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ORTHOGONAL COMPLETIONS OF REDUCED RINGS WITH RESPECT TO ABIAN ORDER

BY

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ABSTRACT. In this paper, it is proved that a reduced ring R has an orthogonal completion if and only if for every idempotent $e \in R$, eR has an orthogonal completion. Every orthogonal subset X of R has a supremum in $Q\max(R)$, the maximal two sided ring of quotients of R, and the orthogonal completion of a reduced ring R, if it exists, is isomorphic to a unique subring of $Q\max(R)$. Hence the orthogonal completion of a reduced ring R, if it exists, is unique upto isomorphism. A reduced ring R has an orthogonal completion if and only if the collection of those elements of $Q\max(R)$ which are supremums of orthogonal subsets of R form a subring of $Q\max(R)$. Furthermore, every projectable ring R has an orthogonal completion $\hat{R}$, which is a Baer ring. It is also proved that $\hat{R} = Q_{\mathcal{F}}(R)$ for projectable rings R, where $\mathcal{F}$ is the idempotent filter of those dense right ideals of R which contain a maximal orthogonal subset of idempotents of R.

Abian [4] showed that the relation "$\leq$" on R, defined by writing $a \leq b$ if $ab = a^2$, is a partial order on a reduced ring R and it makes R a partially ordered multiplicative semigroup. Reduced rings, under this relation "$\leq$" were studied by Abian [2], [3] and Chacron [8] to characterise the direct product of integral domains, division rings and fields. This involved the concept of supremums of orthogonal subsets and orthogonal completions of reduced rings R. Concepts of orthogonal completions and orthogonal completeness on their own merit were studied by Burgess, Raphael and Stephenson ([6], [7], [15]). In their study, they were mainly concerned with characterising reduced rings which are either orthogonally complete or have an orthogonal completion. This paper may be considered as a continuation of their work.

In what follows, all rings referred to will have 1, the identity element. A ring R is called a *reduced ring* if R has no nonzero nilpotent element. Such rings were studied by Abian [1], Andrunakievic and Rjabuhin [5] and some of their findings are quoted here for convenience. All idempotents of R are central and for any subset X of R, right and left annihilators of X in R coincide. Furthermore, R has a collection Π of minimal *completely prime* ideals P (for every $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$) of R such that $\cap\{P : P \in \Pi\} = \{0\}$. Thus every reduced ring is a subdirect product of rings without zero divisors.

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For $X \subseteq R$, we define the *support* of X in Π ($\mathrm{supp}_\Pi X$ for short) by $\mathrm{supp}_\Pi X = \{P \in \Pi : X \not\subseteq P\}$. It is obvious from this definition $\mathrm{supp}_\Pi(ab) = \mathrm{supp}_\Pi(a) \cap \mathrm{supp}_\Pi(b)$ and $ab = 0$ if and only if $\mathrm{supp}_\Pi(a) \cap \mathrm{supp}_\Pi(b) = \varnothing$. In what follows, *$R$ will always denote a reduced ring.*

A subset X of R is called an *orthogonal subset* of R if for every $x, y \in X$, $x \neq y$ implies $xy = 0$.

Let $X \subseteq R$ be an orthogonal set. An element $a \in R$ is called an *upper bound* of X in R if $x \leq a$ for every $x \in X$. An upper bound a of X in R is called a *supremum* of X in R if for every upper bound b of X in R, $a \leq b$. It is obvious from this definition that a supremum of an orthogonal subset of R, if it exists, is unique. We shall denote the supremum of an orthogonal subset X in R by $\sup_R X$. The following proposition which is quoted here for convenience, provides another way of looking at this concept.

PROPOSITION 1. *Let X be an orthogonal subset of R. An element $a \in R$ is the supremum of X if and only if a is an upper bound of X with respect to "$\leq$" and* $\mathrm{ann}_R X \subseteq \mathrm{ann}_R(a)$. (*in fact we have* $\mathrm{ann}_R X = \mathrm{ann}_R(a)$).

Proof. For a proof Raphael and Stephenson [15].

It can be easily verified that the supremum of any orthogonal subset of idempotents of R, if it exists, is an idempotent of R and vice-versa. We shall make use of the following lemma at a later stage.

LEMMA 2. *Let $X = \{x_i : i \in I\}$, $Y = \{y_j : j \in J\}$ be two orthogonal subsets of R such that $\sup_R X = x$ and $\sup_R Y = y$. Let $Z = \{x_i y_j : i \in I, j \in J\}$. Then Z is an orthogonal subset of R and $\sup_R Z = xy$.*

Proof. It is obvious that Z is an orthogonal subset of R. Since $x_i \leq x$ and $y_j \leq y$, $x_i y_j \leq xy$ for every $i \in I$ and $j \in J$. Hence xy is an upper bound of Z. Further, if $rx_i y_j = 0$ for every $i \in I$ and $j \in J$, then by the proposition above, $rx_i \in \mathrm{ann}_R Y \subseteq \mathrm{ann}_R(y)$. Hence $rx_i y = 0$ for every $i \in I$. Since R is reduced, this implies that $ry \in \mathrm{ann}_R X \subseteq \mathrm{ann}_R(x)$. Therefore, $(ry)x = 0 = r(xy)$. Thus $\mathrm{ann}_R Z \subseteq \mathrm{ann}_R(xy)$. Hence by the proposition above, $xy = \sup_R Z$ and this completes the proof.

COROLLARY 3 (*Abian* [4]). *Let $X \subseteq R$ be an orthogonal subset of R and $a \in R$ be the supremum of X. Then for every $r \in R$, $\sup_R rX = ra$ where $rX = \{rx : x \in X\}$.*

Proof. It follows immediately from the lemma above.

