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EXTENSIONS OF *AH* ALGEBRAS WITH THE IDEAL PROPERTY

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In this note we show that if we have an exact sequence of AH algebras (AH stands for "approximately homogeneous") $0 \to I \to A \to B \to 0$, then A has the ideal property (i.e., any ideal is generated by its projections) if and only if I and B have the ideal property. Also, we prove that an extension of two $A\mathbb{T}$ algebras ($A\mathbb{T}$ stands for "approximately circle") with the ideal property is an $A\mathbb{T}$ algebra with the ideal property if and only if the extension is quasidiagonal.

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1. Introduction

An AH algebra is a C^*-algebra which is the inductive limit of a sequence:

$$A_1 \to A_2 \to A_3 \to \cdots$$

with $A_n = \oplus_{i=1}^{k_n} P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i}$, where $X_{n,i}$ are finite, connected CW complexes, k_n, $[n, i]$ are positive integers and $P_{n,i} \in C(X_{n,i}, M_{[n,i]})$ are projections. The problem of finding suitable topological invariants for these C^*-algebras was raised by Effros [**6**] and now it is included in Elliott's project of the classification of the amenable, separable C^*-algebras by invariants including K-theory ([**8**]).

A C^*-algebra is said to have *the ideal property* if its ideals are generated (as ideals) by their projections (here by an ideal we shall mean a closed two-sided ideal).

In this paper we are dealing essentially with extensions of AH algebras with the ideal property. The AH algebras with the ideal property, which have been studied previously in [**14**] and [**12**], are important because they represent a common generalization of the simple AH algebras and of the real rank zero AH algebras ([**3**]). The extension problem for AH algebras is important and highly non-trivial. The C^*-algebras which are extensions of AH algebras should be included on the list of basic building blocks of local approximations of nuclear C^*-algebras([**2**]); hence this problem is related to Elliott's project ([**8**]). While any extension of two AF algebras is an AF algebra – as proved by L. G. Brown in a "pioneering paper" ([**1**]) – it is not true in general that an

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extension of AH algebras is an AH algebra (e.g., the Toeplitz algebra is an extension of $C(\mathbb{T})$ by $\mathcal{K}$, the compact operators on an infinite dimensional, separable Hilbert space, but it is not an AH algebra because it is not finite). The presence of torsion in K-theory produces situations that cannot occur in the extension theory of AF algebras or of $A\mathbb{T}$ algebras (see **[2]**). Brown and Dadarlat constructed in **[2]** examples of extensions A of AH algebras such that even though A is nuclear, stably-finite, of real rank zero and stable rank one, A is not isomorphic to any inductive limit of subhomogeneous C^*-algebras (in particular, A is not an AH algebra). Hence the extension problem for AH algebras is complicated and it is important to find a class of AH algebras which "behaves well" with respect to extensions.

Motivated by a question raised to us by T. Loring we proved that if we have an exact sequence of AH algebras:

$$0 \to I \to A \to B \to 0$$

then A has the ideal property if and only if I and B have the same property (see Theorem 3.1).

Lin and Rørdam gave in **[10]** two necessary and sufficient conditions for an extension of $A\mathbb{T}$ algebras (i.e., inductive limits of circle algebras (see Definition 2.5)) of real rank zero to be also an $A\mathbb{T}$ algebra of real rank zero. One condition is that the extension has real rank zero and stable rank one (**[13]**) and the other one is that the index maps in K-theory associated with the given exact sequence of C^*-algebras are both zero. We considered in this paper the analogue problem in the setting of $A\mathbb{T}$ algebras with the ideal property and we proved that an extension of $A\mathbb{T}$ algebras with the ideal property is an $A\mathbb{T}$ algebra with the ideal property if and only if the extension is quasidiagonal (see Theorem 3.6). The proof is inspired by the proof of the above quoted result given in **[10]** but it uses also other techniques, some of them taken from **[12]**, **[2]**, **[3]** and **[15]**, and also Theorem 3.1.

Dadarlat and Loring proved in **[5]** a partial generalization of the above result of Lin and Rørdam to the $A\mathcal{D}$ case. Might be that the ideas in this paper could be used to extend their result in **[5]**.

