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On Knörrer Periodicity for Quadric Hypersurfaces in Skew Projective Spaces

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Abstract. We study the structure of the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f))$ of graded maximal Cohen–Macaulay module over $S/(f)$ where S is a graded (± 1)-skew polynomial algebra in n variables of degree 1, and $f = x_1^2 + \cdots + x_n^2$. If S is commutative, then the structure of $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f))$ is well known by Knörrer's periodicity theorem. In this paper, we prove that if $n \leqslant 5$, then the structure of $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f))$ is determined by the number of irreducible components of the point scheme of S which are isomorphic to $\mathbb{P}^1$.

1 Introduction

Throughout this paper, we fix an algebraically closed field k of characteristic 0.

Knörrer's periodicity theorem ([5, Theorem 3.1]) plays an essential role in Cohen–Macaulay representation theory of Gorenstein rings. As a special case of Knörrer's periodicity theorem, the following result is well known (see also [3]).

Theorem 1.1 *Let $S = k[x_1, \dots, x_n]$ be a graded polynomial algebra generated in degree 1 and let $f = x_1^2 + x_2^2 + \cdots + x_n^2$. Let $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f))$ denote the stable category of graded maximal Cohen-Macaulay module over $S/(f)$.*

(i) *If n is odd, then $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f)) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(k[x]/(x^2)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k)$.*
(ii) *If n is even, then $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f)) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(k[x,y]/(x^2+y^2)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2)$.*

The purpose of this paper is to study a "(± 1)-skew" version of Theorem 1.1.

Definition 1.2 Let $n \in \mathbb{N}^+$.

(i) We say that S is a graded skew polynomial algebra if

$$S = k\langle x_1, \dots, x_n\rangle/(x_i x_j - \alpha_{ij} x_j x_i)_{1 \le i,j \le n},$$

where $\alpha_{ii} = 1$ for every $1 \le i \le n$, $\alpha_{ij}\alpha_{ji} = 1$ for every $1 \le i, j \le n$, and $\deg x_i = 1$ for every $1 \le i \le n$.
(ii) We say that S is a graded (± 1)-skew polynomial algebra if

$$S = k\langle x_1, \dots, x_n\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)_{1 \le i,j \le n}$$

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is a graded skew polynomial algebra such that ε_{ij} equals either 1 or -1 for every $1 \le i, j \le n, i \ne j$.

Clearly, a graded polynomial algebra $k[x_1, \dots, x_n]$ generated in degree 1 is an example of a graded (± 1)-skew polynomial algebra. Consider the element

$$f = x_1^2 + x_2^2 + \cdots + x_n^2$$

of a graded skew polynomial algebra $S = k\langle x_1, \dots, x_n\rangle/(x_i x_j - \alpha_{ij} x_j x_i)$. Then we notice that f is normal if and only if f is central if and only if S is a (± 1)-skew polynomial algebra.

Let $S = k\langle x_1, \dots, x_n\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ be a graded (± 1)-skew polynomial algebra so that $f = x_1^2 + x_2^2 + \cdots + x_n^2 \in S$ is a homogeneous regular central element. Let A be the graded quotient algebra $S/(f)$. Since S is a noetherian AS-regular algebra of dimension n and A is a noetherian AS-Gorenstein algebra of dimension $n-1$, A is regarded as a homogeneous coordinate ring of a quadric hypersurface in a (± 1)-skew projective space. The main focus of this paper is to determine the structure of $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ from a geometric data associated with S, called the point scheme of S. Based on our experiments, we propose the following conjecture.

Conjecture 1.3 Let $S = k\langle x_1, \dots, x_n\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ be a graded (± 1)-skew polynomial algebra, let $f = x_1^2 + x_2^2 + \cdots + x_n^2 \in S$, and let $A = S/(f)$. Let ℓ be the number of irreducible components of the point scheme of S that are isomorphic to $\mathbb{P}^1$.

(i) If n is odd, then

$$\binom{2m-1}{2} < \ell \le \binom{2m+1}{2} \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k^{2^{2m}})$$

for $m \in \mathbb{N}$ where we consider $\binom{-1}{2} = -\infty$, $\binom{1}{2} = 0$.

(ii) If n is even, then

$$\binom{2m}{2} < \ell \le \binom{2m+2}{2} \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k^{2^{2m+1}})$$

for $m \in \mathbb{N}$ where we consider $\binom{0}{2} = -\infty$.

We prove the following result.

Theorem 1.4 (Theorem 3.10) *Let $S = k\langle x_1, \dots, x_n\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ be a graded (± 1)-skew polynomial algebra, let $f = x_1^2 + x_2^2 + \cdots + x_n^2 \in S$, and let $A = S/(f)$. Assume that $n \le 5$. Let ℓ be the number of irreducible components of the point scheme of S that are isomorphic to $\mathbb{P}^1$.*

(i) *If n is odd, then $\ell \le 10$ and*

$$\ell = 0 \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k),$$
$$0 < \ell \le 3 \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k^4),$$
$$3 < \ell \le 10 \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k^{16}).$$

(ii) *If n is even, then $\ell \le 6$ and*

$$0 \le \ell \le 1 \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k^2),$$
$$1 < \ell \le 6 \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k^8).$$

This theorem asserts that Conjecture 1.3 is true if $n \le 5$.

2 Preliminaries

2.1 Notation

For an algebra A, we denote by $\mathsf{Mod}\, A$ the category of right A-modules, and by $\mathsf{mod}\, A$ the full subcategory consisting of finitely generated modules. The bounded derived category of $\mathsf{mod}\, A$ is denoted by $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, A)$.

For a connected graded algebra A, that is, $A = \bigoplus_{i \in \mathbb{N}} A_i$ with $A_0 = k$, we denote by $\mathsf{GrMod}\, A$ the category of graded right A-modules with A-module homomorphisms of degree zero, and by $\mathsf{grmod}\, A$ the full subcategory consisting of finitely generated graded modules.

Let A be a noetherian AS-Gorenstein algebra of dimension n (see [4, Section 1] for the definition). We define the local cohomology modules of $M \in \mathsf{grmod}\, A$ by $\mathrm{H}^i_{\mathfrak{m}}(M) := \lim_{n\to\infty} \mathrm{Ext}^i_A(A/A_{\ge n}, M)$. It is well known that $\mathrm{H}^i_{\mathfrak{m}}(A) = 0$ for all $i \ne n$. We say that $M \in \mathsf{grmod}\, A$ is graded maximal Cohen–Macaulay if $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i \ne n$. We denote by $\mathsf{CM}^{\mathbb{Z}}(A)$ the full subcategory of $\mathsf{grmod}\, A$ consisting of graded maximal Cohen–Macaulay modules.

