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BROWDER'S THEOREMS AND SPECTRAL CONTINUITY

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Abstract. Let X be a complex infinite dimensional Banach space. We use $\sigma_a(T)$ and $\sigma_{ea}(T)$, respectively, to denote the approximate point spectrum and the essential approximate point spectrum of a bounded operator T on X. Also, $\pi_a(T)$ denotes the set iso $\sigma_a(T) \setminus \sigma_{ea}(T)$. An operator T on X obeys the a-Browder's theorem provided that $\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_a(T)$. We investigate connections between the Browder's theorems, the spectral mapping theorem and spectral continuity.

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1. Introduction. Let X be a complex infinite-dimensional Banach space and let $B(X)$ and $K(X)$ denote respectively the algebra of bounded operators and the ideal of compact operators on X. If $T \in B(X)$, then $\sigma(T)$ denotes the spectrum of T and $\rho(T)$ denotes the resolvent set of T. It is well known that the following sets form semigroups of semi–Fredholm operators on X:

$$\Phi_+(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and } \dim \mathcal{N}(T) < \infty\}$$

and

$$\Phi_-(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and} \dim X/\mathcal{R}(T) < \infty\}.$$

The semigroup of Fredholm operators is $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. If T is semi–Fredholm and $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim X/\mathcal{R}(T)$, then we define the index by $i(T) = \alpha(T) - \beta(T)$. We also consider the sets

$$\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\} \text{ (Weyl operators)},$$
$$\Phi_+^-(X) = \{T \in \Phi_+(X) : i(T) \leq 0\}, \quad \Phi_-^+(X) = \{T \in \Phi_-(X) : i(T) \geq 0\}.$$

The following definitions are well known: the essential spectrum of T is $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi(X)\}$, the Weyl spectrum of T is $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_0(X)\}$ and the Browder spectrum of T is $\sigma_b(T) = \cap\{\sigma(T + K) : TK = KT, K \in K(X)\}$. $\sigma_a(T)$ denotes the approximate point spectrum of $T \in B(X)$. Let $\pi_{00}(T)$ be the set of all $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma(T)$ and $0 < \dim \mathcal{N}(T - \lambda) < \infty$, and let $\pi_0(T)$ be the set of all normal eigenvalues of T; that is the set of all isolated points of $\sigma(T)$ for which the corresponding spectral projection has finite-dimensional range. It is well known that, for all $T \in B(X)$ the next inclusion $\pi_0(T) \subset \pi_{00}(T)$ holds. We say that T *obeys Weyl's theorem* ([**6**],[**8**]), if

$$\sigma_w(T) = \sigma(T) \setminus \pi_{00}(T),$$

and we say that T *obeys Browder's theorem* ([**6**],[**7**]), if

$$\sigma_w(T) = \sigma(T) \setminus \pi_0(T).$$

Let π_{a0} denote the set of all $\lambda \in \mathbb{C}$ such that λ is isolated in $\sigma_a(T)$ and $0 < \alpha(T - \lambda) < \infty$. Also, by definition, $\sigma_{ea}(T) = \cap\{\sigma_a(T + K) : K \in K(X)\}$ is the essential approximate point spectrum ([**9**]) and

$$\sigma_{ab}(T) = \cap\{\sigma_a(T + K) : AK = KA, K \in K(X)\}$$

is the Browder essential approximate point spectrum ([**10**]). It is well known that $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(X)\}$. We say that T obeys *a-Weyl's theorem* ([**3**],[**11**]), if

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_{a0}(T).$$

It is well known that if $T \in B(X)$ obeys a-Weyl's theorem, then it obeys Weyl's theorem also ([**11**]).

Let $\Gamma_{0e}(T)$ be the union of all trivial components of the set ([**1**]): that is

$$\Gamma_{0e}(T) = (\sigma_e(T) \setminus [\rho_{s-F}^{\pm}(T)]^-) \cup (\cup_{-\infty<n<\infty}\{[\rho_{s-F}^{n}(T)]^- \setminus \rho_{s-F}^{n}(T)\}),$$

where

$$\rho_{s-F}^{\pm}(T) = \{\lambda \in \mathbb{C} : T - \lambda \in \Phi_+(X) \cup \Phi_-(X), i(T - \lambda) \neq 0\},$$

$$\rho_{s-F}^{n}(T) = \{\lambda \in \mathbb{C} : T - \lambda \in \Phi_+(X) \cup \Phi_-(X), i(T - \lambda) = n\}.$$