2. Preliminaries

In what follows we shall need the following definitions and results:

Theorem 2.1 (see **[12**, Theorem 3.1**]**). *Let $A = \varinjlim(A_n, \Phi_{n,m})$ be an AH algebra, with $A_n = \oplus_{i=1}^{k_n} A_n^i$, $A_n^i = P_{n,i}C(X_{n,i}, M_{[n,i]})P_{n,i}$ where $X_{n,i}$ are connected, finite CW complexes and $P_{n,i} \in C(X_{n,i}, M_{[n,i]})$ are projections. Then the following are equivalent:*

(1) *A has the ideal property (see Introduction).*

(2) *For any fixed n and fixed $F = \overline{F} \subset U = \mathring{U} \subset SP(A_n) = \sqcup_{i=1}^{k_n} X_{n,i}$, there is $m_0 > n$ such that for any $m \geq m_0$ any partial map $\Phi_{n,m}^j : A_n \to A_m^j$ satisfies either*

$$SP(\Phi^j_{n,m})_y \cap F = \phi \text{ for all } y \in X_{m,j}$$

or

$$SP(\Phi^j_{n,m})_y \cap U \neq \phi \text{ for all } y \in X_{m,j}.$$

(3) *For any fixed n, i and $\delta > 0$ there is $m_0 > n$ such that the following is true:*
For any $F = \overline{F} \subset X_{n,i}$ and any $m \geq m_0$ we have that any partial map $\Phi^{i,j}_{n,m}$ satisfies either

$$SP(\Phi^{i,j}_{n,m})_y \cap F = \phi \text{ for all } y \in X_{m,j}$$

or

$$SP(\Phi^{i,j}_{n,m})_y \cap B_\delta(F) \neq \phi \text{ for all } y \in X_{m,j}.$$

(Here we used the standard notation $B_\delta(M) = \{x \in X_{n,i} : dist(x, M) < \delta\}$ for any subset M of $X_{n,i}$).

(4) *Any ideal of A has a countable approximate unit consisting of projections.*

(5) *For any ideal I of A we have:*
for any integer n, any $\varepsilon > 0$ and any $x \in A_n \cap I$ there is $m > n$ and a projection $p \in A_m \cap I$ such that:

$$\|\Phi_{n,m}(x) - p \cdot \Phi_{n,m}(x)\| \leq \varepsilon.$$

(6) *For any ideal I of A we have:*
for any integer n, any $\varepsilon > 0$ and any $x \in A_n \cap I$ there is $m > n$ and a projection $p \in A_m \cap I$ such that

$$\|\Phi_{n,m}(x) - p\Phi_{n,m}(x)p\| \leq \varepsilon.$$

(Above we used the notation $A_k \cap I = \{y \in A_k : \Phi_{k,\infty}(y) \in I\}$).

The notation used in the above theorem is the one from [9].

Definition 2.2 (see [11]). An extension of C^*-algebras

$$0 \to I \to A \to B \to 0$$

is called *quasidiagonal* if there is an approximate unit $(p_n)_{n=1}^\infty$ of I consisting of projections, which is quasicentral in A, i.e.,

$$\lim_{n\to\infty} \|ap_n - p_n a\| = 0$$

for all $a \in A$.

Theorem 2.3 (see [**2**, Theorem 8]). *Let*

$$0 \to I \xrightarrow{j} A \xrightarrow{\pi} B \to 0$$

be a quasidiagonal extension of C^*-algebras. Then the index maps $\delta_i : K_i(B) \to K_{i+1}(I)$, $i = 0, 1$ are zero and the extensions

$$0 \to K_i(I) \xrightarrow{j_*} K_i(A) \xrightarrow{\pi_*} K_i(B) \to 0$$

$i = 0, 1$ are pure.

Proposition 2.4 (see [**2**, Proposition 11]). *Let A be an AH algebra. Suppose that I is a closed ideal in A and I has an approximate unit of projections. Then the extension*

$$0 \to I \to A \to B \to 0$$

is quasidiagonal.

Definition 2.5. A C^*-algebra is called a *circle algebra* if it is isomorphic to

$$\oplus_{j=1}^{r} C(\mathbf{T}, M_{n_j})$$

for some positive integers $r, n_1, n_2, \cdots, n_r$, where $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$.

A C^*-algebra is called *an* AT *algebra* if it is isomorphic to the inductive limit of a sequence:

$$A_1 \to A_2 \to A_3 \to \cdots$$

where the A_i's are circle algebras (the connecting homomorphisms may or may not be injective).