A reduced ring R is said to be *orthogonally complete* if every orthogonal subset of R has a supremum in R. For reduced rings R, $\hat{R}$ such that $R \subseteq \hat{R}$, we say that $\hat{R}$ is an *orthogonal extension* of R if every element of $\hat{R}$ is the supremum of an orthogonal subset of R with respect to the order relation "$\leq$" on $\hat{R}$. An orthogonal extension $\hat{R}$ is said to be an *orthogonal completion* of R if

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$\hat{R}$ is orthogonally complete. The following proposition is a generalisation of a result proved by Burgess and Raphael [6] for the class of reduced rings in which $aR \cap bR = \{0\}$ implies $ab = 0$.

PROPOSITION 4. *Let R, S, T be reduced rings such that S is an orthogonal extension of R and T is an orthogonal extension of S. Then T is an orthogonal extension of R.*

Proof. Let $t \in T$. Then since T is an orthogonal extension of S, there exists an orthogonal subset $X \subseteq S$ such that $t = \sup_T X$. Now for each $x \in X$, there exists an orthogonal subset $Y_x \subseteq R$ such that $x = \sup_R Y_x$. Now consider $Z = \cup\{Y_x : x \in X\}$. Let $a, b \in Z$, $a \neq b$. If there exists an $x \in X$ such that $a, b \in Y_x$ then $ab = 0$. Hence assume $a \in Y_x$, $b \in Y_{x'}$, and $x \neq x'$. Then $a \leq x$, $b \leq x'$ and hence $ab \leq xx' = 0$. Hence Z is an orthogonal subset of R.

Now, if $a \in Z = \cup\{Y_x : x \in X\}$, then for some $x \in X$, $a \in Y_x$ and consequently $a \leq x \leq t$. Hence t is an upper bound of Z. Further, if $y \in \operatorname{ann}_T Z$ then $y \in \operatorname{ann}_T X$ and hence $y \in \operatorname{ann}_T(t)$. It now follows from proposition 1 that $t = \sup_T Z$ and this completes the proof.

LEMMA 5. *Let R be a reduced ring and $\Pi = \{P_i : i \in I\}$ a collection of minimal completely prime ideals of R such that $\cap\{P_i : i \in I\} = \{0\}$. Let $\{a_j : j \in J\}$, $\{b_j : j \in J\}$ be orthogonal subsets of R such that $\sup_R\{a_j : j \in J\} = a$ and $\sup_R\{b_j : j \in J\} = b$. Suppose there is a disjoint collection $\{\Pi_j : j \in J\}$ of subsets of Π such that $\operatorname{supp}_\Pi(a_j) \subseteq \Pi_j$, $\operatorname{supp}_\Pi(b_j) \subseteq \Pi_j$ for every $j \in J$. Then $\sup_R\{(a_j + b_j) : j \in J\} = a + b$.*

Proof. Clearly $\{(a_j + b_j) : j \in J\}$ is an orthogonal subset of R. Since $\operatorname{supp}_\Pi(a_j) \subseteq \Pi_j$, $\operatorname{supp}_\Pi(b_j) \subseteq \Pi_j$ for every $j \in J$, if $j \neq j'$ then $\operatorname{supp}_\Pi(a_j b_{j'}) = \operatorname{supp}_\Pi(a_j) \cap \operatorname{supp}_\Pi(b_{j'}) \subseteq \Pi_j \cap \Pi_{j'} = \varnothing$. Hence $a_j b_{j'} = 0$ for $j \neq j'$. Thus, for every $j_0 \in J$,

$$a_{j_0} b = a_{j_0} \sup_{j} {}_R\, b_j = \sup_{j} {}_R\, a_{j_0} b_j = a_{j_0} b_{j_0}.$$

Similarly, $b_{j_0} a = b_{j_0} a_{j_0}$ for every $j_0 \in J$. Hence

$$\begin{aligned}(a_j + b_j)(a + b) &= a_j^2 + b_j^2 + a_j b + b_j a \\ &= a_j^2 + b_j^2 + a_j b_j + b_j a_j \\ &= (a_j + b_j)^2\end{aligned}$$

and therefore $a_j + b_j \leq a + b$ for every $j \in J$. Thus, $a + b$ is an upper bound of the orthogonal subset $\{a_j + b_j : j \in J\}$ of R.

Further, if $x \in \operatorname{ann}_R\{a_j + b_j : j \in J\}$ then $x(a_j + b_j) = 0$ and hence $xa_j = -xb_j$ for all $j \in J$. Therefore,

$$xa = x \sup_{j} {}_R\, a_j = \sup_{j} {}_R\, xa_j = \sup_{j} {}_R (-x) b_j = -x \sup_{j} {}_R\, b_j = -xb;$$

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that is, $x(a+b)=0$. Thus $\operatorname{ann}_R\{a_j+b_j : j\in J\}\subseteq \operatorname{ann}_R(a+b)$ and hence by proposition 1., we have $\sup_R\{a_j+b_j : j\in J\}=a+b$, as was to be proved.

Corollary 6. Let $\{a_j : j\in J\}, \{b_j : j\in J\}$ be two orthogonal subsets of R such that $\sup_{R\,j} a_j = a$ and $\sup_{R\,j} b_j = b$. Suppose there exists an orthogonal subset $\{x_j : j\in J\}$ such that $a_j = x_j r_j$ and $b_j = x_j s_j$ for every $j\in J$, where $r_j, s_j\in R$. Then $\sup_R\{(a_j+b_j) : j\in J\}=a+b$.

