta_{\boldsymbol{l}_4}(\alpha)$ for $\alpha \in [0, 1]$. Therefore, the orderings among the optimal policy limits can be opposite in problem (2) for different values of deductibles.

As per Example 3.1, the orderings among optimal allocations of policy limits can be opposite when the deductibles are different from each other, which will bring difficulties in seeking for the explicit ordering configuration of optimal allocation policies under such setting. On account of technicality in the proof, we shall study the allocation problem of policy limits by fixing d_i's as $\bar{d}$ such that $d_i = \bar{d} = d/n$.

The following result depicts the orderings among the best allocations of policy limits, which extends the result of Proposition 1 in Cheung [11] to the case of dependent risks under $\bar{d} = 0$.

Theorem 3.2. *Suppose that $\boldsymbol{X}$ is RWSAI. Let $\boldsymbol{l}^* = (l_1^*, \ldots, l_n^*)$ be the solution to the problem (2). Then, we have $l_i^* \leq l_j^*$ for any $1 \leq i < j \leq n$.*

Proof. Let $\boldsymbol{l} = (l_1, \ldots, l_i, \ldots, l_j, \ldots l_n)$ be any admissible allocation vector with $l_i \leq l_j$, for $1 \leq i < j \leq n$. Denote the objective function in (2) as $\pi(\boldsymbol{l})$. Then, it suffices to show that $\pi(\boldsymbol{l}) \leq \pi(\tau_{ij}(\boldsymbol{l}))$. The RWSAI property of $\boldsymbol{X}$ guarantees that $[(X_i, X_j) \mid \boldsymbol{X}_{\{i,j\}}]$ is also RWSAI. Given $\boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}$, let us consider the following two functions:

$$g_2(x_i, x_j) = \tilde{u}(x_i \wedge \bar{d} + (x_i - (\bar{d} + l_j))_+ + x_j \wedge \bar{d} + (x_j - (\bar{d} + l_i))_+ + T_{ij})$$

and

$$g_1(x_i, x_j) = \tilde{u}(x_i \wedge \bar{d} + (x_i - (\bar{d} + l_i))_+ + x_j \wedge \bar{d} + (x_j - (\bar{d} + l_j))_+ + T_{ij}),$$

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where $T_{ij} = \sum_{r\neq i,j}^{n}(x_r \wedge \bar{d} + (x_r - (\bar{d}+l_r))_+)$. For $x_j' \geq x_j \geq x_i$, we denote

$$L_1 = x_i \wedge \bar{d} + (x_i - (\bar{d}+l_j))_+ + x_j \wedge \bar{d} + (x_j - (\bar{d}+l_i))_+ + T_{ij},$$
$$L_2 = x_i \wedge \bar{d} + (x_i - (\bar{d}+l_i))_+ + x_j \wedge \bar{d} + (x_j - (\bar{d}+l_j))_+ + T_{ij},$$
$$L_1' = x_i \wedge \bar{d} + (x_i - (\bar{d}+l_j))_+ + x_j' \wedge \bar{d} + (x_j' - (\bar{d}+l_i))_+ + T_{ij},$$
$$L_2' = x_i \wedge \bar{d} + (x_i - (\bar{d}+l_i))_+ + x_j' \wedge \bar{d} + (x_j' - (\bar{d}+l_j))_+ + T_{ij}.$$

It is obvious that $L_1' \geq L_1$. By Lemma 2.5, we know that $(x_i - (\bar{d}+l_i))_+ + (x_j - (\bar{d}+l_j))_+$ is an *AD* function in $((x_i, x_j), (l_i, l_j))$, which implies that $L_1' \geq L_2'$ and $L_1' + L_2 \geq L_2' + L_1$. Thus, we have $(L_1', L_2) \geq_{\rm w} (L_2', L_1)$. Upon applying Lemma 2.2, we have $(\tilde{u}(L_1'), \tilde{u}(L_2)) \geq_{\rm w} (\tilde{u}(L_2'), \tilde{u}(L_1))$ according to the increasingness and convexity of $\tilde{u}$. Hence,

$$(g_2(x_i, x_j'), g_1(x_i, x_j)) \geq_{\rm w} (g_1(x_i, x_j'), g_2(x_i, x_j)),$$

which implies that $g_2(x_i, x_j') - g_1(x_i, x_j') \geq g_2(x_i, x_j) - g_1(x_i, x_j)$, that is, $g_2(x_i, x_j) - g_1(x_i, x_j)$ is increasing in $x_j \geq x_i$.

On the other hand, it is plain that $g_2(x_i, x_j) + g_2(x_j, x_i) = g_1(x_i, x_j) + g_1(x_j, x_i)$ for any $x_j \geq x_i$. Thus, from Lemma 2.7, it follows that

$$\begin{aligned}
&\mathbb{E}[\tilde{u}(X_i \wedge \bar{d} + (X_i - \bar{d} - l_i)_+ + X_j \wedge \bar{d} + (X_j - \bar{d} - l_j)_+ \\
&\quad + (\boldsymbol{X} \wedge \bar{\boldsymbol{d}} + (\boldsymbol{X} - \bar{\boldsymbol{d}} - \boldsymbol{l})_+)_{\{i,j\}}) \mid \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}] \\
&\leq \mathbb{E}[\tilde{u}(X_i \wedge \bar{d} + (X_i - \bar{d} - l_j)_+ + X_j \wedge \bar{d} + (X_j - \bar{d} - l_i)_+ \\
&\quad + (\boldsymbol{X} \wedge \bar{\boldsymbol{d}} + (\boldsymbol{X} - \bar{\boldsymbol{d}} - \boldsymbol{l})_+)_{\{i,j\}}) \mid \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}],
\end{aligned} \tag{3}$$

where $\bar{\boldsymbol{d}} = (\bar{d}, \ldots, \bar{d})$. By applying double expectation formula on inequality (3), we have

$$\begin{aligned}
\pi(\boldsymbol{l}) &= \mathbb{E}[\tilde{u}(X_i \wedge \bar{d} + (X_i - \bar{d} - l_i)_+ + X_j \wedge \bar{d} + (X_j - \bar{d} - l_j)_+ \\
&\quad + (\boldsymbol{X} \wedge \bar{\boldsymbol{d}} + (\boldsymbol{X} - \bar{\boldsymbol{d}} - \boldsymbol{l})_+)_{\{i,j\}})] \\
&\leq \mathbb{E}[\tilde{u}(X_i \wedge \bar{d} + (X_i - \bar{d} - l_j)_+ + X_j \wedge \bar{d} + (X_j - \bar{d} - l_i)_+ \\
&\quad + (\boldsymbol{X} \wedge \bar{\boldsymbol{d}} + (\boldsymbol{X} - \bar{\boldsymbol{d}} - \boldsymbol{l})_+)_{\{i,j\}})] \\
&= \pi(\tau_{ij}(\boldsymbol{l})),
\end{aligned}$$

which yields the desired inequality. □

The next example illustrates the result of Theorem 3.2.

Example 3.3. Under the setup of Example 3.1, we consider the following two polices $\boldsymbol{l}_5 = (2, 10)$, $\boldsymbol{l}_6 = (10, 2)$ by fixing $\bar{d} = 6$. As observed in Figure 2, it holds that $\hat{\Delta}_{\boldsymbol{l}_5}(\alpha) \leq \hat{\Delta}_{\boldsymbol{l}_6}(\alpha)$ for $\alpha \in [0, 1]$, which indicates that $\Delta_{\boldsymbol{l}_5}(\alpha) \leq \Delta_{\boldsymbol{l}_6}(\alpha)$ for $\alpha \in [0, 1]$ by applying the strong law of large numbers, which validates the finding in Theorem 3.2.