3. Results

We want to prove first the following:

Theorem 3.1. *Let*

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$$

be an exact sequence of AH algebras. Then, the following are equivalent:

(1) *A has the ideal property.*

(2) *I and B have the ideal property.*

To prove this theorem we need some preparation.

Definition 3.2. Let A and B be finite direct sums of C^*-algebras of the form $PC(X, M_n)P$, where X is a compact space and P is a projection in $C(X, M_n)$ (X, n and P may vary), with $B = \oplus_{j=1}^{m} B^j$. Let $F = \overline{F} \subset U = \mathring{U} \subset SP(A)$. We say that a homomorphism $\Phi : A \to B$ satisfies <u>the condition $(F - U)$</u> if any partial homomorphism $\Phi^j : A \to B^j$ satisfies either

$$SP(\Phi^j)_y \cap F = \phi \text{ for all } y \in SP(B^j)$$

or

$$SP(\Phi^j)_y \cap U \neq \phi \text{ for all } y \in SP(B^j).$$

Lemma 3.3. *Let A, B and C be finite direct sums of C^*-algebras of the form $PC(X, M_k)P$ as in Definition 3.2 and let $\Phi : A \to B$ and $\Psi : B \to C$ be two homomorphisms.*

Let $F = \overline{F} \subset U = \mathring{U} \subset SP(A)$. If Φ satisfies the condition $(F - U)$ then $\Psi \circ \Phi$ also satisfies the condition $(F - U)$.

Proof. We may suppose that A and C have only one direct summand. Let $B = \oplus_{i=1}^{n} B^i$. Then Φ and Ψ induce a diagram of partial homomorphisms:

$$\begin{tikzcd}
 & B^1 \arrow[dddr, "\Psi^1"] & \\
 & \oplus & \\
 & B^2 \arrow[dr, "\Psi^2"] & \\
A \arrow[uuur, "\Phi^1"] \arrow[ur, "\Phi^2"] \arrow[dddr, "\Phi^n"'] & \oplus & C \\
 & \vdots & \\
 & \oplus & \\
 & B^n \arrow[uuur, "\Psi^n"'] &
\end{tikzcd}$$

If

$$SP(\Psi \circ \Phi)_{z_0} \cap F = SP(\oplus_{i=1}^{n} \Psi^i \circ \Phi^i)_{z_0} \cap F \neq \phi$$

for some $z_0 \in SP(C)$, then, since

$$SP(\oplus_{i=1}^{n}\Psi^{i}\circ\Phi^{i})_{z}=\cup_{i=1}^{n}SP(\Psi^{i}\circ\Phi^{i})_{z},\ z\in SP(C)$$

it follows that there is $1 \leq i_0 \leq n$ such that

$$SP(\Psi^{i_0}\circ\Phi^{i_0})_{z_0}\cap F\neq\phi$$

and we may suppose that $i_0 = 1$, i.e.,

$$SP(\Psi^{1}\circ\Phi^{1})_{z_0}\cap F\neq\phi.$$

Since

$$SP(\Psi^{1}\circ\Phi^{1})_{z_0}=\bigcup_{y\in SP(\Psi^{1})_{z_0}}SP(\Phi^{1})_{y}$$

it follows that there is $y_0 \in SP(\Psi^1)_{z_0}$ such that

$$SP(\Phi^{1})_{y_0}\cap F\neq\phi.$$

Since Φ^1 satisfies the condition $(F - U)$ it follows that for any $y \in SP(B^1)$ we have:

$$SP(\Phi^{1})_{y}\cap U\neq\phi.$$

In particular, this is satisfied for any $y \in SP(\Psi^1)_z$, where z is an arbitrary element in $SP(C)$. Hence

$$SP(\Psi^{1}\circ\Phi^{1})_{z}\cap U\neq\phi$$

for any $z \in SP(C)$, which implies that

$$SP(\oplus_{i=1}^{n}\Psi^{i}\circ\Phi^{i})_{z}\cap U\neq\phi$$

for any $z \in SP(C)$. □

Remark 3.4. The above Lemma 3.3 has the following useful consequence:

With the notation from Theorem 2.1 (i.e., [**12**, Theorem 3.1]) it follows that the conditions (1)–(6) in this theorem are also equivalent to the following:

(7) For any fixed n and fixed $F = \overline{F} \subset U = \mathring{U} \subset SP(A_n)$ there is $m_0 > n$ such that $\Phi_{n,m_0} : A_n \to A_{m_0}$ satisfies the condition $(F - U)$.

