ON A THEOREM OF HALMOS CONCERNING UNBIASED ESTIMATION OF MOMENTS

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1. Introduction

In [4] Halmos considers the following situation. Let $\mathscr{D}$ be a class of distribution functions over a given (Borel) subset E of the real line, and F a function over $\mathscr{D}$. He investigates which functions F admit estimates that are unbiased over $\mathscr{D}$ and what are all possible such estimates for any given F. In particular he shows that on the basis of a sample (of size n) one can always obtain an estimate of the first moment which is unbiased in $\mathscr{D}$ and that the central moments $\bar{F}_m$ of order $m \geqq 2$ have estimates which are unbiased in $\mathscr{D}$ if and only if $n \geqq m$, provided $\mathscr{D}$ satisfies the following properties: F_m exists and is finite for all distributions in $\mathscr{D}$ and $\mathscr{D}$ includes all distributions which assign probability one to a finite number of points of E. Halmos also finds that symmetric estimates which are unbiased on $\mathscr{D}$ are unique [1] and have smaller variances on $\mathscr{D}$ than unsymmetric unbiased estimates.

He recognizes that his assumptions are too restrictive for most applications and mentions in particular the case where $\mathscr{D}$ is the class of all normal distributions. The present paper addresses itself to that case.

2. Statement of results

If $\mathscr{D}$ is the class of all nondegenerate univariate normal distributions, then, on the basis of a sample (of size n), an estimate of the first moment which is unbiased over $\mathscr{D}$ exists (and is unique when $n = 1$); and a central moment of order $2r \geqq 2$ has estimates which are unbiased over $\mathscr{D}$ if and only if $n \geqq 2$, and has a unique symmetric unbiased estimate when $n = 2$, but not when $n > 2$.

Specifically, this means the following:

Let $z_1, \cdots, z_n$ be a sample from a normal distribution with mean ν and variance $\omega^2 > 0$. Let $\bar{z} = n^{-1} \sum z_i$, $S^2 = \sum (z_i - \bar{z})^2$. Recall that the even

[1] It will be convenient to call a function on a k-dimensional Euclidean space the unique function satisfying a certain property if any other function on this space satisfying the property may differ from it only on a set of k-dimensional Lebesgue measure zero.

central moments $\bar{F}_{2r}$ equal $\omega^{2r}2^{-r}(2r)!/r!$ and the odd ones vanish.

(a) If $n = 1$, $\bar{z}$ is the unique unbiased estimate of ν, and no unbiased estimate of $\bar{F}_{2r}$ exists for $r = 1, 2, \cdots$ [2] In [5] this seemingly uninteresting fact turns out to be the key to a quite practical question.

(b) If $n \geqq 2$,

$$\bar{f}_{2r} = \frac{\{(n-3)/2\}!(2r)!}{\{(n+2r-3)/2\}!r!} (S/2)^{2r}$$

is an unbiased estimate of $\bar{F}_{2r}$ $(r = 1, 2, \cdots)$, and is the unique symmetric unbiased estimate if $n = 2$, but not if $n > 2$. It then follows from [6] that $\bar{z}$ and $\bar{f}_{2r}$

(c) are the unique unbiased estimates of ν and $\bar{F}_{2r}$, respectively, which depend only on the sufficient statistic $(\bar{z}, S^2)$ and

(d) have the smallest variance among all unbiased estimates.

Note that $\bar{z}$ and S^2 are symmetric functions of the observations. The usual symmetric estimate $\bar{f}'_{2r}$ for $\bar{F}_{2r}$, which is unbiased for all distribution functions for which $\bar{F}_{2r}$ exists, is defined only when $n \geqq 2r$. When $r = 1$ it coincides with $\bar{f}_2$, when $r = 2$ it equals [2, 27.6]

$$\bar{f}'_4 = (n!)^{-1}(n-4)!\{n(n^2-2n+3)\sum(z_i-\bar{z})^4-3(2n-3)S^4\} \qquad (n \geqq 4).$$

For any family $\mathscr{D}$ as first mentioned in the introduction or mentioned in the final section $\bar{f}'_{2r}$ is the only symmetric estimate which is unbiased for all distributions of $\mathscr{D}$. But, if for $\mathscr{D}$ we take the class of nondegenerate univariate normal distributions, our results imply that the symmetric estimate $\bar{f}_{2r}$ is also unbiased over this class and has a smaller variance than $\bar{f}'_{2r}$ for $r > 1$.

In the next two sections we prove the parts of (a) and (b) which are not immediate.

3. Nonexistence of an unbiased estimate of $\bar{F}_{2r}$ in a sample of one

In this section denote z_1 by z. If $h(z)$ is an unbiased estimate of $\bar{F}_{2r}$ then

$$\int_{-\infty}^{\infty} \{h(z+\nu)-z^{2r}\} \exp(-\tfrac{1}{2}z^2\omega^{-2})dz$$

should vanish for all ν and all $\omega > 0$. This integral can be written as an

[2] It has been remarked that it is obvious that from a sample of one it is not possible to obtain an unbiased estimate of two independent parameters (that is, two functions F_1 and F_2 on a class of distributions such that there exists no function g in the plane with $g\{F_1(D), F_2(D)\} = 0$ for all distributions D in the class). That this is not so is easily shown by an example. Let $\theta^2 = \nu^2+\omega^2$, where ν and ω^2, the mean and variance, are independent parameters when, e.g., the class is the normal class. Then ν and θ^2 are also independent parameters over that class with unbiased estimates z_1 and z_1^2.