According to Theorem 3.2, one may ask that whether or not the explicit configuration of the optimal allocation vector in $\mathcal{I}_+^n$ could be obtained in terms of the majorization order. The following example provides a negative answer.

Example 3.4. Under the setup of Example 3.1, we consider $\boldsymbol{l}_7 = (3, 5) \in \mathcal{I}_+^2$, $\boldsymbol{l}_8 = (1, 7) \in \mathcal{I}_+^2$, $\boldsymbol{l}_9 = (49, 51) \in \mathcal{I}_+^2$ and $\boldsymbol{l}_{10} = (1, 99) \in \mathcal{I}_+^2$. It is plain that $\boldsymbol{l}_7 \overset{\rm m}{\leq} \boldsymbol{l}_8$ and $\boldsymbol{l}_9 \overset{\rm m}{\leq} \boldsymbol{l}_{10}$. As observed from Figure 3(a), $\hat{\Delta}_{\boldsymbol{l}_7}$ is larger than $\hat{\Delta}_{\boldsymbol{l}_8}$, while $\hat{\Delta}_{\boldsymbol{l}_9}$ is smaller than $\hat{\Delta}_{\boldsymbol{l}_{10}}$ in accordance with Figure 3(b). Thus,

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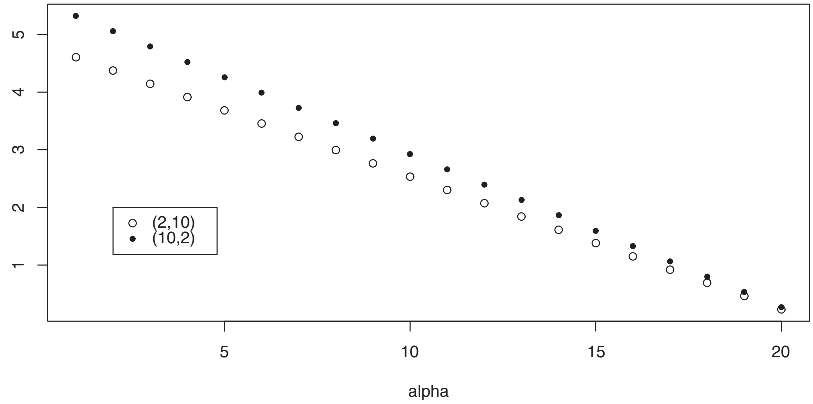
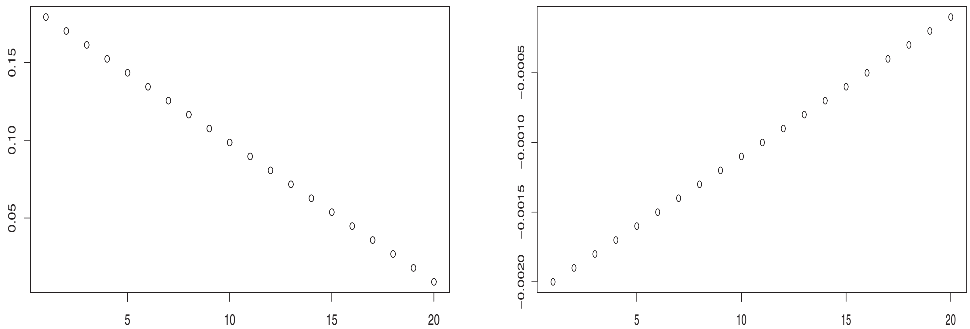


Figure 2. *Plots of* $\hat{\Delta}_{\boldsymbol{l}_r}(\alpha_j)$ *for* $j = 1, 2, \ldots, 20$ *and* $r = 5, 6$.



(a) (b)

Figure 3. *Plots of* $\hat{\Delta}_{\boldsymbol{l}_r}(\alpha_j)$ *for* $j = 1, 2, \ldots, 20$ *and* $r = 7, 8, 9, 10$. *(a)* $\hat{\Delta}_{\boldsymbol{l}_7} - \hat{\Delta}_{\boldsymbol{l}_8}$ *and (b)* $\hat{\Delta}_{\boldsymbol{l}_9} - \hat{\Delta}_{\boldsymbol{l}_{10}}$.

the best allocation of policy limits in $\boldsymbol{l} \in \mathcal{I}_+^n$ cannot be obtained in terms of the majorization order. The optimal allocation strategy might highly depend on the total amount of policy limit as well as the joint distribution. Therefore, some numerical algorithms should be developed to obtain the exact solution of problem (2), which is left for future research.

4. Allocation of policy limits with randomized reinsurance treaties

Albrecher and Cani [1] discussed randomized reinsurance treaties for insurers to conduct efficient risk management. They argued that a randomized reinsurance treaty might be interesting for the insurer to transfer the risk to the reinsurer. In this section, we study the allocation problem of policy limits in randomized layer reinsurance contracts.

Through paying a fixed amount of premium to the reinsurer, the insurer will be compensated by means of the randomized layer reinsurance contract, that is, the amount of $I_i \min\{(X_i - d_i)_+, l_i\}$ will be transferred to the reinsurer if the ith claim is realized, where I_i is a Bernoulli random variable indicating the willingness to provide the compensation for the ith business line, for $i = 1, \ldots, n$. Suppose that $\boldsymbol{X}$ is RWSAI and $\boldsymbol{I}$ is a LWSAI Bernoulli random vector. Consider the following optimization problem

$$\max_{\boldsymbol{d} \in \mathcal{S}_n(d), \boldsymbol{I} \in \mathcal{S}_n(l)} \mathbb{E}\left[u\left(\omega - \left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} I_i \min\{(X_i - d_i)_+, l_i\}\right)\right)\right], \tag{4}$$

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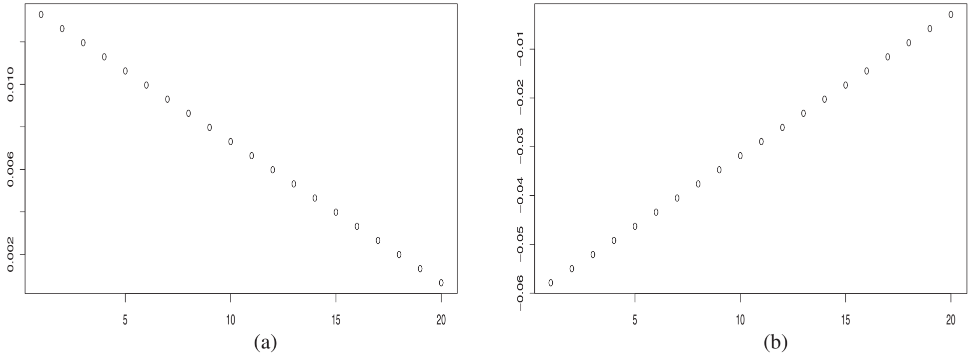

Figure 4. *Plots of (a)* $\hat{\Delta}^I_{\boldsymbol{l}_1}(\alpha_j) - \hat{\Delta}^I_{\boldsymbol{l}_2}(\alpha_j)$ *and (b)* $\hat{\Delta}^I_{\boldsymbol{l}_3}(\alpha_j) - \hat{\Delta}^I_{\boldsymbol{l}_4}(\alpha_j)$*, for* $j = 1, 2, \ldots, 20$*.*

where u is the utility function of the insurer and is increasing and concave. Let $\tilde{u}(x) = -u(\omega - x)$. Then, the problem (4) boils down to solving

$$\min_{\boldsymbol{d} \in \mathcal{S}_n(d), \boldsymbol{l} \in \mathcal{S}_n(l)} \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^n X_i - \sum_{i=1}^n I_i \min\{(X_i - d_i)_+, l_i\}\right)\right], \tag{5}$$

where $\tilde{u}$ is increasing and convex.