Proof of Theorem 3.1. (1) $\Longrightarrow$ (2). This implication is clearly true for any C^*-algebras I, A, B. Indeed, any ideal of I is an ideal of A (since I is an ideal of A) and hence it is generated by its projections, since A has the ideal property (here we identified I with $i(I) \subset A$). Now let J be an ideal of B. Then $\pi^{-1}(J)$ is an ideal of A and since A has the ideal property, it is generated by its projections. Then $\pi(\pi^{-1}(J)) = J$ is also generated by its projections.

(2) $\Longrightarrow$ (1). Let $A = \varinjlim(A_n, \Phi_{n,m})$ with $A_n = \oplus_{i=1}^{k_n} A_n^i = \oplus_{i=1}^{k_n} P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i}$ where $X_{n,i}$ are connected, finite CW complexes and $P_{n,i} \in C(X_{n,i}, M_{[n,i]})$ are projections. (Here we shall identify I with $i(I) \subset A$). Let $I_k = \{x \in A_k : \Phi_{k,\infty}(x) \in I\}$, $k \in \mathbb{N}$. Then $I = \varinjlim(I_n, \Phi_{n,m|I_n})$. Now we shall repeat the construction from **[12]** of a "canonical" approximate unit for I consisting of projections. Let $G_k = \overline{G}_k \subset SP(A_k)$ be such that $I_k = \{f \in A_k : f_{|G_k} = 0\}$, $k \geq 1$. Define a projection $p_m = \oplus_{j=1}^{k_m} p_m^j \in \oplus_{j=1}^{k_m} A_m^j$ for any $m \geq 1$ such that

$$p_m^j = \begin{cases} P_{m,j} & \text{if } G_m^j = \phi \\ 0 & \text{if } G_m^j \neq \phi \end{cases}$$

(here, obviously, G_m^j is the component of G_m in $X_{m,j}$). It is clear that $\Phi_{n,n+1}(p_n) \leq p_{n+1}$. Then $(\Phi_{n,\infty}(p_n))_{n=1}^{\infty}$ is an increasing approximate unit for I consisting of projections, since the $X_{n,i}$'s are connected (**[12]**) (note that I is generated by its projections).

Then

$$I = \varinjlim(p_n A_n p_n, \Phi_{n,m|p_n A_n p_n}).$$

By construction, $J_n = p_n A_n p_n$ is a direct summand of A_n as well as $B_n := (1_{A_n} - p_n) A_n (1_{A_n} - p_n)$. Obviously $A_n = J_n \oplus B_n$.

Then, each $\Phi_{n,m}$ induces partial homomorphisms:

$$\begin{tikzcd}
A_n \arrow[d, "\Phi_{n,m}"'] & = & J_n \arrow[d, "\Psi_{n,m}"'] & \oplus & B_n \arrow[d, "\Gamma_{n,m}"] \arrow[lld, "\Lambda_{n,m}"'] \\
A_m & = & J_m & \oplus & B_m
\end{tikzcd}$$

Clearly

$$I = \varinjlim(J_n, \Psi_{n,m})$$

and

$$B = \varinjlim(B_n, \Gamma_{n,m}).$$

Let us fix $n \in \mathbb{N}$ and $F = \overline{F} \subset U = \mathring{U} \subset SP(A_n)$. Let $F_1 = F \cap SP(J_n)$, $F_2 = F \cap SP(B_n)$, $U_1 = U \cap SP(J_n)$, $U_2 = U \cap SP(B_n)$. Since I and B have the ideal property, then, by Theorem 2.1 (i.e., [**12**, Theorem 3.1]) there is $m > n$ such that $\Psi_{n,m}$ satisfies the condition $(F_1 - U_1)$ and $\Gamma_{n,m}$ satisfies the condition $(F_2 - U_2)$. On the other hand, since the C^*-inductive limit associated with:

$$B_n \xrightarrow{\Lambda_{n,m}} J_m \xrightarrow{\Psi_{m,m+1}} J_{m+1} \xrightarrow{\Psi_{m+1,m+2}} J_{m+2} \xrightarrow{\Psi_{m+2,m+3}} \cdots$$

