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CONTRACTION GROUPS AND EQUIVALENT NORMS*

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Using the notation in [1], the Lumer-Phillips theorem (3. 1 of [2]) is refined to single parameter groups in real Banach space and real Hilbert space. The theory can be extended to complex spaces.

DEFINITION 1.

Let X be a B-space with norm $\|\cdot\|_1$ and let $[\cdot, \cdot]_1$ be a corresponding semi-scalar product on X. Then the semi-scalar product $[\cdot, \cdot]$ is said to be equivalent to $[\cdot, \cdot]_1$ on X iff $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms on X.

THEOREM 1.

Let A be a linear operator with $D(A)$ and $R(A)$ both contained in a B-space $(X, \|\cdot\|_1)$ such that $D(A)$ is dense in X. Then A generates a group $\{T_t; -\infty < t < \infty\}$ in X such that $\{T_t;\ t > 0\}$ is a negative contractive semi-group with respect to an equivalent norm $\|\cdot\|$ iff

$$-\delta\|x\|^2 < [Ax, x] < -\gamma\|x\|^2 \qquad (x \in D(A)) \tag{1}$$

where $\infty > \delta > \gamma > 0$ and $[\cdot, \cdot]$ is an equivalent scalar product consistent with $\|\cdot\|$, and

$$R(I(1-\gamma) - A) = X \qquad R(I(1+\delta) + A) = X. \tag{2}$$

Proof.

The sufficiency of conditions (1) and (2) follows immediately from the results in Yosida [1], pp. 250–254.

Conversely suppose that A generates a group such that $\|T_t\| < e^{-\beta t}$ $(t \geqq 0)$ where $\beta > 0$. It is known that for a group $\|T_t^{-1}\| < Me^{\alpha t}$, where

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$M > 1$ and α can be chosen such that $\alpha > \beta$ [1]. Define $S_t = T_t^{-1} e^{-\alpha t}$ and define $\|\cdot\|_2$ by

$$\|x\|_2 = \sup_{t>0} \|S_t x\|.$$

This yields an equivalent semi-scalar product and the left side of inequality (1) with $\delta = \alpha$. To show the right side is also valid consider

$$[T_s e^{\beta s} x - x, x]_2 \leqq \|T_s e^{\beta s} x\|_2 \|x\|_2 - \|x\|_2^2. \tag{3}$$

Next estimate $\|T_s e^{\beta s} x\|_2$ as follows

$$\|T_s e^{\beta s} x\|_2 = \sup_{t\geqq 0} \|T_{s-t} e^{\alpha(s-t)} x\| \leqq \max(\|x\|, \ e^{(\beta-\alpha)s}\|x\|_2) \leqq \|x\|_2.$$

Hence, (3) yields $[T_s e^{\beta s} x - x, x]_2 \leqq 0$ which in turn implies the right side of (1) with $\gamma = \beta$.

Finally (2) follows from theorem 3. 1 of [2] applied to the contraction operators $T_{-t} e^{\alpha t}$ (with respect to $\|\cdot\|_2$) and $T_t e^{\beta t}$ (with respect to $\|\cdot\|_1$).

Remark.

Theorem 1 is valid for $(H, [\cdot, \cdot]_1)$ a Hilbert space and $[\cdot, \cdot]$ an equivalent scalar product.

Proof.

Using the results of theorem 1, it need only be shown that there exists a scalar product $[\cdot, \cdot]$ equivalent to the scalar product $[\cdot, \cdot]_1$ such that (1) holds. Define $[\cdot, \cdot]$, for any group $\{T_t; -\infty < t < \infty\}$ which is negative with respect to $\|\cdot\|_1$, by

$$[x, y] = \int_0^\infty [T_t x, T_t y]_1 dt. \tag{4}$$

By hypothesis, $\|T_t\|_1 \leqq M e^{-\beta t}$ $(t \geqq 0)$, where $\beta > 0$ and $M \geqq 1$; hence

$$[x, x] = \leqq (M^2/2\beta) \|x\|_1^2. \tag{5}$$

Since $\{T_t\}$ is a group, there exist constants $\alpha \geqq \beta$ and $1/k \geqq 1$ such that $\|T_t^{-1}\|_1 \leqq (1/k) e^{\alpha t}$ for $\gamma \geqq 0$. By using the fact that $\|T_t x\|_1 \geqq \|T_t^{-1}\|_1^{-1} \|x\|_1$ it follows from (4) that

$$[x, x] \geqq (k^2/2\alpha) \|x\|_1^2. \tag{6}$$

We leave it to the reader to verify that $[\cdot, \cdot]$ is a scalar product. The equivalence of the two scalar products follows from (5) and (6).

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To show that an equation of the form (1) is valid we consider

$$[T_t x,\ T_t x]-[x,x]=\lim_{n\to\infty}\left\{\int_0^n [T_s T_t x,\ T_s T_t x]_1 ds-\int_0^n [T_s x,\ T_s x]_1 ds\right\}$$

$$=-\int_0^t [T_s x,\ T_s x]_1 ds, \qquad (t>0).$$

Since $\lim_{t\to 0^+} t^{-1}([T_t x,\ T_t x]-[x,x])=2[Ax,x]$ the last equality implies that

$$2[Ax,x]=-\|x\|_1^2 \qquad (x\in D(A)). \tag{7}$$

Equations (5), (6), and (7) yield (1) with $\gamma=\beta/M^2$ and $\delta=\alpha/k^2$.

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