First, we present an example to show that there is no certain answer on the orderings among the optimal allocations of policy limits when the deductibles are fixed and different from each other.

Example 4.1. Set $p((0,0)) = 0.15$, $p((0,1)) = 0.5$, $p((1,0)) = 0.2$, and $p((1,1)) = 0.15$. It can be verified that (I_1, I_2) is LWSAI. Under the setup of Example 3.1, we consider $\boldsymbol{d} = (1, 15) \in \mathcal{I}^2_+$, $\boldsymbol{l}_1 = (1, 2) \in \mathcal{I}^2_+$, $\boldsymbol{l}_2 = (2, 1) \in \mathcal{D}^2_+$, $\boldsymbol{l}_3 = (2, 8) \in \mathcal{I}^2_+$, and $\boldsymbol{l}_4 = (8, 2) \in \mathcal{D}^2_+$. Let

$$U^I_{\boldsymbol{l}_r} = X_1 - I_1((X_1 - d_1)_+ \wedge l_{r1}) + X_2 - I_2((X_2 - d_2)_+ \wedge l_{r2}),$$

where $\boldsymbol{l}_r = (l_{r1}, l_{r2})$, for $r = 1, 2, 3, 4$. The values of $\hat{\Delta}^I_{\boldsymbol{l}_1}(\alpha_j)$, $\hat{\Delta}^I_{\boldsymbol{l}_2}(\alpha_j)$, $\hat{\Delta}^I_{\boldsymbol{l}_3}(\alpha_j)$, and $\hat{\Delta}^I_{\boldsymbol{l}_4}(\alpha_j)$ are plotted with respect to α_j, for $j = 1, 2, \ldots, 20$. As seen in Figure 4(a), $\hat{\Delta}^I_{\boldsymbol{l}_1}$ is larger than $\hat{\Delta}^I_{\boldsymbol{l}_2}$, while it is observed from Figure 4(b) that $\hat{\Delta}^I_{\boldsymbol{l}_3}$ is smaller than $\hat{\Delta}^I_{\boldsymbol{l}_4}$. By using the strong law of large numbers, the orderings among the optimal policy limits cannot be determined when the deductibles are different.

In the sequel, we shall consider optimal allocation strategies of policy limits when the deductibles are fixed as $d_i = \bar{d} = d/n$.

Theorem 4.2. *Suppose that* $\boldsymbol{X}$ *is RWSAI and* $\boldsymbol{I}$ *is LWSAI. Let* $\boldsymbol{l}^* = (l^*_1, \ldots, l^*_n)$ *be the solution of problem (5). Then, we have* $l^*_i \le l^*_j$ *for any* $1 \le i < j \le n$*.*

Proof. First, let us introduce some notations to simplify our subsequent discussion:

$$\mathbf{1} \circ \boldsymbol{X} - \boldsymbol{\lambda} \circ ((\boldsymbol{X} - \bar{\boldsymbol{d}})_+ \wedge \boldsymbol{l}) = \sum_{i=1}^n x_i - \sum_{i=1}^n \lambda_i((x_i - \bar{d})_+ \wedge l_i),$$

$$A_{i,j} = (\mathbf{1} \circ \boldsymbol{X} - \boldsymbol{\lambda} \circ ((\boldsymbol{X} - \bar{\boldsymbol{d}})_+ \wedge \boldsymbol{l}))_{\{i,j\}} = \sum_{r \ne i,j} X_r - \sum_{r \ne i,j} \lambda_r((X_r - \bar{d})_+ \wedge l_r),$$

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$$a_{i,j} = (\mathbf{1} \circ \boldsymbol{x} - \boldsymbol{\lambda} \circ ((\boldsymbol{x} - \bar{\boldsymbol{d}})_+ \wedge \boldsymbol{l}))_{\{i,j\}} = \sum_{r \neq i,j}^{n} x_r - \sum_{r \neq i,j}^{n} \lambda_r ((x_r - \bar{d})_+ \wedge l_r),$$

$$C_{i,j} = (\mathbf{1} \circ \boldsymbol{X} - \mathbf{1} \circ ((\boldsymbol{X} - \bar{\boldsymbol{d}})_+ \wedge \boldsymbol{l}))_{\{i,j\}} = \sum_{r \neq i,j}^{n} X_r - \sum_{r \neq i,j}^{n} ((X_r - \bar{d})_+ \wedge l_r),$$

$$c_{i,j} = (\mathbf{1} \circ \boldsymbol{x} - \mathbf{1} \circ ((\boldsymbol{x} - \bar{\boldsymbol{d}})_+ \wedge \boldsymbol{l}))_{\{i,j\}} = \sum_{r \neq i,j}^{n} x_r - \sum_{r \neq i,j}^{n} ((x_r - \bar{d})_+ \wedge l_r),$$

$$b_{1,2} = (\boldsymbol{\lambda} \circ (\boldsymbol{x} \wedge \boldsymbol{l}))_{\{1,2\}} = \sum_{r \neq 1,2}^{n} \lambda_r (x_r \wedge l_r).$$

Denote $\eta(\boldsymbol{l}) := \mathbb{E}[\tilde{u}(\mathbf{1} \circ \boldsymbol{X} - \boldsymbol{\lambda} \circ ((\boldsymbol{X} - \bar{\boldsymbol{d}})_+ \wedge \boldsymbol{l}))]$. It suffices to show that $\eta(\boldsymbol{l}) \leq \eta(\tau_{ij}(\boldsymbol{l}))$. Denote

$$\begin{aligned}
B_1 &= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} ((X_i - \bar{d})_+ \wedge l_i), \\
B_2 &= X_i + X_j - ((X_i - \bar{d})_+ \wedge l_j) - ((X_j - \bar{d})_+ \wedge l_i) + C_{i,j}, \\
B_3 &= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \lambda_i ((X_i - \bar{d})_+ \wedge l_i), \\
B_4 &= X_i + X_j - \lambda_i ((X_i - \bar{d})_+ \wedge l_j) - \lambda_j ((X_j - \bar{d})_+ \wedge l_i) + A_{i,j}.
\end{aligned}$$

It can be checked that

$$\begin{aligned}
\eta(\boldsymbol{l}) &= \sum_{k=0}^{n} \sum_{\boldsymbol{\lambda} \in \Lambda_k} \mathbb{E}[\tilde{u}(\mathbf{1} \circ \boldsymbol{X} - \boldsymbol{I} \circ ((\boldsymbol{X} - \bar{\boldsymbol{d}})_+ \wedge \boldsymbol{l})) \mid \boldsymbol{I} = \boldsymbol{\lambda}] p(\boldsymbol{\lambda}) \\
&= p(\mathbf{0}) \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i\right)\right] + p(\mathbf{1}) \mathbb{E}[\tilde{u}(B_1)] + \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda}) \mathbb{E}[\tilde{u}(B_3))]
\end{aligned} \tag{6}$$

and

$$\eta(\tau_{ij}(\boldsymbol{l})) = p(\mathbf{0}) \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i\right)\right] + p(\mathbf{1}) \mathbb{E}[\tilde{u}(B_2)] + \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda}) \mathbb{E}[\tilde{u}(B_4)].$$