is isomorphic to $\varinjlim_{r \geq m}(J_r, \Psi_{r,k}) = \varinjlim(J_r, \Psi_{r,k}) = I$ which has the ideal property, it follows that there is $p > m$ such that $\Psi_{m,p} \circ \Lambda_{n,m}$ satisfies the condition $(F_2 - U_2)$. Then, by Lemma 3.3, $\Psi_{m,p} \circ \Psi_{n,m}$ satisfies the condition $(F_1 - U_1)$ (since $\Psi_{n,m}$ satisfies it) and $\Lambda_{m,p} \circ \Gamma_{n,m}$ and $\Gamma_{m,p} \circ \Gamma_{n,m}$ also satisfies the condition $(F_2 - U_2)$ (since $\Gamma_{n,m}$ satisfies it). In conclusion, $\Phi_{n,p}$ satisfies the condition $(F - U)$. Then, by Remark 3.4, it follows that A has the ideal property. □

Remark 3.5. The above proof of Theorem 3.1 shows in particular that the following assertion is true:

Let

$$0 \to I \to A \to B \to 0$$

be an extension of C^*-algebras such that A is an AH algebra with the ideal property.

Then, I and B are also AH algebras with the ideal property (see also [**4**, Proposition 4.3]).

We prove now the following:

Theorem 3.6. *Let*

$$0 \to I \to A \to B \to 0 \tag{$*$}$$

be an extension of C^*-algebras, where I and B are $A\mathbb{T}$ algebras with the ideal property.

Then, the following conditions are equivalent:

(1) *A is an $A\mathbb{T}$ algebra with the ideal property.*

(2) *The extension $(*)$ is quasidiagonal (see Definition 2.2).*

The proof of Theorem 3.6 will need several auxiliary results.

Lemma 3.7. *Let*

$$0 \to I \to A \to B \to 0$$

be a quasidiagonal extension of C^*-algebras. Let $p \in A$ be a projection and let q be its image in B. Then:

(1) *The "reduced" extension*

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$$0 \to pIp \to pAp \to qBq \to 0$$

is quasidiagonal.

(2) *If I is an AH algebra, then pIp is an AH algebra with the ideal property.*

Proof. (1) Let $(p_n)_{n=1}^{\infty}$ be an approximate unit of I consisting of projections which is quasicentral in A. Observe first that $(pp_np)_{n=1}^{\infty}$ is an approximate unit of pIp. Indeed, for any $x \in I$ we have

$$pp_np \cdot pxp = pp_n(pxp) \to p \cdot (pxp) = pxp$$

since $pxp \in I$ and $(p_n)_{n=1}^{\infty}$ is an approximate unit of I.

Also, one can see that $(pp_np)_{n=1}^{\infty}$ is quasicentral in pAp. Indeed, for any $a \in A$ we have

$$pp_np \cdot pap - pap \cdot pp_np = p(p_n \cdot pap - pap \cdot p_n) - (pap \cdot p_n - p_n \cdot pap)p$$
$$+ (pap \cdot p_n - p_n \cdot pap) \to 0$$

since $(p_n)_{n=1}^{\infty}$ is quasicentral in A.

On the other hand, using again the fact that $(p_n)_{n=1}^{\infty}$ is quasicentral in A we have

$$(pp_np)^2 - pp_np = p(p_np - pp_n)p_np \to 0$$

(since $\|p_n\| \leq 1, n \in \mathbb{N}$) and obviously $(pp_np)^* = pp_np, n \in \mathbb{N}$. Hence, by functional calculus, we find projections r_n in pIp, $n \in \mathbb{N}$, such that

$$pp_np - r_n \to 0.$$

It follows that $(r_n)_{n=1}^{\infty}$ is an approximate unit of pIp consisting of projections which is quasicentral in pAp. This ends the proof.

(2) By (1) in this Lemma it follows that pIp has a countable approximate unit consisting of projections. Then, by **[11]** (or **[15]**) it follows that pIp has an increasing approximate unit $(r_n)_{n=1}^{\infty}$ consisting of projections. Let $I = \varinjlim(I_n, \varphi_n)$ with $I_n = \oplus_{i=1}^{k_n} P_{n,i}C(X_{n,i}, M_{[n,i]})P_{n,i}$ where $X_{n,i}$ are connected, finite CW complexes and $P_{n,i}$ are projections in $C(X_{n,i}, M_{[n,i]})$. Then, after passing to a subsequence of $(I_n)_{n=1}^{\infty}$, we may suppose that there are projections $s_n \in I_n$ such that $\varphi_{n,\infty}(s_n) = r_n$ and $\varphi_n(s_n) \leq s_{n+1}$ (here $\varphi_{n,\infty} : I_n \to I = \varinjlim(I_m, \varphi_m)$ is the canonical map).