If (τ_n) is a sequence of compact subsets of $\mathbb{C}$, then, by the definition, its limit inferior is $\liminf \tau_n = \{\lambda \in \mathbb{C} :$ there are $\lambda_n \in \tau_n$ with $\lambda_n \to \lambda\}$ and its limit superior is $\limsup \tau_n = \{\lambda \in \mathbb{C} :$ there are $\lambda_{n_k} \in \tau_{n_k}$ with $\lambda_{n_k} \to \lambda\}$. If $\liminf \tau_n = \limsup \tau_n$, then $\lim \tau_n$ is defined by this common limit. A mapping p, defined on $B(X)$, whose values are compact subsets of $\mathbb{C}$, is said to be *upper (lower) semi-continuous at* A, provided that if $A_n \to A$ (in the norm topology) then $\limsup p(A_n) \subset p(A)$ ($p(A) \subset \liminf p(A_n)$). If p is both upper and lower semi-continuous at A, then it is said to be *continuous at* A and in this case $\lim p(A_n) = p(A)$.

2. Browder's theorem. Let H be a separable complex infinite-dimensional Hilbert space. It was shown in ([**6**]) that, for an operator $T \in B(X)$, Browder's theorem holds if and only if $\sigma(T) = \sigma_w(T) \cup \pi_{00}(T)$, or equivalently $\sigma_w(T) = \sigma_b(T)$.

THEOREM 2.1. *If the Browder spectrum σ_b is continuous at $T \in B(H)$, then Browder's theorem holds for T.*

Proof. Since σ_b is continuous at $T \in B(H)$, we have by [**1**, Theorem 14.17] that $\sigma_w(T) = \sigma_b(T)$. Now, by an argument of Harte and Lee [**6**, Theorem 2], Browder's theorem holds for T. $\square$

Our next theorem gives a connection between Browder's theorem and spectral continuity.

THEOREM 2.2. *If Browder's theorem holds for $T \in B(H)$, then the following conditions are equivalent:*
(i) *σ is continuous at T;*
(ii) *σ_w is continuous at T;*
(iii) *σ_b is continuous at T.*

Proof. (ii)$\Leftrightarrow$(iii). Since for $T \in B(H)$ Browder's theorem holds, or equivalently $\sigma_b(T) = \sigma_w(T)$, by [**1**, Theorem 14.17], it follows that σ_b is continuous at T if and only if σ_w is continuous at T.

(i)$\Leftrightarrow$(ii) Since T obeys Browder's theorem we have $\sigma_w(T) = \sigma(T) \setminus \pi_0(T)$. Now,

$$\sigma_e(T) \cap \overline{\pi_0(T)} \subset \sigma_w(T) \cap \overline{\pi_0(T)} = (\sigma(T) \setminus \pi_0(T)) \cap \overline{\pi_0(T)} \subset \overline{\Gamma_{0e}(T)}.$$

It follows from [**1**, Theorem 14.17] that σ is continuous at T if and only if σ_w is continuous at T. $\square$

3. *a*-Browder's theorem. Let $a(T)$ be the ascent of T; i.e. the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$. Let $b(T)$ be the descent of T; i.e., the smallest non-negative integer n such that $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$.

The following is basically due to V. Rakočević [**10**].

LEMMA 3.1. *If $T \in B(X)$, the following are equivalent:*
(i) $\lambda \notin \sigma_{ab}(T)$;
(ii) $T - \lambda \in \Phi_+(X)$ *and* $a(T-\lambda) < \infty$;
(iii) $T - \lambda \in \Phi_+(X)$ *and* $\lambda \notin \operatorname{acc} \sigma_a(T)$.