Then, we have

$$\eta(\boldsymbol{l}) - \eta(\tau_{ij}(\boldsymbol{l})) = \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda}) \{\mathbb{E}[\tilde{u}(B_3)] - \mathbb{E}[\tilde{u}(B_4)]\} + p(\mathbf{1}) \{\mathbb{E}[\tilde{u}(B_1)] - \mathbb{E}[\tilde{u}(B_2)]\}. \tag{7}$$

By Theorem 3.2, we have

$$\mathbb{E}[\tilde{u}(B_1)] \leq \mathbb{E}[\tilde{u}(B_2)]. \tag{8}$$

On the other hand, for any $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0,0)$, $k = 1, 2, \ldots, n-1$, it holds that

$$\mathbb{E}[\tilde{u}(B_3)] = \mathbb{E}[\tilde{u}(B_4)]. \tag{9}$$

For $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(1,1)$, we have

$$\mathbb{E}[\tilde{u}(B_3)] \leq \mathbb{E}[\tilde{u}(B_4)]. \tag{10}$$

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Denote

$$B_5 = X_i + X_j - ((X_j - \bar{d})_+ \wedge l_j) + A_{i,j}, \quad B_6 = X_i + X_j - ((X_j - \bar{d})_+ \wedge l_i) + A_{i,j},$$
$$B_7 = X_i + X_j - ((X_i - \bar{d})_+ \wedge l_i) + A_{i,j}, \quad B_8 = X_i + X_j - ((X_i - \bar{d})_+ \wedge l_j) + A_{i,j}.$$

By applying (8), (9), and (10) to (7), we have

$$\begin{aligned}\eta(\boldsymbol{l}) - \eta(\tau_{ij}(\boldsymbol{l}))&\\ \le \sum_{k=1}^{n-1} \Bigg\{ &\sum_{\lambda \in \Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})(\mathbb{E}[\tilde{u}(B_3)] - \mathbb{E}[\tilde{u}(B_4)]) + \sum_{\lambda \in \Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})(\mathbb{E}[\tilde{u}(B_3)] - \mathbb{E}[\tilde{u}(B_4)]) \Bigg\}\\ = \sum_{k=1}^{n-1} \Bigg\{ &\sum_{\lambda \in \Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})(\mathbb{E}[\tilde{u}(B_5)] - \mathbb{E}[\tilde{u}(B_6)]) + \sum_{\lambda \in \Lambda_k^{i,j}(0,1)} p(\tau_{ij}(\boldsymbol{\lambda}))(\mathbb{E}[\tilde{u}(B_7)] - \mathbb{E}[\tilde{u}(B_8)]) \Bigg\}.\end{aligned}$$

Since $\boldsymbol{X}$ is RWSAI, we know that $[(X_i, X_j) \mid \boldsymbol{X}_{\{i,j\}}]$ is RWSAI. Given $\boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}$, for $x_j' \ge x_j \ge x_i$, $l_i \le l_j$, we denote

$$b_5 = x_i + x_j - ((x_j - \bar{d})_+ \wedge l_j) + a_{i,j}, \quad b_6 = x_i + x_j - ((x_j - \bar{d})_+ \wedge l_i) + a_{i,j},$$
$$b_7 = x_i + x_j - ((x_i - \bar{d})_+ \wedge l_i) + a_{i,j}, \quad b_8 = x_i + x_j - ((x_i - \bar{d})_+ \wedge l_j) + a_{i,j},$$
$$b_5' = x_i + x_j' - ((x_j' - \bar{d})_+ \wedge l_j) + a_{i,j}, \quad b_6' = x_i + x_j' - ((x_j' - \bar{d})_+ \wedge l_i) + a_{i,j},$$
$$b_7' = x_i + x_j' - ((x_i - \bar{d})_+ \wedge l_i) + a_{i,j}, \quad b_8' = x_i + x_j' - ((x_i - \bar{d})_+ \wedge l_j) + a_{i,j},$$

$h_2(x_i, x_j) = \tilde{u}(b_6)$ and $h_1(x_i, x_j) = \tilde{u}(b_5)$. Then, it is easy to check that $h_2(x_j, x_i) = \tilde{u}(b_7)$ and $h_1(x_j, x_i) = \tilde{u}(b_8)$. By using Lemma 2.5, we know $-\sum_{i=1}^n ((x_i - \bar{d})_+ \wedge l_i)$ is an *AD* function, which implies that

$$-((x_j' - \bar{d})_+ \wedge l_i) - ((x_j - \bar{d})_+ \wedge l_j) \ge -((x_j' - \bar{d})_+ \wedge l_j) - ((x_j - \bar{d})_+ \wedge l_i). \tag{11}$$

Hence, one has $b_6' + b_5 \ge b_5' + b_6$. By using $x = (x \wedge d) + (x - d)_+$, we have $x_i + x_j' - ((x_j' - \bar{d})_+ \wedge l_i) + a_{i,j} = (x_j' - \bar{d} - l_i)_+ + (x_j' \wedge \bar{d}) + x_i + a_{i,j}$, which further implies $b_6' \ge b_5'$, $b_6' \ge b_6$, and thus $(b_6', b_5) \ge_{\mathrm{w}} (b_5', b_6)$. Now, upon applying Lemma 2.2, we have $(\tilde{u}(b_6'), \tilde{u}(b_5)) \ge_{\mathrm{w}} (\tilde{u}(b_5'), \tilde{u}(b_6))$, which further implies $\tilde{u}(b_6') - \tilde{u}(b_5') \ge \tilde{u}(b_6) - \tilde{u}(b_5)$. Then, $h_2(x_i, x_j) - h_1(x_i, x_j)$ is increasing in $x_j \ge x_i$ for any x_i.

Since $x_j \ge x_i$ and $l_j \ge l_i$, we have $b_6 \le b_7$, $b_5 \le b_8$, $b_7 \ge b_8$, and $b_6 + b_7 \ge b_5 + b_8$, which indicates that $(b_6, b_7) \ge_{\mathrm{w}} (b_5, b_8)$. Upon applying Lemma 2.2, we have $(\tilde{u}(b_6), \tilde{u}(b_7)) \ge_{\mathrm{w}} (\tilde{u}(b_5), \tilde{u}(b_8))$, which further implies $\tilde{u}(b_6) + \tilde{u}(b_7) \ge \tilde{u}(b_5) + \tilde{u}(b_8)$. Thus $h_2(x_i, x_j) + h_2(x_j, x_i) \ge h_1(x_i, x_j) + h_1(x_j, x_i)$ for any $x_j \ge x_i$.

By Lemma 2.7, we have $\mathbb{E}[\tilde{u}(h_1(x_i, x_j))] \le \mathbb{E}[\tilde{u}(h_2(x_i, x_j]$, that is, $\mathbb{E}[\tilde{u}(B_5)] \le \mathbb{E}[\tilde{u}(B_6)]$. Then, from Lemma 2.8, we have

$$\eta(\boldsymbol{l}) - \eta(\tau_{ij}(\boldsymbol{l})) \le \sum_{k=1}^{n-1} \sum_{\lambda \in \Lambda_k^{i,j}(0,1)} p(\tau_{ij}(\boldsymbol{\lambda}))\{\mathbb{E}[\tilde{u}(B_5)] - \mathbb{E}[\tilde{u}(B_6)] + \mathbb{E}[\tilde{u}(B_7)] - \mathbb{E}[\tilde{u}(B_8)]\}. \tag{12}$$

In what follows, we show the non-positivity of

$$\Delta_1 := \mathbb{E}[\tilde{u}(B_5)] - \mathbb{E}[\tilde{u}(B_6)] + \mathbb{E}[\tilde{u}(B_7)] - \mathbb{E}[\tilde{u}(B_8)]. \tag{13}$$