Therefore

$$pIp = \overline{\bigcup_{n\geq 1} r_n(pIp)r_n} = \overline{\bigcup_{n\geq 1} r_nIr_n} = \varinjlim(J_n = s_nI_ns_n, \varphi_{n|J_n})$$

and hence pIp is an AH algebra. The fact that pIp has the ideal property follows now

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from the equivalence (1) $\Leftrightarrow$ (2) in Theorem 2.1 (i.e., [**12**, Theorem 3.1]). □

Lemma 3.8 (compare with [**3**, Lemma 3.13]; see also [**15**, 2.5]). *Let*

$$0 \to I \to A \to E \to 0$$

be a quasidiagonal extension of C^*-algebras. If B is a hereditary C^*-subalgebra of A, then every projection in $B/B \cap I(= B + I/I)$ that lifts to a projection in A can be lifted to a projection in B.

Proof. This follows as in the proof of [**3**, Lemma 3.13] but using the above Lemma 3.7(1) instead of [**3**, Theorem 2.6 (iii)]. □

Lemma 3.9 (compare with [**3**, Lemma 3.15], [**15**, Lemma 2.5] and [**6**, Lemma 9.8]). *Let*

$$0 \to I \to A \to B \to 0$$

be a quasidiagonal extension of C^*-algebras. Then:

(1) *Any projection in B lifts to a projection in A.*

(2) *Any two mutually orthogonal projections in B lift to two mutually orthogonal projections in A.*

(3) *Any set of matrix units $\{f_{ij}\}_{i,j=1}^n$ in B lifts to a set of matrix units $\{e_{ij}\}_{i,j=1}^n$ in A.*

Proof. (1) This follows as in the proof of [**3**, Lemma 3.15] but using the above Lemma 3.8 instead of [**3**, Lemma 3.13] and using also Theorem 2.3 (i.e., [**2**, Theorem 8]) to deduce that the induced homomorphism from $K_0(A)$ to $K_0(B)$ is surjective (or, equivalently, that the index map $\delta_0 : K_0(B) \to K_1(I)$ is zero).

(2) This follows as in the proof of [**15**, Lemma 2.5 (1)] (using Lemma 3.7 (1)).

(3) This follows as in the proof of [**6**, Lemma 9.8] but using Lemma 3.9 (1) instead of [**6**, Lemma 9.7] and exploiting Lemma 3.7 (1). □

Lemma 3.10 (compare with [**10**, Lemma 6]). *Let*

$$0 \to I \to A \to B \to 0 \qquad (**)$$

be a quasidiagonal extension of C^*-algebras, where I and B have stable rank one. Let B_0 be a C^*-subalgebra of B which is a quotient of a circle algebra.

Then, there is a homomorphism $B_0 \to A$ which composed with the epimorphism $A \to B$ is the identity map on B_0.

Proof. This follows as in the proof of [**10**, Lemma 6] but using Lemma 3.9 instead of [**6**, Lemma 9.8] and using also Lemma 3.7 (1) and [**10**, Proposition 4]. (Note that since the extension (∗∗) is quasidiagonal, Theorem 2.3 (i.e., [**2**, Theorem 8]) implies that the index map $\delta_1 : K_1(B) \to K_0(I)$ is zero and hence A has stable rank one by [**10**, Proposition 4 (ii)]). □

Proof of Theorem 3.6. (1) ⇒ (2) follows from Proposition 2.4 (i.e., [**2**, Proposition 11]) and Theorem 2.1 (i.e., [**12**, Theorem 3.1]).

(2) ⇒ (1) The proof of the fact that A is an $A\mathbf{T}$ algebra relies on [**7**, Theorem 4.3] and it follows as in the proof of [**10**, Theorem 5] using the above Lemma 3.10 instead of [**10**, Lemma 6] (note that I and B have stable rank one) and observing that since the extension (∗) is quasidiagonal then I has a countable approximate unit of projections which is quasicentral in A, and hence, in particular, it commutes asymptotically with any C^*-subalgebra of A which is isomorphic to a quotient of a circle algebra. The fact that the $A\mathbf{T}$ algebra A has the ideal property follows from Theorem 3.1.

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