The stable category of graded maximal Cohen–Macaulay modules, denoted by $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$, has the same objects as $\mathsf{CM}^{\mathbb{Z}}(A)$ and the morphism set is given by

$$\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(A)}(M, N) = \mathrm{Hom}_{\mathsf{grmod}\, A}(M, N)/P(M, N)$$

for any $M, N \in \mathsf{CM}^{\mathbb{Z}}(A)$, where $P(M, N)$ consists of degree zero A-module homomorphisms that factor through a projective module in $\mathsf{grmod}\, A$. Since A is AS-Gorenstein, $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ is a triangulated category with respect to the translation functor $M[-1] = \Omega M$ (the syzygy of M) by [8, Theorem 3.1].

2.2 The Algebra $C(A)$

The method we use is due to Smith and Van den Bergh [8]; it was originally developed by Buchweitz, Eisenbud, and Herzog [3].

Let S be an n-dimensional noetherian AS-regular algebra with the Hilbert series $H_S(t) = (1-t)^{-n}$. Then S is Koszul by [7, Theorem 5.11]. Let $f \in S$ be a homogeneous regular central element of degree 2, and let $A = S/(f)$. Then A is Koszul by [8, Lemma 5.1 (1)], and there exists a central regular element $w \in A^!_2$ such that $A^!/(w) \cong S^!$ by [8, Lemma 5.1 (2)]. We can define the algebra

$$C(A) := A^![w^{-1}]_0.$$

By [8, Lemma 5.1 (3)], we have $\dim_k C(A) = \dim_k (S^!)^{(2)} = 2^{n-1}$.

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Theorem 2.1 ([8, Proposition 5.2]) *Let the notation be as above. Then* $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, C(A))$.

2.3 The Point Schemes of Skew Polynomial Algebras

Let S be a quantum polynomial algebra of dimension n (see [6, Definition 2.1] for the definition).

Definition 2.2 A graded module $M \in \mathsf{GrMod}\, S$ is called a *point module* if M is cyclic, generated in degree 0, and $H_M(t) = (1-t)^{-1}$.

If $M \in \mathsf{GrMod}\, S$ is a point module, then M is written as a quotient $S/(g_1 S + g_2 S + \cdots + g_{n-1} S)$ with linearly independent $g_1, \dots, g_{n-1} \in S_1$ by [6, Corollary 5.7, Theorem 3.8], so we can associate it with a unique point $p_M := \mathcal{V}(g_1, \dots, g_{n-1})$ in $\mathbb{P}(S_1^*) = \mathbb{P}^{n-1}$. Then the subset

$$E := \{p_M \in \mathbb{P}^{n-1} \mid M \in \mathsf{GrMod}\, S \text{ is a point module}\}$$

has a k-scheme structure by [1], and it is called the point scheme of S. Point schemes have a pivotal role in noncommutative algebraic geometry.

Thanks to the following result, we can compute the point scheme of a graded skew polynomial algebra.

Theorem 2.3 ([9, Proposition 4.2], [2, Theorem 1 (1)]) *Let*

$$S = k\langle x_1, \dots, x_n\rangle/(x_i x_j - \alpha_{ij} x_j x_i)$$

be a graded skew polynomial algebra. Then the point scheme of S is given by

$$E = \bigcap_{\substack{1 \le i < j < k \le n \\ \alpha_{ij}\alpha_{jk}\alpha_{ki} \ne 1}} \mathcal{V}(x_i x_j x_k) \quad \subset \mathbb{P}^{n-1}.$$

For $1 \le i_0, \dots, i_s \le n$, we define the subspace

$$\mathbb{P}(i_1, \dots, i_s) := \bigcap_{\substack{1 \le j \le n \\ j \ne i_1, \dots, j \ne i_s}} \mathcal{V}(x_j) \ \subset \mathbb{P}^{n-1}.$$

It is easy to see that the point scheme of a graded skew polynomial algebra in three variables is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}(2,3) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2)$. The following is the classification of the point schemes of graded skew polynomial algebras in four variables.

Proposition 2.4 ([9, Corollary 5.1], [2, Section 4.2]) *Let*

$$S = k\langle x_1, x_2, x_3, x_4\rangle/(x_i x_j - \alpha_{ij} x_j x_i)$$

be a graded skew polynomial algebra in four variables. Then the point scheme of S is isomorphic one of the following:

- $\mathbb{P}^3$;
- $\mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(3,4)$;
- $\mathbb{P}(2,3,4) \cup \mathbb{P}(1,4) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2)$;
- $\mathbb{P}(3,4) \cup \mathbb{P}(2,4) \cup \mathbb{P}(2,3) \cup \mathbb{P}(1,4) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2)$.

3 Results

Throughout this section,

- $S = k\langle x_1, \dots, x_n\rangle/(x_ix_j - \varepsilon_{ij}x_jx_i)$ is a graded (±1)-skew polynomial algebra,
- E is the point scheme of S,
- $f = x_1^2 + x_2^2 + \cdots + x_n^2 \in S_2$ (a regular central element of S), and
- $A = S/(f)$.

Note that $\varepsilon_{ij} = \varepsilon_{ji}$ holds for every $1 \le i, j \le n$.

Lemma 3.1 (i) *$A^!$ is isomorphic to*

$$k\langle x_1, \dots, x_n\rangle/(\varepsilon_{ij}x_ix_j + x_jx_i, x_n^2 - x_i^2)_{1\le i,j\le n, i\ne j}.$$

(ii) *$w = x_n^2 \in A^!_2$ is a central regular element such that $A^!/(w) \cong S^!$.*
(iii) *$C(A) := A^![w^{-1}]_0$ is isomorphic to*

$$k\langle t_1, \dots, t_{n-1}\rangle/(t_it_j + \varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn}t_jt_i,\ t_i^2 - 1)_{1\le i,j\le n-1, i\ne j}.$$

Proof (i) and (ii) follow from direct calculation.

(iii) Since S has a k-basis $\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \mid i_1, i_2, \dots, i_n \ge 0\}$, and

$$(x_nx_iw^{-1})(x_nx_jw^{-1}) = x_nx_ix_nx_jw^{-2} = -\varepsilon_{ni}x_n^2x_ix_jw^{-2} = -\varepsilon_{ni}x_ix_jw^{-1}$$

in $C(A)$ for $1 \le i, j \le n-1$, $i \ne j$, it follows that $\{x_nx_1w^{-1}, \dots, x_nx_{n-1}w^{-1}\}$ is a set of generators of $C(A)$. Put $t_i := x_nx_iw^{-1}$ for $1 \le i \le n-1$. Since

$$\begin{aligned} t_it_j &= (x_nx_iw^{-1})(x_nx_jw^{-1}) = -\varepsilon_{ni}x_ix_jw^{-1} = \varepsilon_{ni}\varepsilon_{ji}x_jx_iw^{-1} \\ &= -\varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn}(-\varepsilon_{nj}x_jx_iw^{-1}) = -\varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn}(x_nx_jw^{-1})(x_nx_iw^{-1}) \\ &= -\varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn}t_jt_i, \end{aligned}$$