Denote $f_2(x_i, x_j) = \tilde{u}(b_6) + \tilde{u}(b_8)$ and $f_1(x_i, x_j) = \tilde{u}(b_5) + \tilde{u}(b_7)$. For ease of presentation, we denote

$$M_1 = b_6',\ M_2 = b_8',\ M_3 = b_5,\ M_4 = b_7,\ M_1' = b_6,\ M_2' = b_8,\ M_3' = b_5',\ M_4' = b_7'.$$

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In light of (11), we firstly have

$$M_1 + M_2 + M_3 + M_4 \geq M_1' + M_2' + M_3' + M_4'. \tag{14}$$

The following three cases are considered:

Case 1: $l_i \leq (x_i - \bar{d})_+ \leq (x_j - \bar{d})_+$. For this case, we have $M_1 \geq M_4 \geq M_3$, $M_1 \geq M_2 \geq M_3$, $M_4' \geq M_1' \geq M_2'$, and $M_4' \geq M_3'$. Based on these relations, we can obtain the following possible inequalities:

$$M_1 \geq M_4 \geq M_2 \geq M_3, \quad M_4' \geq M_1' \geq M_3' \geq M_2'; \tag{15a}$$
$$M_1 \geq M_2 \geq M_4 \geq M_3, \quad M_4' \geq M_3' \geq M_1' \geq M_2'; \tag{15b}$$
$$M_1 \geq M_4 \geq M_2 \geq M_3, \quad M_4' \geq M_1' \geq M_2' \geq M_3'; \tag{15c}$$
$$M_1 \geq M_2 \geq M_4 \geq M_3, \quad M_4' \geq M_1' \geq M_3' \geq M_2'; \tag{15d}$$
$$M_1 \geq M_2 \geq M_4 \geq M_3, \quad M_4' \geq M_1' \geq M_2' \geq M_3'; \tag{15e}$$
$$M_1 \geq M_4 = M_2 \geq M_3, \quad M_4' \geq M_3' = M_1' \geq M_2'. \tag{15f}$$

Note that $M_1 = M_4'$, $M_4 = M_1'$, $M_2 \geq M_3'$, and $M_2 \geq M_2'$. Then, $M_1 + M_2 \geq M_4' + M_3'$, $M_1 + M_4 + M_2 \geq M_4' + M_1' + M_3'$, and $M_1 + M_4 + M_2 \geq M_4' + M_1' + M_2'$. Then by (14), we know that (15a), (15b), (15c), and (15f) contribute to $(M_1, M_2, M_3, M_4) \succeq_w (M_1', M_2', M_3', M_4')$. For (15d) and (15e), it can be seen that $M_2 \geq M_4 = M_1'$, and thus $M_1 + M_2 \geq M_4' + M_1'$, $M_1 + M_2 + M_4 \geq M_4' + M_1' + M_3'$, and $M_1 + M_2 + M_4 \geq M_4' + M_1' + M_2'$. By (14), we can also get that (15d) and (15e) lead to $(M_1, M_2, M_3, M_4) \succeq_w (M_1', M_2', M_3', M_4')$.

Case 2: $(x_i - \bar{d})_+ \leq l_i \leq (x_j - \bar{d})_+$. Under this case, we have $M_2 \geq M_4 \geq M_3$, $M_2 \geq M_1 \geq M_3$, $M_4' \geq M_2' \geq M_1'$, and $M_4' \geq M_3'$. The possible inequalities are summarized as follows:

$$M_2 \geq M_4 \geq M_1 \geq M_3, \quad M_4' \geq M_2' \geq M_1' \geq M_3';$$
$$M_2 \geq M_1 \geq M_4 \geq M_3, \quad M_4' \geq M_2' \geq M_1' \geq M_3';$$
$$M_2 \geq M_1 \geq M_4 \geq M_3, \quad M_4' \geq M_3' \geq M_2' \geq M_1';$$
$$M_2 \geq M_4 \geq M_1 \geq M_3, \quad M_4' \geq M_2' \geq M_3' \geq M_1';$$
$$M_2 \geq M_1 \geq M_4 \geq M_3, \quad M_4' \geq M_2' \geq M_3' \geq M_1';$$
$$M_2 \geq M_4 = M_1 \geq M_3, \quad M_4' \geq M_3' = M_2' \geq M_1'.$$

Note that $M_1 \geq M_1'$, $M_1 \geq M_3'$, $M_2 = M_4'$, $M_3 \leq M_1'$, and $M_4 = M_2'$. Similar with the discussions in Case 1, one can get $(M_1, M_2, M_3, M_4) \succeq_w (M_1', M_2', M_3', M_4')$.

Case 3: $(x_i - \bar{d})_+ \leq (x_j - \bar{d})_+ \leq l_i$. In this case, we have $M_2 \geq M_4 \geq M_3$, $M_2 \geq M_1$, $M_4' \geq M_2' \geq M_1'$, and $M_4' \geq M_3'$. Then, we have the following several possible inequalities:

$$M_2 \geq M_4 \geq M_1 \geq M_3, \quad M_4' \geq M_2' \geq M_3' \geq M_1';$$
$$M_2 \geq_1 \geq M_4 \geq M_3, \quad M_4' \geq M_3' \geq M_2' \geq M_1';$$
$$M_2 \geq M_1 \geq M_4 \geq M_3, \quad M_4' \geq M_2' \geq M_3' \geq M_1';$$
$$M_2 \geq M_4 \geq M_1 \geq M_3, \quad M_4' \geq M_2' \geq M_1' \geq M_3';$$
$$M_2 \geq M_1 \geq M_4 \geq M_3, \quad M_4' \geq M_2' \geq M_1' \geq M_3';$$
$$M_2 \geq M_4 \geq M_3 \geq M_1, \quad M_4' \geq M_2' \geq M_1' \geq M_3';$$
$$M_2 \geq M_4 = M_1 \geq M_3, \quad M_4' \geq M_3' = M_2' \geq M_1';$$
$$M_2 \geq M_4 \geq M_3 = M_1, \quad M_4' \geq M_2' \geq M_3' = M_1';$$
$$M_2 \geq M_4 = M_3 = M_1, \quad M_4' \geq M_3' = M_2' = M_1'.$$

Observe that $M_1 \geq M_3'$, $M_2 = M_4'$, $M_3 = M_1'$, and $M_4 = M_2'$. Similarly, it can be checked that all of the above inequalities result in $(M_1, M_2, M_3, M_4) \succeq_w (M_1', M_2', M_3', M_4')$.

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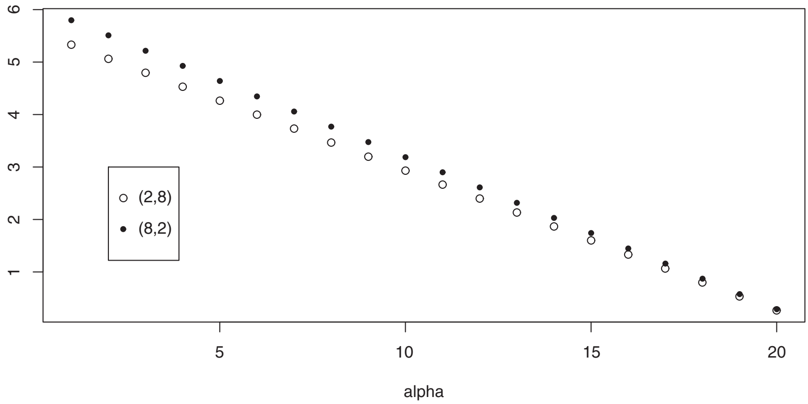

Figure 5. *Plots of* $\hat{\Delta}^{I}_{\boldsymbol{l}_r}(\alpha_j)$ *for* $j = 1, 2, \ldots, 20$ *and* $r = 3, 4$.