Proof. $\{(a_j+b_j) : j\in J\}$ is obviously an orthogonal subset of R. Let Π be a collection of minimal completely prime ideals of R such that $\bigcap_{P\in\Pi} P=\{0\}$. Let $\Pi_j=\{P\in\Pi : x_j\notin P\}$ for every $j\in J$. Then $\Pi_j=\operatorname{supp}_\Pi(x_j)$ and since $\{x_j : j\in J\}$ is an orthogonal subset of R, $\Pi_j\cap\Pi_{j'}=\varnothing$ for $j\neq j'$. Since $a_j=x_jr_j$, $b_j=x_js_j$ for every $j\in J$, $\operatorname{supp}_\Pi(a_j)\subseteq\Pi_j$ and $\operatorname{supp}_\Pi(b_j)\subseteq\Pi_j$. Hence by lemma 5., $\sup_{R\,j}(a_j+b_j)=a+b$. This proves the corollary.

Lemma 5 and its corollary 6 give us a sufficient condition under which the supremum of the "sum" of two orthogonal subsets of R is the sum of their supremums. We use these two results to prove the following important result.

Proposition 7. *Let R be a reduced ring. Then R has an orthogonal completion $\hat{R}$ if and only if for every nonzero idempotent e of R, eR has an orthogonal completion. In this case, $e\hat{R}$ is an orthogonal completion of eR. Further, whenever these completions exist, $\hat{R}=\prod_i e_i\hat{R}$, where $\{e_i : i\in I\}$ is any maximal orthogonal subset of idempotents of R.*

Proof. Suppose R has an orthogonal completion $\hat{R}$ and e is an idempotent of R. Let $\{er_j : j\in J\}$ be an orthogonal subset of eR. Then $\{er_j : j\in J\}$ is an orthogonal subset of R and hence there exists an $r\in\hat{R}$ such that $\sup_{\hat{R}}\{er_j : j\in J\}=r$. Since

$$\begin{aligned} er &= e\cdot \sup_{\hat{R}}\{er_j : j\in J\}=\sup_{\hat{R}}\{e^2r_j : j\in J\}\\ &=\sup_{\hat{R}}\{er_j : j\in J\}=r, \end{aligned}$$

$r\in e\hat{R}$. Hence every orthogonal subset of eR has a supremum in $e\hat{R}$. But, since every element of $e\hat{R}$ is the supremum of an orthogonal subset of eR, we see that $e\hat{R}$ is an orthogonal completion of eR.

The converse is trivial since $1\in R$.

Now consider the diagram:

$$\begin{tikzcd} \hat{R} \arrow[r, "p_i"] & e_i\hat{R} \\ S \arrow[u, dashed, "f"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

where $p_i : \hat{R} \to e_i\hat{R}$ is given by $p_i(r) = e_ir$, S is any other ring and $f_i : S \to e_i\hat{R}$ are homomorphisms for all $i \in I$. Define $f : S \to \hat{R}$ by $f(s) = \sup_{\hat{R}} f_i(s)$ (sup over i). Since for every element s of S and for every $i \in I$, $e_if_i(s) = f_i(s)$, we have,

$$\begin{aligned} f(s+t) &= \sup_{\hat{R}}\{f_i(s+t) : i \in I\} \\ &= \sup_{\hat{R}}\{f_i(s)+f_i(t) : i \in I\} \\ &= \sup_{\hat{R}}\{e_if_i(s)+e_if_i(t) : i \in I\} \\ &= \sup_{\hat{R}}\{f_i(s) : i \in I\} \\ &\quad + \sup_{\hat{R}}\{f_i(t) : i \in I\} \text{ (by } \textit{coro. } 6.) \\ &= f(s)+f(t). \end{aligned}$$

Also,

$$\begin{aligned} f(st) &= \sup_{\hat{R}}\{f_i(st) : i \in I\} = \sup_{\hat{R}}\{f_i(s)f_i(t) : i \in I\} \\ &= f(s)f(t) \text{ (by } \textit{lemma } 2.) \end{aligned}$$

Thus $f : s \to \hat{R}$ is a ring homomorphism. Further, since for every $s \in S$,

$$\begin{aligned} p_i(f(s)) &= p_i(\sup_{\hat{R}}\{f_j(s) : j \in I\}) = e_i(\sup_{\hat{R}}\{f_j(s) : j \in I\}) \\ &= \sup_{\hat{R}}\{e_if_j(s) : j \in I\} = e_if_i(s) = f_i(s), \end{aligned}$$

the above diagram commutes.

Now, since $\{e_i : i \in I\}$ is a maximal orthogonal subset of idempotents of R, $\sup_{\hat{R}} e_i = 1$. For, let $\sup_{\hat{R}} e_i = e$. Then $e \in \hat{R}$ and $e^2 = e$. Consider now $1-e \in \hat{R}$. It is an idempotent of $\hat{R}$, the orthogonal completion of R. Hence, there exists an orthogonal subset $\{f_j : j \in J\}$ of idempotents of R such that $1-e = \sup_{\hat{R}}\{f_j : j \in J\}$. If $1-e \neq 0$ then $f_{j_0} \neq 0$ for some $j_0 \in J$ and hence

$$e_if_{j_0} \leq e_i(1-e) = e_i - e_i^2 = 0.$$

Thus, $\{e_i : i \in J\} \cup \{f_{j_0}\}$ is an orthogonal subset of idempotents of R properly containing $\{e_i : i \in I\}$. But this contradicts the maximality of the orthogonal set $\{e_i : i \in I\}$ and hence we have $\sup_{\hat{R}} e_i = 1$.