Proof. An argument of Rakočević [**10**, Theorem 2.1] gives that $\lambda \notin \sigma_{ab}(T)$ if and only if $T - \lambda \in \Phi_+^-(X)$ and $a(T-\lambda) < \infty$, giving the implication (i) $\Rightarrow$ (ii). For the converse, it suffices to show that $T - \lambda \in \Phi_+(X)$ and $a(T-\lambda) < \infty$ imply $T - \lambda \in \Phi_+^-(X)$. Indeed, if $T - \lambda \in \Phi_+(X)$, but $T - \lambda \notin \Phi(X)$, then evidently $i(T-\lambda) \leq 0$. If instead $T - \lambda \in \Phi(X)$ and $a(T-\lambda) < \infty$, then either $b(T-\lambda) < \infty$ whence $\lambda \notin \sigma_b(T)$, so that $i(T-\lambda) = 0$, or $b(T-\lambda) = \infty$ and we have

$$n\, i(T-\lambda) = i((T-\lambda)^n) = \dim \mathcal{N}((T-\lambda)^n) - \dim X/\mathcal{R}((T-\lambda)^n) \xrightarrow{n\to\infty} -\infty,$$

which implies that $i(T-\lambda) < 0$. This proves the implication (ii) $\Rightarrow$ (i).

The implication (i) $\Leftrightarrow$ (iii) follows from the proof of [**10**, Theorem 2.1]. $\square$

We denote

$$\pi_a(T) := \sigma_a(T) \setminus \sigma_{ab}(T). \tag{3.1}$$

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By Lemma 3.1,

$$\pi_a(T) = \{\lambda \in \sigma_a(T) : \ T - \lambda \in \Phi_+(X) \text{ and } a(T-\lambda) < \infty\}. \tag{3.2}$$

Evidently,

$$\operatorname{iso}\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_a(T) \subset \pi_{a0}(T). \tag{3.3}$$

Definition 3.2. We say that *a-Browder's theorem* holds for $T \in B(X)$ if

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_a(T). \tag{3.4}$$

Evidently, a-Weyl's theorem implies a-Browder's theorem. However, the converse is not true in general: for example, consider the operator $T \in B(\ell_2)$ defined by

$$T : (x_1, x_2, x_3, \cdots) \mapsto (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots).$$

Then the operator T does not obey a-Weyl's theorem [**6**, Example 3] and T obeys a-Browder's theorem.

Theorem 3.3. *If $T \in B(X)$, the following are equivalent:*
(i) *a-Browder's theorem holds for T;*
(ii) $\sigma_{ea}(T) = \sigma_{ab}(T)$;
(iii) $\sigma_a(T) = \sigma_{ea}(T) \cup \pi_a(T)$;
(iv) $\pi_a(T) = \Delta(T)$, *where* $\Delta(T) := \{\lambda \in \mathbb{C} : T - \lambda \in \Phi_+^-(X)$ *and* $\alpha(T-\lambda) > 0\}$;
(v) $\operatorname{acc}\sigma_a(T) \subset \sigma_{ea}(T)$;
(vi) $\pi_a(T) \cup \Delta(T)$ *is a subset of discontinuities of* $\gamma_T(\lambda) := \gamma(T-\lambda)$, *where* $\gamma(\cdot)$ *denotes the reduced minimum modulus.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from (3.1) and (3.4).
(ii) $\Leftrightarrow$ (iii). If $\sigma_{ea}(T) = \sigma_{ab}(T)$, then

$$\sigma_{ea}(T) \cup \pi_a(T) = \sigma_{ab}(T) \cup [\sigma_a(T) \setminus \sigma_{ab}(T)] = \sigma_a(T).$$

Conversely, if $\sigma_a(T) = \sigma_{ea}(T) \cup \pi_a(T)$, then $\sigma_a(T) \setminus \sigma_{ea}(T) \subset \pi_a(T)$. Since $\pi_a(T) = \sigma_a(T) \setminus \sigma_{ab}(T)$, we have that $\sigma_a(T) \setminus \sigma_{ea}(T) \subset \sigma_a(T) \setminus \sigma_{ab}(T)$. Thus $\sigma_{ab}(T) \subset \sigma_{ea}(T)$, and hence $\sigma_{ea}(T) = \sigma_{ab}(T)$.