To sum up, we always have $(M_1, M_2, M_3, M_4) \geq_{\mathrm{w}} (M'_1, M'_2, M'_3, M'_4)$, that is, $(b'_6, b'_8, b_5, b_7) \geq_{\mathrm{w}} (b_6, b_8, b'_5, b'_7)$. Upon applying Lemma 2.2, we have

$$(\tilde{u}(b'_6), \tilde{u}(b'_8), \tilde{u}(b_5), \tilde{u}(b_7)) \geq_{\mathrm{w}} (\tilde{u}(b_6), \tilde{u}(b_8), \tilde{u}(b'_5), \tilde{u}(b'_7)),$$

which in turn implies that

$$\tilde{u}(b'_6) + \tilde{u}(b'_8) - \tilde{u}(b'_5) - \tilde{u}(b'_7) \geq \tilde{u}(b_6) + \tilde{u}(b_8) - \tilde{u}(b_5) - \tilde{u}(b_7).$$

Thus, $f_2(x_i, x_j) - f_1(x_i, x_j)$ is increasing in $x_j \geq x_i$ for any x_i. Note that $f_2(x_i, x_j) + f_2(x_j, x_i) = f_1(x_i, x_j) + f_1(x_j, x_i)$ for any $x_j \geq x_i$. Upon using Lemma 2.7, we can conclude that $\Delta_1 \leq 0$. This invokes $\eta(\boldsymbol{a}) \leq \eta(\tau_{ij}(\boldsymbol{a}))$, yielding the desired result. □

The next example illustrates the result of Theorem 4.2.

Example 4.3. Under the setup of Example 4.1, we consider $\boldsymbol{l}_3 = (2, 8)$, $\boldsymbol{l}_4 = (8, 2)$, and $\bar{d} = 6$. As displayed in Figure 5, it holds that $\hat{\Delta}^{I}_{\boldsymbol{l}_3}(\alpha) \leq \hat{\Delta}^{I}_{\boldsymbol{l}_4}(\alpha)$ for $\alpha \in [0, 1]$. Therefore, the result of Theorem 4.2 is validated by applying the strong law of large numbers.

The next numerical example states that the best allocation vector in $\mathcal{I}^n_+$ cannot be given in terms of the majorization order under the setting of Theorem 4.2.

Example 4.4. Under the setup of Example 4.3, we consider $\boldsymbol{l}_7 = (3, 5) \in \mathcal{I}^2_+$, $\boldsymbol{l}_8 = (1, 7) \in \mathcal{I}^2_+$, $\boldsymbol{l}_9 = (49, 51) \in \mathcal{I}^2_+$, and $\boldsymbol{l}_{10} = (1, 99) \in \mathcal{I}^2_+$. It is plain that $\boldsymbol{l}_7 \overset{\mathrm{m}}{\preceq} \boldsymbol{l}_8$ and $\boldsymbol{l}_9 \overset{\mathrm{m}}{\preceq} \boldsymbol{l}_{10}$. As seen from Figure 6(a), $\hat{\Delta}^{I}_{\boldsymbol{l}_7}$ is larger than $\hat{\Delta}^{I}_{\boldsymbol{l}_8}$, while it is observed in Figure 6(b) that $\hat{\Delta}^{I}_{\boldsymbol{l}_9}$ is smaller than $\hat{\Delta}^{I}_{\boldsymbol{l}_{10}}$. Therefore, by applying the strong law of large numbers, the best allocation vector in $\boldsymbol{l} \in \mathcal{I}^n_+$ cannot be given in the sense of the majorization order. Numerical algorithms might be resorted to dealing with the optimal allocations.

In some practical situations, the exact dependence structure among the losses is usually unknown for the insurer. Therefore, one may consider the following robust optimization problem:

$$\min_{\boldsymbol{l} \in \mathcal{S}(l)} \max_{\boldsymbol{X} \in \mathcal{R}(F_1, \ldots, F_n)} \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} I_i((X_i - \bar{d})_+ \wedge l_i)\right)\right]. \tag{16}$$

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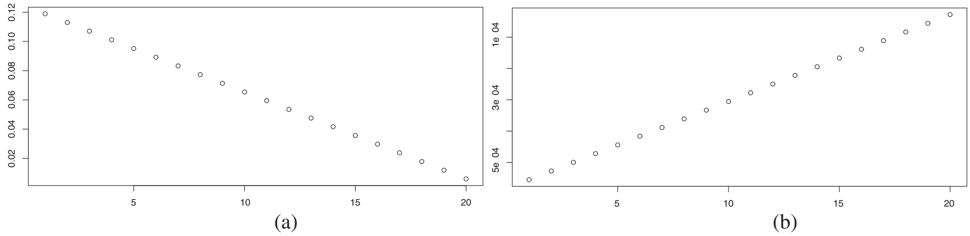

Figure 6. *Plots of* $\hat{\Delta}^I_{\boldsymbol{l}_7}(\alpha_j) - \hat{\Delta}^I_{\boldsymbol{l}_8}(\alpha_j)$ *and* $\hat{\Delta}^I_{\boldsymbol{l}_9}(\alpha_j) - \hat{\Delta}^I_{\boldsymbol{l}_{10}}(\alpha_j)$ *for* $j = 1, 2, \ldots, 20$*. (a)* $\hat{\Delta}^I_{\boldsymbol{l}_7} - \hat{\Delta}^I_{\boldsymbol{l}_8}$ *and (b)* $\hat{\Delta}^I_{\boldsymbol{l}_9} - \hat{\Delta}^I_{\boldsymbol{l}_{10}}$.

Being the strongest positive dependence, comonotonicity is commonly used to model the worst dependency structure from the viewpoint of the insurer. The next result shows that the retained loss $\sum_{i=1}^n X_i - \sum_{i=1}^n I_i((X_i - \bar{d})_+ \wedge l_i)$ is maximized according to the convex order when the claim severities are comonotonic.

Lemma 4.5. *Suppose* $\boldsymbol{X}^c = (X_1^c, \ldots, X_n^c)$ *is the comonotonic version of* $\boldsymbol{X}$*, and* $\boldsymbol{I}$ *is a Bernoulli random vector. Then, we have*

$$\sum_{i=1}^n X_i - \sum_{i=1}^n I_i((X_i - \bar{d})_+ \wedge l_i) \le_{\rm cx} \sum_{i=1}^n X_i^c - \sum_{i=1}^n I_i((X_i^c - \bar{d})_+ \wedge l_i),$$

for any $\boldsymbol{l} = (l_1, \ldots, l_n) \in \mathcal{S}_n(l)$.

Proof. Note that

$$\sum_{i=1}^n x_i - \sum_{i=1}^n I_i((x_i - \bar{d})_+ \wedge l_i)$$
$$= \begin{cases} \sum_{j\neq i}^n x_j - \sum_{j\neq i}^n I_j((x_j - \bar{d})_+ \wedge l_j) + (x_i \wedge \bar{d}) + (x_i - \bar{d} - l_i), & I_i = 1 \\ \sum_{j\neq i}^n x_j + x_i, & I_i = 0. \end{cases}$$

It is evident that $\sum_{i=1}^n x_i - \sum_{i=1}^n I_i((x_i - \bar{d})_+ \wedge l_i)$ is increasing in x_i, for $i = 1, \ldots, n$. Then by Lemma 2.10, we can get the desired result. □

In light of Lemma 4.5, problem (16) reduces to

$$\min_{\boldsymbol{l}\in\mathcal{S}(l)} \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^n X_i - \sum_{i=1}^n I_i((X_i - \bar{d})_+ \wedge l_i)\right)\right], \tag{17}$$

where $\boldsymbol{X}$ is comonotonic. As per Example 4.4, the best allocation in $\mathcal{I}_+^n$ cannot be given. In the sequel, we consider the worst allocation of policy limits in $\mathcal{D}_+^n$.