We use this fact to prove that the map $f : S \to \hat{R}$ which makes the above diagram commute is unique. Let $f' : s \to \hat{R}$ be another map which makes the above diagram commute. Then for every $s \in S$ and $i \in I$,

$$p_i(f(s)-f'(s)) = e_i(f(s)-f'(s)) = 0.$$

Hence

$$\begin{aligned} f(s)-f'(s) \in \operatorname{ann}_{\hat{R}}\{e_i : i \in I\} &= \operatorname{ann}_{\hat{R}}(\sup_{R} e_i) \\ &= \operatorname{ann}_{\hat{R}}(1) = 0 \end{aligned}$$

and therefore $f(s) = f'(s)$ for every $s \in S$. Thus f is unique and hence $\hat{R} = \prod_i e_i\hat{R}$,

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where $\{e_i : i \in I\}$ is a maximal orthogonal subset of idempotents of R. This completes the proof.

As a corollary to this we have the following result.

COROLLARY 8. *Let $R = \prod_{i \in I} R_i$ where each R_i is reduced. Then R is orthogonally complete if and only if each R_i is orthogonally complete.*

Proof. Suppose R is orthogonally complete. Then in the notation of the theorem above, we may take $\hat{R} = R$. Let e_i denote the identity element of R_i. Then by proposition 7., $e_i R = R_i$ has an orthogonal completion $e_i \hat{R}$. But, since $e_i \hat{R} = e_i R = R_i$, each R_i is orthogonally complete.

Conversely, suppose each R_i is orthogonally complete and let $\{a_j : j \in J\}$ be an orthogonal subset of R. Then $\{\pi_i(a_j) : j \in J\}$ is an orthogonal subset of R_i where $\pi_i : R \to R_i$ is the canonical projection of R onto R_i. Let $a_i \in R$ be such that $a_i = \sup_{R_i, j} \pi_i(a_j)$. Since each R_i is orthogonally complete, such an $a_i \in R_i$ exists for every $i \in I$. Now consider the element $a \in R$ which is such that $\pi_i(a) = a_i$ for every $i \in I$. Since,

$$\pi_i(aa_j) = a_i \pi_i(a_j) = \pi_i(a_j)^2$$

for every $i \in I$, $aa_j = a^2$ for every $j \in J$. Hence $a_j \le a$ for every $j \in J$. Also,

$$\begin{aligned} r \in \operatorname{ann}_R\{a_j : j \in J\} &\Rightarrow \pi_i(r) \in \operatorname{ann}_{R_i}\{\pi_i(a_j) : j \in J\} \text{ for all } i \in I \\ &\Rightarrow \pi_i(r) \in \operatorname{ann}_{R_i}(a_i) \text{ for all } i \in I \\ &\Rightarrow \pi_i(r) \in \operatorname{ann}_{R_i}(a) \text{ for all } i \in I \\ &\Rightarrow r \in \operatorname{ann}_R(a). \end{aligned}$$

Hence $\operatorname{ann}_R\{a_j : j \in J\} \subseteq \operatorname{ann}_R(a)$. Thus $a = \sup_{R, j} a_j$ and therefore R is orthogonally complete.

Now we shall need some information about rings of quotients of reduced rings R. For this, good references are Faith [10], Lambek [13], [14], Steinberg [17] and Stenström [18]. In a reduced ring R a right (resp. left) ideal D is dense if and only if it is an essential right (resp. left) ideal of R. A two sided ideal of R is dense as a right ideal of R if its annihilator in R is 0. A ring $S \supseteq R$ is said to be a *ring of right quotients* of R if for every $s \in S$, $s^{-1}R = \{r \in R : sr \in R\}$ is a dense right ideal of R and $s(s^{-1}R) \neq \{0\}$. Similarly one defines *ring of left quotients* of R. We shall denote the right (resp. left) quotient ring of R with respect to the idempotent filter of all dense right (resp. left) ideals of R by $Q\,max{-}r(R)$ (resp. $Q\,max{-}1(R)$). It is proved in Lambek [13] (proposition 8, page 99) that for every ring of right quotients S of R there exists a ring monomorphism $f : S \to Q\,\max{-}r(R)$ such that $f \mid R$ is the identity map on R. Hence every ring of right quotients may be regarded as a subring of

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Q max$-r(R)$. Similar result holds good when the word "right" in the above is replaced with the word "left". Let $Q\,max(R)=\{q\in Q\,max{-}r(R): Dq\subseteq R$ for some dense left ideal D of $R\}$. Then obviously $R\subseteq Q\max(R)$ and it is proven by Wong and Johnson [20] that $Q\max(R)$ is a subring of Q max$-r(R)$ and it is the unique (upto isomorphism over R) *maximal ring of two sided quotients* of R. Also, for every reduced ring R, $Q\max(R)$ is reduced (see Steinberg [17]).

It is well known that if S is a ring of right (resp. left) quotients of R and T is a ring of right (resp. left) quotients of S then T is a ring of right (resp. left) quotients of R (see for instance Lambek [13], exercise 7, page 100). It follows from this that $Q\max(Q\max(R))$ is a two sided ring of quotients of R. Since, $Q\max(R)$ is the maximal ring of two sided quotients of R, we have $Q\max(Q\max(R)\subseteq Q\max(R)$. Obviously $Q\max(R)\subseteq Q\max(Q\max(R))$. Hence we have $Q\max(R)=Q\max(Q\max(R))$ for every reduced ring R. We now prove the following result.