for $1 \le i, j \le n-1$, $i \ne j$, and

$$t_i^2 = (x_nx_iw^{-1})(x_nx_iw^{-1}) = -\varepsilon_{ni}x_i^2w^{-1} = -\varepsilon_{ni}x_n^2w^{-1} = -\varepsilon_{ni}$$

for $1 \le i \le n-1$, we have a surjection $k\langle t_1, \dots, t_{n-1}\rangle/(t_it_j + \varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn}t_jt_i,\ t_i^2 + \varepsilon_{ni}) \to C(A)$. This is an isomorphism, because the algebras have the same dimension. Since $\varepsilon_{ni} \ne 0$ for $1 \le i \le n-1$, the homomorphism defined by $t_i \to \sqrt{-\varepsilon_{ni}}t_i$ induces the isomorphism

$$\begin{aligned} &k\langle t_1, \dots, t_{n-1}\rangle/(t_it_j + \varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn}t_jt_i,\ t_i^2 + \varepsilon_{ni}) \xrightarrow{\sim} \\ &\qquad k\langle t_1, \dots, t_{n-1}\rangle/(t_it_j + \varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn}t_jt_i,\ t_i^2 - 1). \end{aligned}$$ ■

Proposition 3.2 (i) *If $E = \mathbb{P}^{n-1}$, then $C(A)$ is isomorphic to*

$$C_+ := k\langle t_1, \dots, t_{n-1}\rangle/(t_it_j + t_jt_i,\ t_i^2 - 1)_{1\le i,j\le n-1, i\ne j}.$$

(ii) *$E = \bigcup_{1\le i<j\le n}\mathbb{P}(i,j)$ if and only if $C(A)$ is isomorphic to*

$$C_- := k\langle t_1, \dots, t_{n-1}\rangle/(t_it_j - t_jt_i,\ t_i^2 - 1)_{1\le i,j\le n-1, i\ne j}.$$

Proof First note that

$$\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} = (\varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn})(\varepsilon_{nj}\varepsilon_{jk}\varepsilon_{kn})(\varepsilon_{nk}\varepsilon_{ki}\varepsilon_{in}) \tag{3.1}$$

for $1 \le i < j < k \le n$.

(i) By Theorem 2.3, (3.1), and Lemma 3.1(iii), it follows that

$$\begin{aligned} E = \mathbb{P}^{n-1} &\iff \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} = 1 \text{ for every } 1 \le i < j < k \le n \\ &\iff \varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn} = 1 \text{ for every } 1 \le i < j \le n \\ &\Longrightarrow C(A) \cong C_+. \end{aligned}$$

(ii) By Theorem 2.3, (3.1), and Lemma 3.1(iii), it follows that

$$\begin{aligned} E = \bigcup_{1\le i<j\le n} \mathbb{P}(i,j) &\iff \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} \ne 1 \text{ for every } 1 \le i < j < k \le n \\ &\iff \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} = -1 \text{ for every } 1 \le i < j < k \le n \\ &\iff \varepsilon_{ni}\varepsilon_{ij}\varepsilon_{jn} = -1 \text{ for every } 1 \le i < j \le n \\ &\iff C(A) \cong C_-. \end{aligned}$$

Here the last $\Longleftarrow$ is by commutativity of $C(A)$. ∎

Theorem 3.3 (i) *If $E = \mathbb{P}^{n-1}$ and n is odd, then $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k)$.*
(ii) *If $E = \mathbb{P}^{n-1}$ and n is even, then $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2)$.*
(iii) *$E = \bigcup_{1\le i<j\le n} \mathbb{P}(i,j)$ if and only if $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^{2^{n-1}})$.*

Proof Since C_+ is a Clifford algebra over k, it is known that

$$C_+ \cong \begin{cases} M_{2^{(n-1)/2}}(k) & \text{if } n \text{ is odd,} \\ M_{2^{(n-2)/2}}(k)^2 & \text{if } n \text{ is even,} \end{cases} \tag{3.2}$$

so

$$\operatorname{mod} C_+ \cong \begin{cases} \operatorname{mod} M_{2^{(n-1)/2}}(k) \cong \operatorname{mod} k & \text{if } n \text{ is odd,} \\ \operatorname{mod} M_{2^{(n-2)/2}}(k)^2 \cong \operatorname{mod} k^2 & \text{if } n \text{ is even.} \end{cases}$$

Thus, (i) and (ii) follow from Theorem 2.1 and Proposition 3.2(i).

We next show (iii). If $E = \bigcup_{1\le i<j\le n} \mathbb{P}(i,j)$, then $C(A) \cong C_-$ by Proposition 3.2(ii). Since C_- is isomorphic to the group algebra of $(\mathbb{Z}_2)^{n-1}$ over k, we have $C_- \cong k^{2^{n-1}}$, so it follows that $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^{2^{n-1}})$ by Theorem 2.1. Conversely, if $\underline{\mathsf{CM}}^{\mathbb{Z}}(S/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^{2^{n-1}})$, then $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} C(A)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^{2^{n-1}})$ by Theorem 2.1. Since $\dim_k C(A) = 2^{n-1}$, it follows that $C(A) \cong k^{2^{n-1}} \cong C_-$. Hence, $E = \bigcup_{1\le i<j\le n} \mathbb{P}(i,j)$ by Proposition 3.2(ii). ∎

Note that Theorem 3.3(i), (ii) recover Theorem 1.1, and Theorem 3.3(iii) shows that a new phenomenon appears in the noncommutative case. We can now give an explicit classification of $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ in the case $n \le 3$ (the case $n = 1$ is clear; see Theorem 1.1(i)).

Corollary 3.4 (i) *If $n = 2$, then $E = \mathbb{P}^1$ and $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2)$.*

(ii) *If $n = 3$, then*

$$E = \mathbb{P}^2 \iff \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k),$$
$$E = \mathbb{P}(2,3) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2) \iff \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k^4).$$

Proof These follow from Theorem 2.3 and Theorem 3.3. ■

As we will see later, the converse of Theorem 3.3(i), (ii) does not hold in general. So, in order to give a classification for the cases $n = 4$ and $n = 5$, we need a precise computation.

For a permutation $\sigma \in \mathfrak{S}_n$, we have an isomorphism

$$S = k\langle x_1, \dots, x_n \rangle/(x_i x_j - \varepsilon_{ij} x_j x_i) \xrightarrow[\varphi]{\sim} k\langle x_1, \dots, x_n \rangle/(x_{\sigma(i)} x_{\sigma(j)} - \varepsilon_{ij} x_{\sigma(j)} x_{\sigma(i)}) =: S_\sigma$$

between graded (± 1)-skew polynomial algebras, which we call a permutation isomorphism. Since φ preserves f, it induces an isomorphism

$$A = S/(f) \xrightarrow{\sim} S_\sigma/(f),$$

which we also call a permutation isomorphism.

Lemma 3.5 *If $n = 4$, then, via a permutation isomorphism, S is isomorphic to a graded (± 1)-skew polynomial algebra whose point scheme is one of the following:*

(4a) $\mathbb{P}^3$;

(4b) $\mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(3,4)$;

(4c) $\mathbb{P}(3,4) \cup \mathbb{P}(2,4) \cup \mathbb{P}(2,3) \cup \mathbb{P}(1,4) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2)$.