(i) $\Leftrightarrow$ (iv). Suppose a-Browder's theorem holds for T, so that $\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_a(T)$. Let $\lambda \in \Delta(T)$. Then $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$, and so $\lambda \in \pi_a(T)$. Thus $\Delta(T) \subset \pi_a(T)$, and evidently the inclusion is reversible. Conversely suppose $\pi_a(T) = \Delta(T)$. If $\lambda \in \sigma_{ea}(T)$, then $T - \lambda \notin \Phi_+^-(X)$ and so $\lambda \notin \Delta(T) = \pi_a(T)$. Thus $\lambda \in \sigma_a(T) \setminus \pi_a(T)$ and hence $\sigma_{ea}(T) \subset \sigma_a(T) \setminus \pi_a(T)$. For the reverse inclusion, observe that if $\lambda \in \sigma_a(T) \setminus \pi_a(T)$ and $\lambda \notin \sigma_{ea}(T)$, then $T - \lambda \in \Phi_+^-(X)$. Since $\lambda \notin \pi_a(T) = \Delta(T)$, we have $\alpha(T - \lambda) = 0$. Thus $T - \lambda$ is bounded below and hence $\lambda \notin \sigma_a(T)$, a contradiction. Thus we should have $\lambda \in \sigma_{ea}(T)$, so that $\sigma_a(T) \setminus \pi_a(T) \subset \sigma_{ea}(T)$. Therefore a-Browder's theorem holds for T.

(ii) $\Leftrightarrow$ (v). By Lemma 3.1, $\sigma_{ab}(T) = \sigma_{ea}(T) \cup \operatorname{acc}\sigma_a(T)$. Thus $\sigma_{ab}(T) = \sigma_{ea}(T)$ if and only if $\operatorname{acc}\sigma_a(T) \subset \sigma_{ea}(T)$.

(iv) $\Leftrightarrow$ (vi). Suppose that $\pi_a(T) = \Delta(T)$. Recall that by [**4**, Lemma 5.52], $\pi_a(T)$ consists of discontinuities of $\gamma_T(\lambda)$ and hence $\pi_a(T) \cup \Delta(T)$. Conversely suppose that $\pi_a(T) \cup \Delta(T)$ consists of discontinuities of $\gamma_T(\lambda)$. Let $\lambda_0 \in \Delta(T)$. Then $\alpha(T - \lambda_0) > 0$ and $\gamma(\lambda_0) > 0$. By the punctured neighborhood theorem, there exists a neighborhood $N(\lambda_0; p)$, for some $p > 0$, such that $\alpha(T - \lambda)$ is constant (say n_0) on $N(\lambda_0; p) \setminus \{\lambda_0\}$ and $0 \leq \alpha(T - \lambda) < \alpha(T - \lambda_0)$. We now claim that $n_0 = 0$. Assume to the contrary that $n_0 \neq 0$. Also by the punctured neighborhood theorem there exists a neighborhood $N(\lambda_0; q)$, for some $q > 0$, such that $\lambda_1 \in N(\lambda_0; q) \setminus \{\lambda_0\}$ implies $\alpha(T - \lambda_1) > 0$ and $T - \lambda_1 \in \Phi_+^-(X)$. Thus we have $\lambda_1 \in \Delta(T)$. Now, by the same reason as for λ_0, there exists a neighborhood $N(\lambda_1; r)$ for some $r > 0$ such that $\alpha(T - \mu)$ is constant (say n_1) and $0 \leq \alpha(T - \mu) < \alpha(T - \lambda_1)$. Thus

$$\lambda \in \Big[N(\lambda_0; q) \cap N(\lambda_1; r)\Big] \setminus \{\lambda_0, \lambda_1\} \Longrightarrow \alpha(T - \lambda) = n_1 < n_0,$$

a contradiction. Therefore $n_0 = 0$ and hence $\lambda_0 \in \operatorname{iso}\sigma_a(T)$. Thus $\lambda_0 \in \Delta(T)$ implies $\lambda_0 \in \operatorname{iso}\sigma_a(T)$ and so $\lambda_0 \in \sigma_a(T) \setminus \sigma_{ab}(T)$. Therefore we have $\lambda_0 \in \pi_a(T)$ and hence $\Delta(T) \subset \pi_a(T)$. The converse is evident. □

COROLLARY 3.4. *Let σ_{ab} be continuous at $T \in B(H)$. Then a-Browder's theorem holds for T.*

Proof. Since σ_{ab} is continuous at T, by [**2**, Theorem 2.2] it follows that $\sigma_{ea}(T) = \sigma_{ab}(T)$. Now, by Theorem 3.3, we have that a-Browder's theorem holds for T. □

COROLLARY 3.5. *a-Browder's theorem holds for quasinilpotent operators and algebraic operators.*

Proof. If $T \in B(X)$ is either quasinilpotent or algebraic then $\operatorname{acc}\sigma_a(T) = \emptyset$. Thus by Theorem 3.3, T obeys a-Browder's theorem. □

In [**8**], it was shown that if $T \in B(X)$ then $\partial\sigma_w(T) \subset \sigma_{ea}(T)$, where $\partial(\cdot)$ denotes the topological boundary. We can prove more.