Theorem 4.6. *Suppose that* $\boldsymbol{X}$ *is comonotonic with* $X_1 \le_{\rm st} \cdots \le_{\rm st} X_n$ *and* $\boldsymbol{I}$ *is LWSAI. For any two allocation vectors* $\boldsymbol{l}_1, \boldsymbol{l}_2 \in \mathcal{D}_+^n$*,* $\boldsymbol{l}_1 \ge_{\rm m} \boldsymbol{l}_2$ *implies that*

$$\sum_{r=1}^n (X_r - I_r((X_r - \bar{d})_+ \wedge l_{1r})) \ge_{\rm sl} \sum_{r=1}^n (X_r - I_r((X_r - \bar{d})_+ \wedge l_{2r})).$$

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Proof. Without loss of generality, we set $\bar{d} = 0$ for ease of the presentation of the proof. By exploiting a similar proof method as in Theorem 4.2, it is enough to prove

$$\eta(\boldsymbol{l}_1) - \eta(\boldsymbol{l}_2) = \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda}) \left\{ \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1) \right) \right] - \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_2) \right) \right] \right\} + p(\mathbf{1}) \left\{ \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (X_i - l_{1i})_+ \right) \right] - \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (X_i - l_{2i})_+ \right) \right] \right\}. \tag{18}$$

By the nature of majorization order, it suffices to prove the non-negativity of (18) under the conditions $l_{11} \geq l_{12}$, $l_{21} \geq l_{22}$, $(l_{11}, l_{12}) \overset{\mathrm{m}}{\geq} (l_{21}, l_{22})$, and $l_{1i} = l_{2i}$ for $i = 3, \ldots, n$.

Owing to comonotonicity of (X_1, X_2) with $X_1 \leq_{\mathrm{st}} X_2$, we have X_1 is smaller than X_2 almost surely given that $\boldsymbol{X}_{\{1,2\}} = \boldsymbol{x}_{\{1,2\}}$. Taking the realizations of X_1 and X_2 as x_1 and x_2, we know $x_1 \leq x_2$ with probability 1. Then, the assumption $(l_{11}, l_{12}) \overset{\mathrm{m}}{\geq} (l_{21}, l_{22})$ implies $(x_1 - l_{11})_+ + (x_2 - l_{12})_+ \geq (x_1 - l_{21})_+ + (x_2 - l_{22})_+$ and $(x_1 \wedge l_{11}) + (x_2 \wedge l_{12}) \leq (x_1 \wedge l_{21}) + (x_2 \wedge l_{22})$ with probability 1. Therefore, upon using double expectation formula, it follows that

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (X_i - l_{1i})_+ \right) \right] \geq \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (X_i - l_{2i})_+ \right) \right]. \tag{19}$$

Besides, for any $\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0, 0)$, $k = 1, 2, \ldots, n-1$, it holds that

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1) \right) \right] = \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_2) \right) \right]. \tag{20}$$

For $\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1, 1)$, we can obtain

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_1 \wedge l_{11}) - (X_2 \wedge l_{12}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \geq \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_1 \wedge l_{21}) - (X_2 \wedge l_{22}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_2))_{\{1,2\}} \right) \right]. \tag{21}$$

By using (19), (20), and (21), it can be reached that

$$\eta(\boldsymbol{l}_1) - \eta(\boldsymbol{l}_2) \geq \sum_{k=1}^{n-1} \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)} p(\boldsymbol{\lambda}) \left(\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1) \right) \right] - \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_2) \right) \right] \right) \right.$$

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$$
\begin{aligned}
&+ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1,0)} p(\boldsymbol{\lambda}) \left(\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1) \right) \right] - \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - \boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_2) \right) \right] \right) \Bigg\} \\
&= \sum_{k=1}^{n-1} \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)} p(\boldsymbol{\lambda}) \left(\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_2 \wedge l_{12}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \right. \right. \\
&\quad - \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_2 \wedge l_{22}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \Bigg) \\
&\quad + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)} p(\tau_{12}(\boldsymbol{\lambda})) \left(\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_1 \wedge l_{11}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \right. \\
&\quad \left. \left. - \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_1 \wedge l_{21}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \right) \right\} \\
&\geq \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)} p(\tau_{12}(\boldsymbol{\lambda})) \Delta_2, \qquad (22)
\end{aligned}
$$

where (22) is due to Lemma 2.8 with $l_{12} \leq l_{22}$, and

$$
\begin{aligned}
\Delta_2 := \;& \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_2 \wedge l_{12}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \\
&- \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_2 \wedge l_{22}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \\
&+ \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_1 \wedge l_{11}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right] \\
&- \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} X_i - (X_1 \wedge l_{21}) - (\boldsymbol{\lambda} \circ (\boldsymbol{X} \wedge \boldsymbol{l}_1))_{\{1,2\}} \right) \right].
\end{aligned}
$$

Next, we need to show the non-negativity of Δ_2. For any $\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)$, if $x_1 \geq l_{12}$, then

$$
\sum_{i=1}^{n} x_i - (x_2 \wedge l_{12}) - b_{1,2} \geq \sum_{i=1}^{n} x_i - (x_1 \wedge l_{21}) - b_{1,2}
$$

and

$$
\sum_{i=1}^{n} x_i - (x_2 \wedge l_{12}) - b_{1,2} \geq \sum_{i=1}^{n} x_i - (x_2 \wedge l_{22}) - b_{1,2};
$$

if $x_1 \leq l_{12}$, then

$$
\begin{aligned}
\sum_{i=1}^{n} x_i - (x_1 \wedge l_{11}) - b_{1,2} &\geq \sum_{i=1}^{n} x_i - (x_2 \wedge l_{12}) - b_{1,2}, \\
\sum_{i=1}^{n} x_i - (x_1 \wedge l_{11}) - b_{1,2} &\geq \sum_{i=1}^{n} x_i - (x_1 \wedge l_{21}) - b_{1,2},
\end{aligned}
$$

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and

$$\sum_{i=1}^{n} x_i - (x_1 \wedge l_{11}) - b_{1,2} \geq \sum_{i=1}^{n} x_i - (x_2 \wedge l_{22}) - b_{1,2}.$$

Moreover,

$$\left(\sum_{i=1}^{n} x_i - (x_1 \wedge l_{11}) - b_{1,2}\right) + \left(\sum_{i=1}^{n} x_i - (x_2 \wedge l_{12}) - b_{1,2}\right)$$
$$\geq \left(\sum_{i=1}^{n} x_i - (x_1 \wedge l_{21}) - b_{1,2}\right) + \left(\sum_{i=1}^{n} x_i - (x_2 \wedge l_{22}) - b_{1,2}\right).$$