Proof First, via a permutation isomorphism, S is isomorphic to one of the following:

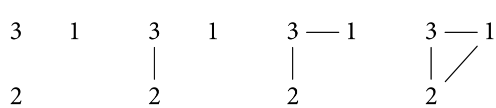
(4i) a graded (± 1)-skew polynomial algebra with $\varepsilon_{41}\varepsilon_{12}\varepsilon_{24} = \varepsilon_{41}\varepsilon_{13}\varepsilon_{34} = \varepsilon_{42}\varepsilon_{23}\varepsilon_{34} = 1$;

(4ii) a graded (± 1)-skew polynomial algebra with $\varepsilon_{41}\varepsilon_{12}\varepsilon_{24} = \varepsilon_{41}\varepsilon_{13}\varepsilon_{34} = 1$, $\varepsilon_{42}\varepsilon_{23}\varepsilon_{34} = -1$;

(4iii) a graded (± 1)-skew polynomial algebra with $\varepsilon_{41}\varepsilon_{12}\varepsilon_{24} = 1$, $\varepsilon_{41}\varepsilon_{13}\varepsilon_{34} = \varepsilon_{42}\varepsilon_{23}\varepsilon_{34} = -1$;

(4iv) a graded (± 1)-skew polynomial algebra with $\varepsilon_{41}\varepsilon_{12}\varepsilon_{24} = \varepsilon_{41}\varepsilon_{13}\varepsilon_{34} = \varepsilon_{42}\varepsilon_{23}\varepsilon_{34} = -1$.

Note that the above follows from (3.1) and the classification of simple graphs of order 3:



(we define $\varepsilon_{4i}\varepsilon_{ij}\varepsilon_{j4} = -1$ if $\{i, j\}$ is an edge in the graph, and $\varepsilon_{4i}\varepsilon_{ij}\varepsilon_{j4} = 1$ otherwise).

The point scheme of an algebra in the case (4i) is $\mathbb{P}^3$, so this is (4a).

The point scheme of an algebra in the case (4iii) is

$$\mathcal{V}(x_1 x_3 x_4) \cap \mathcal{V}(x_2 x_3 x_4) = \mathcal{V}(x_3) \cup \mathcal{V}(x_4) \cup \mathcal{V}(x_1, x_2),$$

so this is (4b). The point scheme of an algebra in the case (4ii) is $\mathcal{V}(x_1x_2x_3) \cap \mathcal{V}(x_2x_3x_4) = \mathcal{V}(x_2) \cup \mathcal{V}(x_3) \cup \mathcal{V}(x_1, x_4)$, so an algebra in the case (4ii) is isomorphic to an algebra in the case (4iii) via the permutation isomorphism induced by $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{smallmatrix}\right)$.

The point scheme of an algebra in the case (4iv) is $\bigcap_{1\le i<j<k\le 4} \mathcal{V}(x_ix_jx_k) = \bigcup_{1\le i<j\le 4} \mathcal{V}(x_i, x_j)$, so this is (4c). ■

Remark 3.6 It follows from Lemma 3.5 that not every point scheme in Proposition 2.4 appears as the point scheme of a graded (±1)-skew polynomial algebra.

Lemma 3.7 *If $n = 5$, then, via a permutation isomorphism, S is isomorphic to a graded (±1)-skew polynomial algebra whose point scheme is one of the following:*

(5a) $\mathbb{P}^4$;
(5b) $\mathbb{P}(1,2,3,5) \cup \mathbb{P}(1,2,3,4) \cup \mathbb{P}(4,5)$;
(5c) $\mathbb{P}(1,2,3,4) \cup \mathbb{P}(3,4,5) \cup \mathbb{P}(1,2,5)$;
(5d) $\mathbb{P}(3,4,5) \cup \mathbb{P}(1,4,5) \cup \mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(2,3,4)$;
(5e) $\mathbb{P}(1,3,5) \cup \mathbb{P}(1,3,4) \cup \mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,4) \cup \mathbb{P}(4,5) \cup \mathbb{P}(2,3)$;
(5f) $\mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(4,5) \cup \mathbb{P}(3,5) \cup \mathbb{P}(3,4)$;
(5g) $\mathbb{P}(4,5) \cup \mathbb{P}(3,5) \cup \mathbb{P}(3,4) \cup \mathbb{P}(2,5) \cup \mathbb{P}(2,4) \cup \mathbb{P}(2,3) \cup \mathbb{P}(1,5) \cup \mathbb{P}(1,4) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2)$.

Proof First, via a permutation isomorphism, S is isomorphic to one of the following:

(5i) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = 1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = 1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = 1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = 1;$$

(5ii) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = 1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = 1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = 1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1;$$

(5iii) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = 1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = 1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1;$$

(5iv) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = -1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = 1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = 1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = 1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1;$$

(5v) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = 1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1;$$

(5vi) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = 1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = 1;$$

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(5vii) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = 1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = 1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1;$$

(5viii) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = 1;$$

(5ix) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1;$$

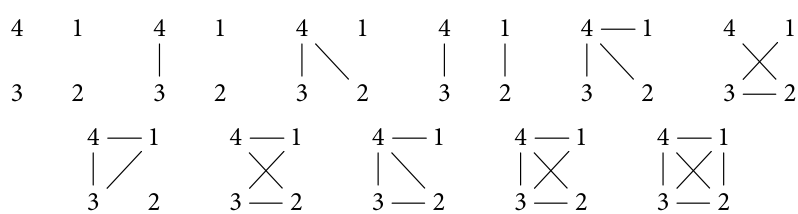
(5x) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1;$$

(5xi) a graded (±1)-skew polynomial algebra with

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = -1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1.$$

Note that the above follows from (3.1) and the classification of simple graphs of order 4:



(we define $\varepsilon_{5i}\varepsilon_{ij}\varepsilon_{j5} = -1$ if $\{i, j\}$ is an edge in the graph, and $\varepsilon_{5i}\varepsilon_{ij}\varepsilon_{j5} = 1$ otherwise).

The point scheme of an algebra in the case (5i) is $\mathbb{P}^4$, so this is (5a).

The point scheme of an algebra in the case (5v) is

$$\mathcal{V}(x_1x_4x_5) \cap \mathcal{V}(x_2x_4x_5) \cap \mathcal{V}(x_3x_4x_5) = \mathcal{V}(x_4) \cup \mathcal{V}(x_5) \cup \mathcal{V}(x_1, x_2, x_3),$$

so this is (5b). The point scheme of an algebra in the case (5ii) is $\mathcal{V}(x_1x_3x_4) \cap \mathcal{V}(x_2x_3x_4) \cap \mathcal{V}(x_3x_4x_5) = \mathcal{V}(x_3) \cup \mathcal{V}(x_4) \cup \mathcal{V}(x_1, x_2, x_5)$, so an algebra in the case (5ii) is isomorphic to an algebra in the case (5v) via the permutation isomorphism induced by $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{smallmatrix}\right)$.