PROPOSITION 9. *Let R be a reduced ring and $X=\{x_i : i\in I\}$ an orthogonal subset of R. Then there exists an $x\in Q\max(R)$ such that $x=\sup_{Q\max(R)}X$.*

Proof. Embed $X=\{x_i : i\in I\}$ into a maximal orthogonal subset $X=\{x_i : i\in J\}$ of R. Let D denote the 2-sided ideal of R generated by $\{x_i : i\in J\}$. Then $\operatorname{ann}_R D=\{0\}$ and hence D is a dense right and dense left ideal of R.

Every element of D is of the form $\sum_{k,i} a_{ki}x_ib_{ki}$ where $a_{ki}, b_{ki}\in R$ and $x_i\in X$. Define $x\in \operatorname{Hom}_R(D_R, R_R)$ by

$$\begin{aligned} x(a_{ki}x_ib_{ki}) &= x_i(a_{ki}x_i)b_{ki} \text{ if } i\in I\\ &= 0 \text{ if } i\in J\setminus I\end{aligned}$$

and extend it over sums in D_R by linearity. Then (by Lambek [13], cor. 3, page 97) x determines an element of Q max$-r(R)$ (which we shall also denote by x) which is such that $xx_i=x_i^2$ for every $i\in I$ and $xx_i=0$ for every $i\in J\setminus I$.

Now, consider $s_{i_0}x_{i_0}r_{i_0}x-s_{i_0}x_{i_0}r_{i_0}x_{i_0}\in Q$ max$-r(R)$, where $i_0\in I$ and $s_{i_0}, r_{i_0}\in R$. By the definition of x,

$$\begin{aligned}(s_{i_0}x_{i_0}r_{i_0}x-s_{i_0}x_{i_0}r_{i_0}x_{i_0})(a_ix_ib_i) &= (s_{i_0}x_{i_0}r_{i_0}x)(a_ix_ib_i)-(s_{i_0}x_{i_0}r_{i_0})(a_ix_ib_i)\\ &= s_{i_0}x_{i_0}r_{i_0}x_ia_ix_ib_i-s_{i_0}x_{i_0}r_{i_0}x_{i_0}a_ix_ib_i\\ &=0.\end{aligned}$$

Hence $(s_{i_0}x_{i_0}r_{i_0}x-s_{i_0}x_{i_0}r_{i_0}x_{i_0})D=0$. Thus $s_{i_0}x_{i_0}r_{i_0}x-s_{i_0}x_{i_0}r_{i_0}x_{i_0}$ annihilates a dense right ideal of R. Hence we have, $s_{i_0}x_{i_0}r_{i_0}x=s_{i_0}x_{i_0}r_{i_0}x_{i_0}\in R$ for every $i_0\in I$. If $i_0\in J\setminus I$, then

$$\begin{aligned} s_{i_0}x_{i_0}r_{i_0}x(a_ix_ib_i) &= s_{i_0}x_{i_0}r_{i_0}x_ia_ix_ib_i=0 \text{ if } i\in I\\ &=0 \text{ if } i\in J\setminus I \text{ by the definition of } x.\end{aligned}$$

So $s_{i_0}x_{i_0}r_{i_0}x=0$ for every $i_0\in J\setminus I$. Hence $Dx\subseteq R$. Since $x\in Q$ max$-r(R)$, it

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follows that $x \in Q\max(R)$ and for every $i \in I$, $x_i x = x_i^2$. Therefore, $x_i \leq x$ for every $i \in I$.

Let $q \in Q\max(R)$ be such that $qx_i = 0$ for every $i \in I$. Consider $qx \in Q\max(R)$. It annihilates the dense ideal D of R. So $qx = 0$ and hence $q \in \operatorname{ann}_{Q\max(R)}(x)$. Thus, $\operatorname{ann}_{Q\max(R)} X \subseteq \operatorname{ann}_{Q\max(R)}(x)$. Therefore, by proposition 1, $x = \sup_{Q\max(R)} X$ and this completes the proof.

COROLLARY 10. *Let R be a reduced ring. Then $Q\max(R)$ is orthogonally complete.*

Proof. Let $X = \{x_i : i \in I\}$ be an orthogonal subset of $Q\max(R)$. Then by the above proposition, there exists an $x \in Q\max(Q\max(R))$ such that $x = \sup_{Q\max(Q\max(R))} X$. But since $Q\max(Q\max(R)) = Q\max(R)$, $x \in Q\max(R)$ and hence $Q\max(R)$ is orthogonally complete.

Burgess and Raphael [6] raised the question which forms the content of this corollary. They were able to prove it for reduced rings R in which $aR \cap bR = \{0\}$ implies $ab = 0$ for every $a, b \in R$. The following result is due to them.

PROPOSITION 11. *Let X be an orthogonal subset of a reduced ring R such that $\sup_R X = a$. Then $\sup_{Q\max(R)} X = a$.*

Proof. For a proof, see Burgess and Raphael [6].