PROPOSITION 3.6. *If $T \in B(X)$, then $\partial\sigma_e(T) \subset \sigma_{ea}(T)$, and hence $\sigma_{ea}(T) \neq \emptyset$.*

Proof. If $\lambda \notin \sigma_{ea}(T)$, then $T - \lambda \in \Phi_+(X)$ and $i(T - \lambda) \leq 0$. Assume to the contrary that $\lambda \in \partial\sigma_e(T)$. Then, by the punctured neighborhood theorem, there exists $p > 0$ such that $\mu \in N(\lambda; p) \setminus \{\lambda\}$ implies that $T - \mu \in \Phi(X)$. Since $T - \lambda \in \Phi_+(X)$, it follows from the continuity of the (semi-Fredholm) index that $T - \lambda \in \Phi(X)$. Hence $\lambda \notin \sigma_e(T)$, a contradiction. □

COROLLARY 3.7. *Suppose that H is a separable Hilbert space and that $T \in B(H)$ is a polynomially Riesz operator, in the sense that there exists a nonzero complex polynomial p such that $p(T)$ is a Riesz operator. Then a-Browder's theorem holds for T.*

Proof. Suppose that T is polynomially Riesz. Then, by [**7**, Lemma 3], $\operatorname{acc}\sigma(T) \subset \sigma_w(T)$ and hence $\operatorname{acc}\sigma_a(T) \subset \sigma_w(T)$. Since $\sigma_w(T)$ is finite it follows that $\sigma_w(T) = \partial\sigma_w(T)$. Now $\partial\sigma_w(T) \subset \partial\sigma_e(T)$ and we have, by Lemma 3.6, $\operatorname{acc}\sigma_a(T) \subset \sigma_{ea}(T)$. Therefore, by Theorem 3.3, T obeys a-Browder's theorem. $\square$

Also, it is easy to see that if $T \in B(H)$ obeys a-Browder's theorem, then σ_{ea} is continuous at T if and only if σ_{ab} is continuous at T.

THEOREM 3.8. *Necessary and sufficient conditions that a-Weyl's theorem holds for $T \in B(X)$ are a-Browder's theorem holds together with one of the following:*
(i) *if $\lambda \in \pi_{a0}(T)$, then $\mathcal{R}(T-\lambda)$ is closed;*
(ii) $\sigma_{ea}(T) \cap \pi_{a0}(T) = \emptyset$;
(iii) $\pi_{a0}(T) = \pi_a(T)$.

Proof. Suppose a-Weyl's theorem holds for $T \in B(X)$. Then evidently a-Browder's theorem holds too and the condition (i) holds by [**9**, Theorem 5.6]. By a-Weyl's theorem, $\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_{00}(T)$; i.e. $\sigma_{ea}(T) \cap \pi_a(T) = \emptyset$. By [**3**, Lemma 2.6] we have that $\sigma_{ea}(T) = \sigma_{ab}(T)$ and now

$$\pi_{a0}(T) = \sigma_a(T) \setminus \sigma_{ea}(T) = \sigma_a(T) \setminus \sigma_{ab}(T) = \pi_a(T).$$

Suppose that T obeys a-Browder's theorem and the condition (i) holds. Let $T-\lambda \in \Phi_+^-(X)$. Then $\lambda \notin \sigma_{ea}(T) = \sigma_{ab}(T)$. Now, by [**10**, Corollary 2.4] it follows that λ is not a limit point of $\sigma_a(T)$ and by [**11**, Theorem 1.1] T obeys a-Weyl's theorem.

If a-Browder's theorem holds for $T \in B(X)$ we have that

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_a(T) \supset \sigma_a(T) \setminus \pi_{a0}(T).$$

By the condition (ii) we have that the opposite inclusion holds; i.e.