The point scheme of an algebra in the case (5viii) is

$$\mathcal{V}(x_1x_3x_5) \cap \mathcal{V}(x_1x_4x_5) \cap \mathcal{V}(x_2x_3x_5) \cap \mathcal{V}(x_2x_4x_5) = \mathcal{V}(x_5) \cup \mathcal{V}(x_1, x_2) \cup \mathcal{V}(x_3, x_4),$$

so this is (5c). The point scheme of an algebra in the case (5iii) is $\mathcal{V}(x_1x_2x_4) \cap \mathcal{V}(x_1x_3x_4) \cap \mathcal{V}(x_2x_4x_5) \cap \mathcal{V}(x_3x_4x_5) = \mathcal{V}(x_4) \cup \mathcal{V}(x_1, x_5) \cup \mathcal{V}(x_2, x_3)$, so an algebra in the case (5iii) is isomorphic to an algebra in the case (5viii) via the permutation isomorphism induced by $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{smallmatrix}\right)$.

The point scheme of an algebra in the case (5vi) is

$$\mathcal{V}(x_1x_2x_4) \cap \mathcal{V}(x_1x_3x_4) \cap \mathcal{V}(x_1x_3x_5) \cap \mathcal{V}(x_2x_3x_5) \cap \mathcal{V}(x_2x_4x_5) = \mathcal{V}(x_1,x_2) \cup \mathcal{V}(x_2,x_3) \cup \mathcal{V}(x_3,x_4) \cup \mathcal{V}(x_4,x_5) \cup \mathcal{V}(x_5,x_1),$$

so this is (5d).

The point scheme of an algebra in the case (5ix) is

$$\mathcal{V}(x_1x_2x_3) \cap \mathcal{V}(x_1x_4x_5) \cap \mathcal{V}(x_2x_3x_4) \cap \mathcal{V}(x_2x_3x_5) \cap \mathcal{V}(x_2x_4x_5) \cap \mathcal{V}(x_3x_4x_5) = \mathcal{V}(x_2,x_4) \cup \mathcal{V}(x_2,x_5) \cup \mathcal{V}(x_3,x_4) \cup \mathcal{V}(x_3,x_5) \cup \mathcal{V}(x_1,x_2,x_3) \cup \mathcal{V}(x_1,x_4,x_5),$$

so this is (5e). The point scheme of an algebra in the case (5iv) is $\mathcal{V}(x_1x_2x_3) \cap \mathcal{V}(x_1x_2x_4) \cap \mathcal{V}(x_1x_2x_5) \cap \mathcal{V}(x_1x_3x_4) \cap \mathcal{V}(x_2x_3x_4) \cap \mathcal{V}(x_3x_4x_5) = \mathcal{V}(x_1,x_3) \cup \mathcal{V}(x_1,x_4) \cup \mathcal{V}(x_2,x_3) \cup \mathcal{V}(x_2,x_4) \cup \mathcal{V}(x_1,x_2,x_5) \cup \mathcal{V}(x_3,x_4,x_5)$, so an algebra in the case (5iv) is isomorphic to an algebra in the case (5ix) via the permutation isomorphism induced by $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{smallmatrix}\right)$.

The point scheme of an algebra in the case (5x) is

$$\begin{aligned} &\mathcal{V}(x_1x_3x_4) \cap \mathcal{V}(x_1x_3x_5) \cap \mathcal{V}(x_1x_4x_5) \cap \mathcal{V}(x_2x_3x_4) \cap \mathcal{V}(x_2x_3x_5) \\ &\qquad \cap \mathcal{V}(x_2x_4x_5) \cap \mathcal{V}(x_3x_4x_5) \\ &\quad = \mathcal{V}(x_3,x_4) \cup \mathcal{V}(x_3,x_5) \cup \mathcal{V}(x_4,x_5) \cup \mathcal{V}(x_1,x_2,x_3) \cup \mathcal{V}(x_1,x_2,x_4) \\ &\qquad \cup \mathcal{V}(x_1,x_2,x_5), \end{aligned}$$

so this is (5f). The point scheme of an algebra in the case (5vii) is $\mathcal{V}(x_1x_2x_3) \cap \mathcal{V}(x_1x_2x_4) \cap \mathcal{V}(x_1x_3x_4) \cap \mathcal{V}(x_1x_3x_5) \cap \mathcal{V}(x_1x_4x_5) \cap \mathcal{V}(x_2x_3x_4) \cap \mathcal{V}(x_3x_4x_5) = \mathcal{V}(x_1,x_3) \cup \mathcal{V}(x_1,x_4) \cup \mathcal{V}(x_3,x_4) \cup \mathcal{V}(x_1,x_2,x_5) \cup \mathcal{V}(x_2,x_3,x_5) \cup \mathcal{V}(x_2,x_4,x_5)$, so an algebra in the case (5vii) is isomorphic to an algebra in the case (5x) via the permutation isomorphism induced by $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{smallmatrix}\right)$.

The point scheme of an algebra in the case (5xi) is $\bigcap_{1\le i<j<k\le 5} \mathcal{V}(x_ix_jx_k) = \bigcup_{1\le i<j<k\le 5} \mathcal{V}(x_i,x_j,x_k)$, so this is (5g). ∎

To describe the algebras $C(A)$ appearing in Lemma 3.1, we show that the following algebras are isomorphic to algebras of the form $M_i(k)^j$.

Lemma 3.8 (i) $C_{\mathrm{i}} := k\langle t_1,t_2,t_3\rangle/(t_1t_2+t_2t_1, t_1t_3+t_3t_1, t_2t_3-t_3t_2, t_1^2-1, t_2^2-1, t_3^2-1)$ *is isomorphic to* $M_2(k)^2$.

(ii) $C_{\mathrm{ii}} := k\langle t_1,t_2,t_3\rangle/(t_1t_2+t_2t_1, t_1t_3-t_3t_1, t_2t_3-t_3t_2, t_1^2-1, t_2^2-1, t_3^2-1)$ *is isomorphic to* $M_2(k)^2$.

(iii) $C_{\mathrm{iii}} := k\langle t_1,t_2,t_3,t_4\rangle/(t_1t_2+t_2t_1, t_1t_3+t_3t_1, t_1t_4-t_4t_1, t_2t_3+t_3t_2, t_2t_4-t_4t_2, t_3t_4-t_4t_3, t_1^2-1, t_2^2-1, t_3^2-1, t_4^2-1)$ *is isomorphic to* $M_2(k)^4$.

(iv) $C_{\mathrm{iv}} := k\langle t_1,t_2,t_3,t_4\rangle/(t_1t_2+t_2t_1, t_1t_3-t_3t_1, t_1t_4-t_4t_1, t_2t_3-t_3t_2, t_2t_4-t_4t_2, t_3t_4+t_4t_3, t_1^2-1, t_2^2-1, t_3^2-1, t_4^2-1)$ *is isomorphic to* $M_4(k)$.