PROPOSITION 12. *Let R be a reduced ring which admits an orthogonal extension $\hat{R}$. Then $\hat{R}$ is isomorphic (over R) to a subring of $Q\max(R)$.*

Proof. Let $\hat{R}$ be an orthogonal extension of R and suppose $x \in \hat{R}$. Then there exists an orthogonal subset $X = x_i : \{i \in I\} \subseteq R$ such that $x = \sup_R x_i$. Consider $D = \{r \in R : xr \in R\}$. Then D is a right ideal of R and $X \subseteq D$. Hence $D \neq \{0\}$. Let $0 \neq a \in R$ and consider $D \cap aR$. If a annihilates each x_i then $aR \subseteq D$. Hence $aR \cap D \neq \{0\}$. So suppose there exists an $i \in I$ such that $ax_i \neq 0$. Then

$$(xax_i - x_iax_i)^2 = xax_i^2ax_i + x_iax_i^2ax_i - xax_i^2ax_i - x_iax_i^2ax_i$$
$$= 0,$$

which implies that $xax_i = x_iax_i$. Hence $ax_i \in D$ and therefore $D \cap aR \neq \{0\}$. Thus, D is a dense right ideal of R and $xD \subseteq R$. Similarly, one can prove that there exists a dense left ideal D' of R such that $D'x \subseteq R$. It follows from this that R is a two sided ring of quotients of R. Hence every orthogonal extension of R is isomorphic (over R) to a subring of $Q\max(R)$ (see Wong and Johnson [20], proposition 8.).

We now prove that orthogonal completions of reduced rings are unique upto isomorphism.

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PROPOSITION 13. *Let R be a reduced ring which admits an orthogonal completion $\hat{R}$. Then $\hat{R}$ is isomorphic over R to a unique subring of $Q\max(R)$.*

Proof. Suppose that $\hat{R}$ and $\hat{R}'$ are two orthogonal completions of R. Then by the proposition above, there exist monomorphisms $f:\hat{R}\to Q\max(R)$, $g:\hat{R}'\to Q\max(R)$ which are inclusion maps when restricted to R. We show that $f(\hat{R})=g(\hat{R}')$.

Let $f(x)\in f(\hat{R})$ where $x\in\hat{R}$. Let $X=\{x_i : i\in I\}$ be an orthogonal subset of R such that $x=\sup_{\hat{R}} X$. Then for every $i\in I$,

$$f(x)x_i=f(x)f(x_i)=f(xx_i)=f(x_i^2)=f(x_i)^2=x_i^2$$

and hence $f(x)$ is an upper bound of X in $Q\max(R)$. If $q\in\operatorname{ann}_{Q\max(R)} X$ then $qx_i=0$ for every $i\in I$. Now consider $qf(x)\in Q\max(R)$. Let K be the ideal generated by X in R and $D=K+\operatorname{ann}_R R$. Then since $\operatorname{ann}_R D=\{0\}$, D is a dense ideal of R and since $qx_i=0$ for every $i\in I$, $qK=\{0\}$. On the other hand if $y\in\operatorname{ann}_R K$, then

$$f(x)y=f(xy)=f(0)=0$$

because $x=\sup_R X$ and $\operatorname{ann}_R(x)=\operatorname{ann}_R X=\operatorname{ann}_R K$. Hence $(qf(x))D=0$. Since D is a dense right ideal of R, this implies that $qf(x)=0$. Thus $\operatorname{ann}_{Q\max(R)} X\subseteq\operatorname{ann}_{Q\max(R)} f(x)$ and hence $f(x)=\sup_{Q\max(R)} X$.

Similarly, if $x'=\sup_{\hat{R}'} X$, then it can be shown that $g(x')=\sup_{Q\max(R)} X$. Since supremums are unique in any reduced ring, we find that $f(\hat{R})\subseteq g(\hat{R}')$. By symmetry, $g(\hat{R}')\subseteq f(\hat{R})$. Hence $g(\hat{R}')=f(\hat{R})$.

It follows from this that orthogonal completions, whenever they exist, are unique upto isomorphism and we from now on consider the orthogonal completion $\hat{R}$ of R, if it exists, as a subring of $Q\max(R)$.

In view of propositions 9, 11, 12, 13, to make our notations less cumbersome, we shall denote the supremum of an orthogonal subset X of R in $Q\max(R)$ by $\sup X$ instead of $\sup_{Q\max(R)} X$. In what follows, this is not likely to cause any confusion.

Every reduced ring R is a subring of a smallest orthogonally complete ring $R'=\cap\{S : S$ is an orthogonally complete subring of $Q\max(R)$ which contains $R\}$. However, not every element of R' need be the supremum of an orthogonal subset of R. If this is the case, then R has no orthogonal completion. This is clear from the following proposition.

PROPOSITION 14. *Let R be a reduced ring and S be the collection of those elements of $Q\max(R)$ which are supremums of orthogonal subsets of R. Then R has an orthogonal completion if and only if S is a subring of $Q\max(R)$.*

Proof. Suppose R has an orthogonal completion $\hat{R}$. Then by the proposition above, we can assume that $\hat{R}\subseteq Q\max(R)$. Since every element of R is the supremem of an orthogonal subset of R, $\hat{R} \subseteq S$. Similarly, $S \subseteq \hat{R}$. Hence $\hat{R} = S$ and therefore S is a subring of $Q\max(R)$.

Conversely, suppose S is a subring of $Q\max(R)$. Then S is an orthogonal extension of R. Let X be an orthogonal subset of S. Then for every $x \in X$, there is an orthogonal subset $Y_x \subseteq R$ such that $\sup Y_x = x$. Take $Z = \bigcup_{x \in X} Y_x$. Then as in the proof of proposition 4., one can easily check that Z is an orthogonal subset of R and $\sup Z = \sup X \in S$. Thus S is an orthogonally complete ring and so it is the orthogonal completion of R.