integral over the positive axis and then we can make the substitution $u = z^{\frac{1}{2}}$ and obtain, setting $\omega' = (2\omega^2)^{-1}$, that

$$\int_0^\infty \{h(-u^{\frac{1}{2}}+\nu)+h(u^{\frac{1}{2}}+\nu)-2u^r\}u^{-\frac{1}{2}} \exp(-u\omega')\,du$$

is zero for all ν and all $\omega' > 0$. This being a Laplace transform of $u^{-\frac{1}{2}}$ times the expression in brackets, it follows that

$$h(-z+\nu)+h(z+\nu)-2z^{2r} = 0$$

for all ν and almost all positive z. For all ν there is a set S_ν on the positive z axis such that the Lebesgue measure l of the positive points z not in S_ν is zero and such that the above equality holds on S_ν. Denote $\bigcap_{k=1,2,4,5} S_{c+ak/2}$ by T.

It is easily shown[3] that there exists a pair of points a and $\frac{1}{2}a$ in T. Choosing $\nu = a$ and $2a$ respectively gives for $z = a$

$$h(0)+h(2a) = 2a^{2r}, \qquad h(a)+h(3a) = 2a^{2r},$$

so that

$$h(0)+h(a)+h(2a)+h(3a) = 4a^{2r}.$$

Choosing $\nu = \frac{1}{2}a$ and $2\frac{1}{2}a$ respectively gives for $z = \frac{1}{2}a$

$$h(0)+h(a) = a^{2r}/2^{2r-1}, \qquad h(2a)+h(3a) = a^{2r}/2^{2r-1},$$

so that

$$h(0)+h(a)+h(2a)+h(3a) = a^{2r}/2^{2r-2}.$$

Since $a \neq 0$, this is a contradiction.

4. Uniqueness of the unbiased symmetric estimate of $\bar{F}_{2r}$ in a sample of two and nonuniqueness in a larger sample

For $n \geqq 2$ (so that S^2 is not identically zero) the sufficiency of the statistic $(\bar{z}, S^2)$ and the completeness of its distribution imply that $\bar{f}_{2r}$ is the unique unbiased estimate of its expectation $\bar{F}_{2r}$ among unbiased estimates depending on $(\bar{z}, S^2)$ only [5]. Now if $n = 2$, $(\bar{z}, S^2)$ determines the set $\{z_1, z_2\}$ of observations, but not their order. Therefore $\bar{f}_{2r}$ is also the unique unbiased estimate of $\bar{F}_{2r}$ among unbiased estimates which are symmetric in the observations.

In general, when $n > 2$, for any $a \neq 0$,

[3] Let a' be in T and let $0 < b < a'$. Define the disjoint intervals I_i from ia' to $i(a'+b)$ for $i = 1, 2$, which have $l(I_i T) = ib$. Denote by $p_j(I_i T)$ the set of points x in $I_j T$ such that ix/j is in $I_i T$; $l\{p_j(I_i T)\} = jb$. Now let

$$T_2 = Tp_2(I_1 T), \qquad T_1 = p_1(I_2 T_2);$$

then, since the T_i are subsets of T of measure ib, there exists $a > 0$ such that $\frac{1}{2}ia$ is in T_i for $i = 1$ and 2. In fact, there exist c such that, for almost all a in T, $\frac{1}{2}a$ is in T. For brevity use $c = 0$.

$$\bar{f}_{2r}+a\{n(n+1)\sum (z_i-\bar{z})^4-3(n-1)S^4\}$$

will be an unbiased symmetric estimate of $\bar{F}_{2r}$ different from $\bar{f}_{2r}$, since the mean of $\sum (z_i-\bar{z})^4$ is $3n^{-1}(n-1)^2\omega^4$ and the mean of S^4 is $(n-1)(n+1)\omega^4$, and since for $n > 2$ the bracket is not identically equal to zero. For example, if $n = 3$, $1\frac{1}{2}\sum (z_i-\bar{z})^4$ has mean $\bar{F}_4+3\bar{F}_2^2$ and, in the normal case, S^4 has mean $8\bar{F}_2^2$, so that $1\frac{1}{2}\{\sum (z_i-\bar{z})^4-S^4/4\}$ and $\frac{3}{4}\sum (z_i-\bar{z})^4$ are unbiased estimates of $\bar{F}_4$ different from $\bar{f}_4 = 3S^4/8$.

5. Remarks

One could similarly discuss unbiased estimation of other functions over the class of normal distributions.

Fraser [3] adapts Halmos' argument to cases where $\mathscr{D}$ is a certain class of distributions that have a density. Some cases of this kind have been found by Lehmann and Scheffé; see [1].

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References

[1] Bell, C. B., Blackwell, D., and Breiman, L., On the Completeness of Order Statistics. *Ann. Math. Statist.*, **31**, (1960), 794–7.

[2] Cramér, H. *Mathematical Methods of Statistics*. Princeton University Press, Princeton, (1946).

[3] Fraser, D. A. S., Completeness of Order Statistics. *Canad. J. of Math.*, **6**, (1954), 42–5.

[4] Halmos, P. R., The Theory of Unbiased Estimation. *Ann. Math. Statist.*, **17**, (1946), 34–43.

[5] Konijn, H. S., Non-Existence of Consistent Estimator Sequences and Unbiased Estimators: A Practical Example. (To appear)

[6] Lehmann, E. L., and Scheffé, H., Completeness, Similar Regions, and Unbiased Estimation. *Sankhya*, **10** (1950) 305–40; **15**, (1955) 219–36.

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