(v) $C_{\mathrm{v}} := k\langle t_1,t_2,t_3,t_4\rangle/(t_1t_2+t_2t_1, t_1t_3-t_3t_1, t_1t_4+t_4t_1, t_2t_3-t_3t_2, t_2t_4-t_4t_2, t_3t_4+t_4t_3, t_1^2-1, t_2^2-1, t_3^2-1, t_4^2-1)$ *is isomorphic to* $M_4(k)$.

(vi) $C_{\mathrm{vi}} := k\langle t_1,t_2,t_3,t_4\rangle/(t_1t_2+t_2t_1, t_1t_3+t_3t_1, t_1t_4-t_4t_1, t_2t_3-t_3t_2, t_2t_4-t_4t_2, t_3t_4-t_4t_3, t_1^2-1, t_2^2-1, t_3^2-1, t_4^2-1)$ *is isomorphic to* $M_2(k)^4$.

(vii) $C_{\mathrm{vii}} := k\langle t_1,t_2,t_3,t_4\rangle/(t_1t_2+t_2t_1, t_1t_3-t_3t_1, t_1t_4-t_4t_1, t_2t_3-t_3t_2, t_2t_4-t_4t_2, t_3t_4-t_4t_3, t_1^2-1, t_2^2-1, t_3^2-1, t_4^2-1)$ *is isomorphic to* $M_2(k)^4$.

Proof (i) Let

$$e_1 = \frac{1}{4}(1 + t_2 + t_3 + t_2t_3), \qquad e_2 = \frac{1}{4}(1 - t_2 + t_3 - t_2t_3),$$
$$e_3 = \frac{1}{4}(1 + t_2 - t_3 - t_2t_3), \qquad e_4 = \frac{1}{4}(1 - t_2 - t_3 + t_2t_3).$$

Then they form a complete set of orthogonal idempotents of C_{i}. Since

$$\begin{aligned} e_1t_1 &= \frac{1}{4}(1 + t_2 + t_3 + t_2t_3)t_1 = \frac{1}{4}t_2(1 - t_1 - t_3 + t_2t_3) = t_1e_4, \\ e_2t_1 &= \frac{1}{4}(1 - t_2 + t_3 - t_2t_3)t_1 = \frac{1}{4}t_2(1 + t_1 - t_3 - t_2t_3) = t_1e_3, \\ e_3t_1 &= \frac{1}{4}(1 + t_2 - t_3 - t_2t_3)t_1 = \frac{1}{4}t_2(1 - t_1 + t_3 - t_2t_3) = t_1e_2, \\ e_4t_1 &= \frac{1}{4}(1 - t_2 - t_3 + t_2t_3)t_1 = \frac{1}{4}t_2(1 + t_1 + t_3 + t_2t_3) = t_1e_1, \end{aligned}$$

it follows that the map $M_2(k)^2 \to C_{\mathrm{i}}$;

$$\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \longmapsto$$
$$\begin{matrix} a_{11}e_1 & +a_{12}e_1t_1e_4 & +b_{11}e_2 & +b_{12}e_2t_1e_3 \\ +a_{21}e_4t_1e_1 & +a_{22}e_4 & +b_{21}e_3t_1e_2 & +b_{22}e_3 \end{matrix}$$

is an isomorphism of algebras.

(ii) Since t_3 commutes with t_1, t_2 in C_{ii}, we have

$$\begin{aligned} C_{\mathrm{ii}} &\cong k\langle t_1, t_2\rangle/(t_1t_2 + t_2t_1, t_1^2 - 1, t_2^2 - 1) \otimes_k k[t_3]/(t_3^2 - 1) \\ &\cong M_2(k) \otimes_k k^2 \cong M_2(k)^2 \end{aligned}$$

by (3.2).

(iii) Since t_4 commutes with t_1, t_2, t_3 in C_{iii}, we have

$$\begin{aligned} C_{\mathrm{iii}} &\cong k\langle t_1, t_2, t_3\rangle/(t_1t_2 + t_2t_1, t_1t_3 + t_3t_1, t_2t_3 + t_3t_2, t_1^2 - 1, t_2^2 - 1, t_3^2 - 1) \\ &\quad \otimes_k k[t_4]/(t_4^2 - 1) \\ &\cong M_2(k)^2 \otimes_k k^2 \cong M_2(k)^4 \end{aligned}$$

by (3.2).

(iv) Since t_3, t_4 commute with t_1, t_2 in C_{iv}, we have

$$\begin{aligned} C_{\mathrm{iv}} &\cong k\langle t_1, t_2\rangle/(t_1t_2 + t_2t_1, t_1^2 - 1, t_2^2 - 1) \\ &\quad \otimes_k k\langle t_3, t_4\rangle/(t_3t_4 + t_4t_3, t_3^2 - 1, t_4^2 - 1) \\ &\cong M_2(k) \otimes_k M_2(k) \cong M_4(k) \end{aligned}$$

by (3.2).

(v) Let

$$e_1 = \frac{1}{4}(1 + t_1 + t_3 + t_1t_3), \qquad e_2 = \frac{1}{4}(1 - t_1 + t_3 - t_1t_3),$$
$$e_3 = \frac{1}{4}(1 + t_1 - t_3 - t_1t_3), \qquad e_4 = \frac{1}{4}(1 - t_1 - t_3 + t_1t_3).$$

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Then they form a complete set of orthogonal idempotents of C_{v}. Similar to the proof of (i), we have

$$\begin{aligned} e_1t_4 &= t_4e_4, & e_1t_2 &= t_2e_2, & e_1t_4t_2 &= t_4t_2e_3,\\ e_2t_4 &= t_4e_3, & e_2t_2 &= t_2e_1, & e_2t_4t_2 &= t_4t_2e_4,\\ e_3t_4 &= t_4e_2, & e_3t_2 &= t_2e_4, & e_3t_4t_2 &= t_4t_2e_1,\\ e_4t_4 &= t_4e_1, & e_4t_2 &= t_2e_3, & e_4t_4t_2 &= t_4t_2e_2, \end{aligned}$$

so it follows that the map $M_4(k) \to C_{\mathrm{v}}$;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \longmapsto$$

$$\begin{matrix} a_{11}e_1 & +a_{12}e_1t_4e_4 & +a_{13}e_1t_2e_2 & +a_{14}e_1t_4t_2e_3\\ +a_{21}e_4t_4e_1 & +a_{22}e_4 & +a_{23}e_4t_4t_2e_2 & +a_{24}e_4t_2e_3\\ +a_{31}e_2t_2e_1 & +a_{32}e_2t_4t_2e_4 & +a_{33}e_2 & +a_{34}e_2t_4e_3\\ +a_{41}e_3t_4t_2e_1 & +a_{42}e_3t_2e_4 & +a_{43}e_3t_4e_2 & +a_{44}e_3 \end{matrix}$$

is an isomorphism of algebras.