Corollary 15. *Let R and S be as above. Then S is closed under multiplication and hence S is a ring if and only if $x, y \in S$ implies $x + y \in S$.*

Proof. We prove that S is closed under multiplication. So the rest of the proof will follow from this. Let $a, b \in S$. Then there exist orthogonal subsets $\{a_i : i \in I\}, \{b_j : j \in J\}$ such that $a = \sup_i a_i$, $b = \sup_j b_j$. It follows now from lemma 2. that $\{a_ib_j : i \in I, j \in J\}$ is an orthogonal subset of R and $ab = \sup\{a_ib_j : i \in I, j \in J\}$. Hence $ab \in S$ proving that S is closed under multiplication. This completes the proof.

Thus, the only way a reduced ring R may fail to have an orthogonal completion is that the sum of supremums of two orthogonal subsets of R may not be a supremum of any orthogonal subset of R. We describe below a class of rings which always have an orthogonal completion.

A reduced ring R is called *projectable* if for every $a \in R$, $\mathrm{ann}_R(a) + \mathrm{ann}_R(\mathrm{ann}_R(a) = R$ (Keimel [11]). It follows immediately from this definition that every reduced *Baer ring* (a ring in which every annihilator is generated by an idempotent) and every regular ring is projectable.

In a reduced ring $R, \mathrm{ann}_R(a) \cap \mathrm{ann}_R(\mathrm{ann}_R(a)) = \{0\}$ for every $a \in R$. Hence the sum $\mathrm{ann}_R(a) + \mathrm{ann}_R(\mathrm{ann}_R(a))$ is direct. If R is projectable, then there are $e \in \mathrm{ann}_R(a)$ and $e' \in \mathrm{ann}_R(\mathrm{ann}_R(a))$ such that $e + e' = 1$. Hence we have $e^2 = e$, $e'^2 = e'$ and $ee' = 0$. Also $\mathrm{ann}_R(a) = eR$ and $\mathrm{ann}_R(\mathrm{ann}_R(a)) = e'R$. Thus in a projectable ring R, for every $0 \neq x \in R$, there exists an idempotent $e_x \in R$ such that $\mathrm{ann}_R(a) = \mathrm{ann}_R(e_x)$. It follows from this that when x and e_x are so related, $e_x x = x$. We now prove the following result.

Proposition 16. *Every projectable ring R has an orthogonal completion.*

Proof. Let S be the collection of all those elements of $Q\max(R)$ which are the supremum of some orthogonal subset of R. Let $a, b \in S$. Then there exist orthogonal subsets $\{a_i : i \in I\}, \{b_j : j \in J\}$ such that $a = \sup_i a_i$, $b = \sup_j b_j$. Let $\{e_i : i \in I\}, \{f_j : j \in J\}$ be subsets of idempotents of R such that $\mathrm{ann}_R(a_i) = \mathrm{ann}_R(e_i)$, $\mathrm{ann}_R(b_j) = \mathrm{ann}_R(f_j)$ for every $i \in I$ and $j \in J$. It can be easily checked that $\{e_i : i \in I\}, \{f_j : j \in J\}$ are orthogonal subsets of R. Let $e, f \in S$ be such that $e = \sup_i e_i$ and $f = \sup_j f_j$. Then e and f are idempotents in S.

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Now pick an orthogonal subset $\{e_k : k \in K\}$ of idempotents of R such that $\{e_i : i \in I\} \cup \{e_k : k \in K\}$ is a maximal orthogonal subset of R. Let $e' \in S$ be such that $e' = \sup_k e_k$. Since $1-e-e'$ annihilates the dense ideal D generated by the maximal orthogonal subset $\{e_i : i \in I\} \cup \{e_k : k \in K\}$, we must have $1-e-e'=0$ and hence $e+e'=1$. Therefore, $e'=1-e \in S$. Similarly $1-f \in S$.

Let $X=\{(1-f)a_i : i \in I\}$, $Y=\{(1-e)b_j : j \in J\}$ and $Z=\{(a_i+b_j)e_if_j : i \in I, j \in J\}$. By corollary 15., S is closed under multiplication. Hence $X \cup Y \cup Z \subseteq S$. Since,

$$[(a_i+b_j)e_if_j][(a_{i'}+b_{j'})e_{i'}f_{j'}]=(a_i+f_j)(a_{i'}+b_{j'})(e_ie_{i'})(f_jf_{j'})$$

and since $\{e_i : i \in I\}, \{f_j : j \in J\}$ are orthogonal subsets of R, Z is an orthogonal subset of R, It follows from this that $X \cup Y \cup Z$ is an orthogonal subset of S.

Now consider the orthogonal subsets $\{a_if_j : (i,j) \in I \times J\}$, $\{b_je_i : (i,j) \in I \times J\}$ and $\{e_if_j : (i,j) \in I \times J\}$ of R. Since $a_if_j = e_if_ja_i$ and $b_je_i = e_if_jb_j$, by corollary 6 and lemma 2, we have

$$\begin{aligned}\sup\{(a_i+b_j)e_if_j : (i,j) \in IxJ\} &= sup\{a_if_j : (i,j) \in IxJ\} + \sup\{b_je_i : (i,j) \in IxJ\}\\ &= af+be.\end{aligned}$$

Hence $\sup(X \cup Y \cup Z) = (1-f)a + (1-e)b + (af+be) = a+b$. Thus $a+b$ is the supremum of the orthogonal subset $X \cup Y \cup Z$ of S. Since every element of S is the supremum of an orthogonal subset of R, it follows that $a+b$ is the supremum of an orthogonal subset of R, Thus $a, b \in S$ implies $a+b \in S$. Hence by proposition 14., R has an orthogonal completion, namely S. This completes the proof.