It then holds that $(\sum_{i=1}^{n} x_i - (x_1 \wedge l_{11}) - b_{1,2}, \sum_{i=1}^{n} x_i - (x_2 \wedge l_{12}) - b_{1,2}) \geq_{\mathrm{w}} (\sum_{i=1}^{n} x_i - (x_1 \wedge l_{21}) - b_{1,2}, \sum_{i=1}^{n} x_i - (x_2 \wedge l_{22}) - b_{1,2})$, which implies $(\tilde{u}(\sum_{i=1}^{n} x_i - (x_1 \wedge l_{11}) - b_{1,2}), \tilde{u}(\sum_{i=1}^{n} x_i - (x_2 \wedge l_{12}) - b_{1,2})) \geq_{\mathrm{w}} (\tilde{u}(\sum_{i=1}^{n} x_i - (x_1 \wedge l_{21}) - b_{1,2}), \tilde{u}(\sum_{i=1}^{n} x_i - (x_2 \wedge l_{22}) - b_{1,2}))$ upon applying Lemma 2.2. Hence, we have

$$\tilde{u}\left(\sum_{i=1}^{n} x_i - (x_1 \wedge l_{11}) - b_{1,2}\right) + \tilde{u}\left(\sum_{i=1}^{n} x_i - (x_2 \wedge l_{12}) - b_{1,2}\right)$$
$$\geq \tilde{u}\left(\sum_{i=1}^{n} x_i - (x_1 \wedge l_{21}) - b_{1,2}\right) + \tilde{u}\left(\sum_{i=1}^{n} x_i - (x_2 \wedge l_{22}) - b_{1,2}\right).$$

By using the double expectation formula, we have $\Delta_2 \geq 0$, and this in turn implies $\eta(\boldsymbol{l}_1) \geq \eta(\boldsymbol{l}_2)$. Hence, the proof is finished. □

Under the setting of Theorem 4.6, it is not hard to see that the worst allocation of policy limits can be determined as $(l, 0, \ldots, 0)$, as summarized in the following corollary.

Corollary 4.7. *Suppose that* $\boldsymbol{X}$ *is comonotonic with* $X_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} X_n$ *and* $\boldsymbol{I}$ *is LWSAI. Then, the solution of problem (17) is* $(l, 0, \ldots, 0)$.

The following example illustrates the results of Theorem 4.6 and Corollary 4.7.

Example 4.8. Under the setup of Example 4.1, we consider $(X_1, X_2) = (-\lambda_1^{-1} \log U, -\lambda_2^{-1} \log U)$, where U is uniformly distributed on $(0, 1)$ and $(\lambda_1, \lambda_2) = (0.6, 0.2)$. It is easy to check that (X_1, X_2) is comonotonic and $X_1 \leq_{\mathrm{st}} X_2$. Consider the following four allocation polices $\boldsymbol{l}_{11} = (8, 0)$, $\boldsymbol{l}_{12} = (7, 1)$, $\boldsymbol{l}_{13} = (5, 3)$, $\boldsymbol{l}_{14} = (4, 4)$ within $\mathcal{D}_+^2$. Let $\bar{d} = 6$. We denote

$$U_{\boldsymbol{l}_r}^{\boldsymbol{I}} = X_1 - I_1(X_1 \wedge l_{r,1}) + X_2 - I_2(X_2 \wedge l_{r,2}),$$

where $\boldsymbol{l}_r = (l_{r,1}, l_{r,2})$ for $r = 11, 12, 13, 14$. It is plain that $\boldsymbol{l}_{11} \overset{\mathrm{m}}{\succeq} \boldsymbol{l}_{12} \overset{\mathrm{m}}{\succeq} \boldsymbol{l}_{13} \overset{\mathrm{m}}{\succeq} \boldsymbol{l}_{14}$. The values of $\hat{\Delta}_{\boldsymbol{l}_{11}}^{\boldsymbol{I}}(\alpha_j)$, $\hat{\Delta}_{\boldsymbol{l}_{12}}^{\boldsymbol{I}}(\alpha_j)$, $\hat{\Delta}_{\boldsymbol{l}_{13}}^{\boldsymbol{I}}(\alpha_j)$, and $\hat{\Delta}_{\boldsymbol{l}_{14}}^{\boldsymbol{I}}(\alpha_j)$ are plotted for different values of α_j, for $j = 1, 2, \ldots, 20$. As displayed in Figure 7, it holds that $\hat{\Delta}_{\boldsymbol{l}_{11}}^{\boldsymbol{I}}(\alpha) \geq \hat{\Delta}_{\boldsymbol{l}_{12}}^{\boldsymbol{I}}(\alpha) \geq \hat{\Delta}_{\boldsymbol{l}_{13}}^{\boldsymbol{I}}(\alpha) \geq \hat{\Delta}_{\boldsymbol{l}_{14}}^{\boldsymbol{I}}(\alpha)$ for $\alpha \in [0, 1]$. Therefore, the results of Theorem 4.6 and Corollary 4.7 are validated by applying the strong law of large numbers.

5. Conclusion

In this paper, for risk-averse insurers exposed to n RWSAI random losses, we have investigated the orderings among the optimal allocations of policy limits in layer reinsurance treaties by maximizing the expected utility of the terminal wealth, where the deductibles are fixed equally. Following this setting, numerical examples are also given to state that the optimal allocations of policy limits cannot

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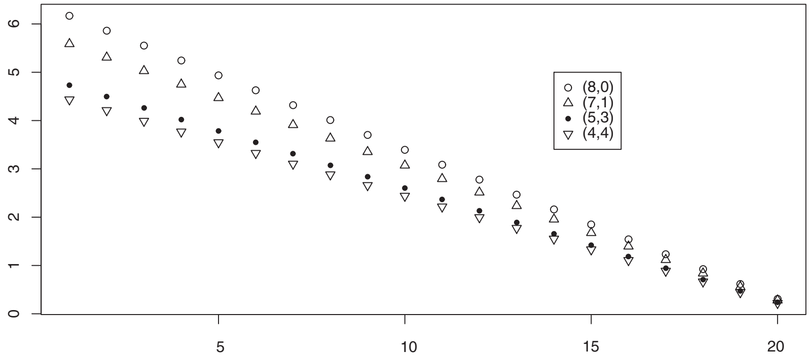

Figure 7. *Plots of* $\hat{\Delta}^{I}_{\boldsymbol{l}_r}(\alpha_j)$ *for* $j = 1, 2, \ldots, 20$ *and* $r = 11, 12, 13, 14$.

be determined with respect to the majorization order. Furthermore, we revisit the problem of optimal allocations of policy limits for randomized layer reinsurance treaties. We also provided the orderings of the optimal limits for risk-averse insurers by maximizing the expected utility of the terminal wealth when the vector of randomization indicators is LWSAI and the potential losses are RWSAI. Since the family of distortion risk measures with concave distortion functions are preserved in the sense of the stop-loss order (cf. [19]), the problems (2), (5), and (17) can be equivalently rephrased by means of minimizing the insurer's risk position depicted by any convex distortion risk measure imposed on the retained loss.

Note that we require the basic assumption that the amount of premium paid to the reinsurance company is fixed as a constant *ex ante*, which might be inconsistent with some real scenarios in insurance context that exchanging the allocations definitely results in different premiums. Therefore, it is of interest to implement optimal allocations of deductibles and policy limits when the premium is a functional of layer reinsurance treaty.

The present paper focuses on the allocations of policy limits when the deductibles are fixed *ex ante*. As a parallel study, it would be of interest to study the optimal allocations of deductibles when the policy limits are predetermined. Furthermore, it is meaningful in practice to assume that the summation of the policy limit and deductible for each loss is fixed (e.g., optimal ceded loss functions derived under VaR or TVaR); see Chi and Tan [15] and Section 6 of Albrecher and Cani [1]. Besides, the study on stochastic properties of extreme claim amounts arising from a set of heterogeneous layer reinsurance contracts is also worth investigating.

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