(vi) Since t_4 commutes with t_1, t_2, t_3 in C_{vi}, we have

$$\begin{aligned} C_{\mathrm{vi}} &\cong k\langle t_1, t_2, t_3\rangle/(t_1t_2 + t_2t_1, t_1t_3 + t_3t_1, t_2t_3 - t_3t_2, t_1^2 - 1, t_2^2 - 1, t_3^2 - 1)\\ &\quad \otimes_k k[t_4]/(t_4^2 - 1)\\ &\cong M_2(k)^2 \otimes_k k^2 \cong M_2(k)^4 \end{aligned}$$

by (i).

(vii) Since t_4 commutes with t_1, t_2, t_3 in C_{vii}, we have

$$\begin{aligned} C_{\mathrm{vii}} &\cong k\langle t_1, t_2, t_3\rangle/(t_1t_2 + t_2t_1, t_1t_3 - t_3t_1, t_2t_3 - t_3t_2, t_1^2 - 1, t_2^2 - 1, t_3^2 - 1)\\ &\quad \otimes_k k[t_4]/(t_4^2 - 1)\\ &\cong M_2(k)^2 \otimes_k k^2 \cong M_2(k)^4 \end{aligned}$$

by (ii). ■

Theorem 3.9 (i) *If $n = 4$, then*

$$\begin{aligned} &E \cong \mathbb{P}^3 \text{ or } \mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(3,4)\\ &\qquad \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2),\\ &E = \mathbb{P}(3,4) \cup \mathbb{P}(2,4) \cup \mathbb{P}(2,3) \cup \mathbb{P}(1,4) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2)\\ &\qquad \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^8). \end{aligned}$$

(ii) *If $n = 5$, then*

$$\begin{aligned} E \cong (5a), (5c), \text{ or } (5d) &\Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k),\\ E \cong (5b), (5e), \text{ or } (5f) &\Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^4),\\ E = (5g) &\Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^{16}), \end{aligned}$$

where
(5a) $\mathbb{P}^4$
(5b) $\mathbb{P}(1,2,3,5) \cup \mathbb{P}(1,2,3,4) \cup \mathbb{P}(4,5)$
(5c) $\mathbb{P}(1,2,3,4) \cup \mathbb{P}(3,4,5) \cup \mathbb{P}(1,2,5)$
(5d) $\mathbb{P}(3,4,5) \cup \mathbb{P}(1,4,5) \cup \mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(2,3,4)$
(5e) $\mathbb{P}(1,3,5) \cup \mathbb{P}(1,3,4) \cup \mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,4) \cup \mathbb{P}(4,5) \cup \mathbb{P}(2,3)$
(5f) $\mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(4,5) \cup \mathbb{P}(3,5) \cup \mathbb{P}(3,4)$
(5g) $\mathbb{P}(4,5) \cup \mathbb{P}(3,5) \cup \mathbb{P}(3,4) \cup \mathbb{P}(2,5) \cup \mathbb{P}(2,4) \cup \mathbb{P}(2,3) \cup \mathbb{P}(1,5) \cup \mathbb{P}(1,4) \cup \mathbb{P}(1,3) \cup \mathbb{P}(1,2)$.

Proof (i) By Lemma 3.5, there exists a graded (±1)-skew polynomial algebra S' such that $A \cong S'/(f)$ and the point scheme E' of S' is $\mathbb{P}^3, \mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(3,4)$, or $\bigcup_{1\le i<j\le 4} \mathbb{P}(i,j)$. (Note that $E \cong E'$.) By Theorem 3.3(ii), (iii), we only consider the case $E' = \mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(3,4)$. In this case,

$$\varepsilon_{41}\varepsilon_{12}\varepsilon_{24} = 1, \quad \varepsilon_{41}\varepsilon_{13}\varepsilon_{34} = -1, \quad \varepsilon_{42}\varepsilon_{23}\varepsilon_{34} = -1$$

(see (4iii) in the proof of Lemma 3.5), so $C(S'/(f))$ is isomorphic to

$$k\langle t_1, t_2, t_3\rangle/(t_1t_2 + t_2t_1, t_1t_3 - t_3t_1, t_2t_3 - t_3t_2, t_i^2 - 1) \cong M_2(k)^2$$

by Lemma 3.8(ii). Thus, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(S'/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2)$ by Theorem 2.1.

(ii) By Lemma 3.7, there exists a graded (±1)-skew polynomial algebra S' such that $A \cong S'/(f)$ and the point scheme E' of S' is (5a), ..., (5f), or (5g). By Theorem 3.3(i), (iii), we only consider the cases (5b) to (5f).

If E is (5b), then

$$\begin{aligned} &\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, && \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, && \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1, \\ &\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = 1, && \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, && \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1, \end{aligned}$$

(see (5v) in the proof of Lemma 3.7), so $C(S'/(f))$ is isomorphic to

$$\begin{aligned} k\langle t_1, t_2, t_3, t_4\rangle/(&t_1t_2 + t_2t_1, t_1t_3 + t_3t_1, \\ &t_1t_4 - t_4t_1, t_2t_3 + t_3t_2, t_2t_4 - t_4t_2, t_3t_4 - t_4t_3, t_i^2 - 1) \cong M_2(k)^4 \end{aligned}$$

by Lemma 3.8(iii). Thus, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(S'/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^4)$ by Theorem 2.1.

If E is (5c), then

$$\begin{aligned} &\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, && \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, && \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1, \\ &\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, && \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, && \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = 1, \end{aligned}$$

(see (5viii) in the proof of Lemma 3.7), so $C(S'/(f))$ is isomorphic to

$$\begin{aligned} k\langle t_1, t_2, t_3, t_4\rangle/(&t_1t_2 + t_2t_1, t_1t_3 - t_3t_1, t_1t_4 - t_4t_1, t_2t_3 - t_3t_2, \\ &t_2t_4 - t_4t_2, t_3t_4 + t_4t_3, t_i^2 - 1) \cong M_4(k) \end{aligned}$$

by Lemma 3.8(iv). Thus, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(S'/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k)$ by Theorem 2.1.

If E is (5d), then

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = 1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = 1,$$

(see (5vi) in the proof of Lemma 3.7), so $C(S'/(f))$ is isomorphic to

$$k\langle t_1, t_2, t_3, t_4\rangle/(t_1t_2 + t_2t_1, t_1t_3 - t_3t_1, t_1t_4 + t_4t_1, t_2t_3 - t_3t_2, t_2t_4 - t_4t_2, t_3t_4 + t_4t_3, t_i^2 - 1) \cong M_4(k)$$

by Lemma 3.8(v). Thus, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(S'/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k)$ by Theorem 2.1.