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_{a0}(T).$$

Now, by a-Browder's theorem together with the condition (iii), it follows directly that a-Weyl's theorem holds for $T \in B(X)$. $\square$

THEOREM 3.9. *If a-Browder's theorem holds for $T \in B(X)$, then T obeys Browder's theorem.*

Proof. Suppose the contrary. Then, by [**6**, Theorem 2] it follows that $\sigma_b(T) \neq \sigma_w(T)$; i.e. there exists $\lambda \in \sigma_b(T) \setminus \sigma_w(T)$. For this λ we have that $T-\lambda \in \Phi_0(X)$ and $a(T-\lambda) = \infty$ by [**5**, Theorem 7.9.3]. Since we have $T-\lambda \in \Phi_0(X)$ and $a(T-\lambda) = \infty$, it follows from [**10**] that $\lambda \in \sigma_{ab}(T)$. By Theorem 3.3, we have that $\sigma_{ea}(T) = \sigma_{ab}(T)$; i.e. $\lambda \in \sigma_{ea}(T) \subset \sigma_w(T)$. This is a contradiction because of our assumption that $\lambda \notin \sigma_w(T)$. $\square$

Let $T \in B(X)$ and let p be a polynomial. It is known that the inclusion $\sigma_{ea}(p(T)) \subset p(\sigma_{ea}(T))$ holds [**9**] and it is known that for σ_{ab} a spectral mapping theorem holds [**10**]. The next theorem gives some sufficient conditions for a spectral mapping theorem for σ_{ea} to hold.

THEOREM 3.10. *If a-Browder's theorem holds for $T \in B(X)$ and if p is a polynomial, then a-Browder's theorem holds for $p(T)$ if and only if $p(\sigma_{ea}(T)) = \sigma_{ea}(p(T))$.*

Proof. If a-Browder's theorem holds for $p(T)$ then

$$\sigma_{ea}(p(T)) = \sigma_{ab}(p(T)) = p(\sigma_{ab}(T)) \supset p(\sigma_{ea}(T)).$$

Since the opposite inclusion holds we have that $p(\sigma_{ea}(T)) = \sigma_{ea}(p(T))$.

Suppose that $p(\sigma_{ea}(T)) = \sigma_{ea}(p(T))$ and let a-Browder's theorem hold for T. Then we have

$$\sigma_{ab}(p(T)) = p(\sigma_{ab}(T)) = p(\sigma_{ea}(T)) = \sigma_{ea}(p(T))$$

that is, by Theorem 3.3, a-Browder's theorem holds for $p(T)$. □

Let S and T be in $B(X)$. Then clearly,

$$\sigma_{ea}(S \oplus T) \subset \sigma_{ea}(S) \cup \sigma_{ea}(T). \qquad (*)$$

By contrast the Browder essential approximate point spectrum of a direct sum is the union of the Browder essential approximate point spectrum of the components. This might suggest that a-Browder's theorem for S and T is sufficient for equality in $(*)$.

THEOREM 3.11. *If a-Browder's theorem holds for S and $T \in B(X)$, then a-Browder's theorem holds for $S \oplus T$ if and only if $\sigma_{ea}(S \oplus T) = \sigma_{ea}(S) \cup \sigma_{ea}(T)$.*

Proof. If $S \oplus T$ obeys a-Browder's theorem, then

$$\sigma_{ea}(S) \cup \sigma_{ea}(T) \subset \sigma_{ab}(S) \cup \sigma_{ab}(T) = \sigma_{ab}(S \oplus T) = \sigma_{ea}(S \oplus T) \subset \sigma_{ea}(S) \cup \sigma_{ea}(T),$$

and so we have

$$\sigma_{ea}(S \oplus T) = \sigma_{ea}(S) \cup \sigma_{ea}(T).$$

Suppose that $\sigma_{ea}(S \oplus T) = \sigma_{ea}(S) \cup \sigma_{ea}(T)$. Then we have

$$\sigma_{ab}(S \oplus T) = \sigma_{ab}(S) \cup \sigma_{ab}(T) = \sigma_{ea}(S) \cup \sigma_{ea}(T) = \sigma_{ea}(S \oplus T);$$

that is, $\sigma_{ab}(S \oplus T) = \sigma_{ea}(S \oplus T)$. By Theorem 3.3, a-Browder's Theorem holds for $S \oplus T$. □

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