In the proof of this proposition we have used the fact that if X and Y are orthogonal subsets of R such that $X \cup Y$ is an orthogonal subsets of R then $\sup(X \cup Y) = \sup X + \sup Y$, which is easy to establish with the help of proposition 1.

Burgess and Raphael [6] proved that if R is a reduced Baer ring in which $aR \cap bR = \{0\}$ implies $ab=0$ then R has an orthogonal completion. Proposition 16 is a generalisation of their result.

It follows from proposition 16 that every reduced Baer ring and every reduced regular ring has an orthogonal completion. However, the class of rings which have an orthogonal completion may be much larger. We now prove the following proposition.

Proposition 17. *Let R be a projectable ring. Then its orthogonal completion $\hat{R}$ (which exists by the proposition above) is a Baer ring.*

Proof. Let $X \subseteq \hat{R}$ by such that $\operatorname{ann}_{\hat{R}} X \neq \{0\}$. Pick a maximal orthogonal subset $\{e_j : j \in J\}$ of idempotents of R orthogonal to X and let $e = \sup_j e_j$. Then for every $x \in X$, $xe = x(\sup_j e_j) = \sup_j xe_j = 0$. Hence $\operatorname{ann}_{\hat{R}} X \subseteq e\hat{R}$.

On the other hand, let $r \in \operatorname{ann}_{\hat{R}} X$. If $er=r$, then $r \in e\hat{R}$. So suppose $er-r \neq 0$. Then since $e(er-r)=0$ and for every $x \in X$, $x(er-r)=0$, $er-r \in$

$\text{ann}_{\hat{R}}(X \cup \{e_j : j \in J\})$. Since $er - r$ is the supremum of an orthogonal subset of R, there exists a $0 \neq s \in R$ such that $s \in \text{ann}_{\hat{R}}(X \cup \{e_j : j \in J\})$. Hence by our hypothesis about R, there exists an idempotent $0 \neq e' \in R$ such that $e' \in \text{ann}_{\hat{R}}(X \cup \{e_j : j \in J\})$. But this contradicts the fact that $\{e_j : j \in J\}$ is a maximal orthogonal subset of idempotents in $\text{ann}_{\hat{R}} X$. Hence $er = r$ for every $r \in \text{ann}_{\hat{R}} X$ and thus $\text{ann}_{\hat{R}} X \subseteq e\hat{R}$. It follows from this that $\text{ann}_{\hat{R}} X = e\hat{R}$ and hence $\hat{R}$ is a Baer ring.

Finally we prove the following proposition.

PROPOSITION 18. *Let R be a projectable ring and $\mathcal{F}$ be the idempotent filter of all those dense right ideals of R which contain a maximal orthogonal subset of idempotents of R. Then $Q_{\mathcal{F}}(R)$ (the ring of right quotients of R with respect to the idempotent filter $\mathcal{F}$) is the orthogonal completion of R.*

Proof. $Q_{\mathcal{F}}(R) \subseteq Q\max{-r}(R)$ is obvious. Let $q \in Q_{\mathcal{F}}(R)$ and $D \in \mathcal{F}$ be such that $qD \subseteq R$. Let $E = \{e_i : i \in I\}$ be a maximal orthogonal subset of idempotents of R contained in D. Then $qE \subseteq R$. Let D' be the ideal of R generated by E. Then $\text{ann}_R D' = \{0\}$ and hence D' is a dense ideal of R. Since every idempotent of R is central, for every $i \in I$ and for every $d \in D'$,

$$(qe_i - e_iq)d = q(e_id) - e_i(qd) = q(de_i) - (qd)e_i = 0.$$

Hence $qe_i = e_iq$ for every $i \in I$ and therefore $D'q \subseteq R$. It follows from this that $Q_{\mathcal{F}}(R) \subseteq Q\max(R)$. Furthermore, $qE = \{qe_i : i \in I\}$ is an orthogonal subset of R and

$$\sup\{qe_i : i \in I\} = q(\sup\{e_i : i \in I\}) = ql = q.$$

Hence every element of $Q_{\mathcal{F}}(R)$ is the supremum of an orthogonal subset of R.

Now to prove that $Q_{\mathcal{F}}(R)$ is an orthogonal completion of R, it is enough to prove that every orthogonal subset $X = \{x_i : i \in I\}$ of R has a supremum in $Q_{\mathcal{F}}(R)$. Let $E = \{e_i : i \in I\}$ be the collection of those idempotents of R for which $\text{ann}_R(x_i) = \text{ann}_R(e_i)$ for every $i \in I$. Let E' be the maximal orthogonal subset of idempotents of R containing E and D be the ideal of R generated by E'. Then $D \in \mathcal{F}$. Let $x \in Q_{\mathcal{F}}(R)$ be determined by

$$\begin{aligned} xe_i &= x_i \text{ if } e_i \in E \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then obviously x is an upper bound of X in $Q_{\mathcal{F}}(R)$. Let $y \in \text{ann}_{Q_{\mathcal{F}}(R)} X$. Then $yx_i = 0$ for every $i \in I$. Consider $yx \in Q_{\mathcal{F}}(R)$. From the definition of y and x, it is obvious that $(yx)E'\{0\}$. Hence $(yx)D = \{0\}$ proving that $yx = 0$. Thus $\text{ann}_{Q_{\mathcal{F}}(R)} X \subseteq \text{ann}_{Q\mathcal{F}(R)}(x)$ and hence x is the supremum of X in $Q_{\mathcal{F}}(R)$. This completes the proof.

A similar result was proved in Burgess and Raphael [6] for the class of reduced Baer rings in which $aR \cap bR = \{0\}$ implies $ab = 0$ for every $a, b \in R$.

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