If E is (5e), then

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = 1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1,$$

(see (5ix) in the proof of Lemma 3.7), so $C(S'/(f))$ is isomorphic to

$$k\langle t_1, t_2, t_3, t_4\rangle/(t_1t_2 + t_2t_1, t_1t_3 + t_3t_1, t_1t_4 - t_4t_1, t_2t_3 - t_3t_2, t_2t_4 - t_4t_2, t_3t_4 - t_4t_3, t_i^2 - 1) \cong M_2(k)^4$$

by Lemma 3.8(vi). Thus, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(S'/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^4)$ by Theorem 2.1.

If E is (5f), then

$$\varepsilon_{51}\varepsilon_{12}\varepsilon_{25} = 1, \quad \varepsilon_{51}\varepsilon_{13}\varepsilon_{35} = -1, \quad \varepsilon_{51}\varepsilon_{14}\varepsilon_{45} = -1,$$
$$\varepsilon_{52}\varepsilon_{23}\varepsilon_{35} = -1, \quad \varepsilon_{52}\varepsilon_{24}\varepsilon_{45} = -1, \quad \varepsilon_{53}\varepsilon_{34}\varepsilon_{45} = -1,$$

(see (5x) in the proof of Lemma 3.7), so $C(S'/(f))$ is isomorphic to

$$k\langle t_1, t_2, t_3, t_4\rangle/\big(t_1t_2 + t_2t_1, t_1t_3 - t_3t_1, t_1t_4 - t_4t_1, t_2t_3 - t_3t_2, t_2t_4 - t_4t_2, t_3t_4 - t_4t_3, t_i^2 - 1\big) \cong M_2(k)^4$$

by Lemma 3.8(vii). Thus, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(S'/(f)) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^4)$ by Theorem 2.1. ∎

Let ℓ denote the number of irreducible components of E that are isomorphic to $\mathbb{P}^1$, that is, the number of irreducible components of the form $\mathbb{P}(i, j)$. Corollary 3.4 and Theorem 3.9 imply the following result, which states that Conjecture 1.3 is true for $n \leq 5$.

Theorem 3.10 *Assume that $n \leq 5$.*

(i) *If n is odd, then $\ell \leq 10$ and*

$$\ell = 0 \iff \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k),$$
$$0 < \ell \leq 3 \iff \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^4),$$
$$3 < \ell \leq 10 \iff \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^{16}).$$

(ii) *If n is even, then $\ell \le 6$ and*

$$0 \le \ell \le 1 \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2),$$
$$1 < \ell \le 6 \Longleftrightarrow \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^8).$$

At the end of paper, we collect some examples when $n = 6$ as further evidence for Conjecture 1.3.

Example 3.11 (i) Let $S = k\langle x_1, \dots, x_6\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ with

$$\begin{array}{lllll} \varepsilon_{12} = 1, & \varepsilon_{13} = -1, & \varepsilon_{14} = 1, & \varepsilon_{15} = -1, & \varepsilon_{16} = 1, \\ \varepsilon_{23} = -1, & \varepsilon_{24} = -1, & \varepsilon_{25} = -1, & \varepsilon_{26} = 1, & \varepsilon_{34} = 1, \\ \varepsilon_{35} = -1, & \varepsilon_{36} = 1, & \varepsilon_{45} = -1, & \varepsilon_{46} = 1, & \varepsilon_{56} = 1. \end{array}$$

Then the point scheme of S is $\mathbb{P}(3,4,5) \cup \mathbb{P}(2,3,4) \cup \mathbb{P}(1,4,5) \cup \mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(3,4,6) \cup \mathbb{P}(1,4,6) \cup \mathbb{P}(1,2,6) \cup \mathbb{P}(5,6)$, so $\ell = 1$. On the other hand, one can check that $C(A) \cong M_4(k)^2$, so we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2)$.

(ii) Let $S = k\langle x_1, \dots, x_6\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ with

$$\begin{array}{lllll} \varepsilon_{12} = 1, & \varepsilon_{13} = -1, & \varepsilon_{14} = -1, & \varepsilon_{15} = -1, & \varepsilon_{16} = 1, \\ \varepsilon_{23} = 1, & \varepsilon_{24} = -1, & \varepsilon_{25} = -1, & \varepsilon_{26} = 1, & \varepsilon_{34} = -1, \\ \varepsilon_{35} = -1, & \varepsilon_{36} = 1, & \varepsilon_{45} = 1, & \varepsilon_{46} = 1, & \varepsilon_{56} = 1. \end{array}$$

Then the point scheme of S is $\mathbb{P}(2,3,4,5) \cup \mathbb{P}(1,2,4,5) \cup \mathbb{P}(2,3,6) \cup \mathbb{P}(1,2,6) \cup \mathbb{P}(4,5,6) \cup \mathbb{P}(1,3)$, so $\ell = 1$. On the other hand, one can check that $C(A) \cong M_4(k)^2$, so we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^2)$.

(iii) Let $S = k\langle x_1, \dots, x_6\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ with

$$\begin{array}{lllll} \varepsilon_{12} = 1, & \varepsilon_{13} = -1, & \varepsilon_{14} = -1, & \varepsilon_{15} = -1, & \varepsilon_{16} = 1, \\ \varepsilon_{23} = 1, & \varepsilon_{24} = -1, & \varepsilon_{25} = -1, & \varepsilon_{26} = 1, & \varepsilon_{34} = -1, \\ \varepsilon_{35} = -1, & \varepsilon_{36} = 1, & \varepsilon_{45} = -1, & \varepsilon_{46} = 1, & \varepsilon_{56} = 1. \end{array}$$

Then the point scheme of S is $\mathbb{P}(2,3,5) \cup \mathbb{P}(2,3,4) \cup \mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,6) \cup \mathbb{P}(2,3,6) \cup \mathbb{P}(4,5) \cup \mathbb{P}(1,3) \cup \mathbb{P}(4,6) \cup \mathbb{P}(5,6)$, so $\ell = 4$. On the other hand, one can check that $C(A) \cong M_2(k)^8$, so we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^8)$.

(iv) Let $S = k\langle x_1, \dots, x_6\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ with

$$\begin{array}{lllll} \varepsilon_{12} = 1, & \varepsilon_{13} = -1, & \varepsilon_{14} = -1, & \varepsilon_{15} = -1, & \varepsilon_{16} = 1, \\ \varepsilon_{23} = -1, & \varepsilon_{24} = -1, & \varepsilon_{25} = -1, & \varepsilon_{26} = 1, & \varepsilon_{34} = -1, \\ \varepsilon_{35} = -1, & \varepsilon_{36} = 1, & \varepsilon_{45} = -1, & \varepsilon_{46} = 1, & \varepsilon_{56} = 1. \end{array}$$

Then the point scheme of S is $\mathbb{P}(1,2,5) \cup \mathbb{P}(1,2,4) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(1,2,6) \cup \mathbb{P}(4,5) \cup \mathbb{P}(3,5) \cup \mathbb{P}(3,4) \cup \mathbb{P}(4,6) \cup \mathbb{P}(3,6) \cup \mathbb{P}(5,6)$, so $\ell = 6$. On the other hand, one can check that $C(A) \cong M_2(k)^8$, so we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} k^8)$.

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