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BRANCHING BROWNIAN MOTION IN A PERIODIC ENVIRONMENT AND UNIQUENESS OF PULSATING TRAVELING WAVES

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Abstract

Using one-dimensional branching Brownian motion in a periodic environment, we give probabilistic proofs of the asymptotics and uniqueness of pulsating traveling waves of the Fisher–Kolmogorov–Petrovskii–Piskounov (F-KPP) equation in a periodic environment. This paper is a sequel to 'Branching Brownian motion in a periodic environment and existence of pulsating travelling waves' (Ren *et al.*, 2022), in which we proved the existence of the pulsating traveling waves in the supercritical and critical cases, using the limits of the additive and derivative martingales of branching Brownian motion in a periodic environment.

Keywords: F-KPP equation; asymptotic behavior; Bessel-3 process; martingale change of measures

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1. Introduction

McKean [15] established the connection between branching Brownian motion (BBM) and the Fisher–Kolmogorov–Petrovskii–Piskounov (F-KPP) reaction-diffusion equation

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \beta(\mathbf{f}(\mathbf{u}) - \mathbf{u}), \tag{1.1}$$

where $\mathbf{f}$ is the generating function of the offspring distribution and β is the (constant) branching rate of BBM. The F-KPP equation has been studied intensively using both analytic techniques (see, for example, Kolmogorov *et al.* [12] and Fisher [6]) and probabilistic methods (see, for instance, McKean [15], Bramson [2, 3], Harris [10], and Kyprianou [13]).

A traveling wave solution of (1.1) with speed c is a solution of the following equation:

$$\frac{1}{2}\Phi_c'' + c\Phi_c' + \beta(\mathbf{f}(\Phi_c) - \Phi_c) = 0. \tag{1.2}$$

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If Φ_c is a solution of (1.2), then $\mathbf{u}(t, x) = \Phi_c(x - ct)$ satisfies (1.1). Using the relation between the F-KPP equation (1.1) and BBM, Kyprianou [13] gave probabilistic proofs of the existence, asymptotics, and uniqueness of traveling wave solutions. In this paper, we study the following more general F-KPP equation, in which the constant β is replaced by a continuous and 1-periodic function $\mathbf{g}$:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g} \cdot (\mathbf{f}(\mathbf{u}) - \mathbf{u}), \tag{1.3}$$

where $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R} \to [0, 1]$. In [16], we have shown that the above equation is related to branching Brownian motion in a periodic environment.

Now we describe branching Brownian motion in a periodic environment. Initially there is a single particle v at the origin of the real line. This particle moves as a standard Brownian motion $B = \{B(t), t \geq 0\}$ and produces a random number of offspring, $1 + L$, after a random time η_v. We assume that L has distribution $\{p_k, k \geq 0\}$ with $m := \sum_{k\geq 0} kp_k \in (0, \infty)$. Let b_v and d_v be the birth time and death time, respectively, of the particle v, and let $X_v(s)$ be the location of the particle v at time s; then $\eta_v = d_v - b_v$, the lifetime of v, satisfies

$$\mathbb{P}_x(\eta_v > t \mid b_v, \{X_v(s) : s \geq b_v\}) = \exp\left\{-\int_{b_v}^{b_v+t} \mathbf{g}(X_v(s))\mathrm{d}s\right\},$$

where we assume the branching rate function $\mathbf{g} \in C^1(\mathbb{R})$ is strictly positive and 1-periodic. Starting from their points of creation, each of these children evolves independently.

Let N_t be the set of particles alive at time t, and let $X_u(s)$ be the position of the particle u or its ancestor at time s for any $u \in N_t$, $s \leq t$. Define

$$Z_t = \sum_{u\in N_t} \delta_{X_u(t)}$$

and $\mathcal{F}_t = \sigma(Z_s : s \leq t)$. Then $\{Z_t : t \geq 0\}$ is called a branching Brownian motion in a periodic environment (BBMPE). Let $\mathbb{P}_x$ be the law of $\{Z_t : t \geq 0\}$ when the initial particle starts at $x \in \mathbb{R}$, that is, $\mathbb{P}_x(Z_0 = \delta_x) = 1$, and let $\mathbb{E}_x$ be expectation with respect to $\mathbb{P}_x$. For simplicity, $\mathbb{P}_0$ and $\mathbb{E}_0$ will be written as $\mathbb{P}$ and $\mathbb{E}$, respectively. Notice that the distribution of L does not depend on the spatial location. In the remainder of this paper, expectations with respect to L will be written as $\mathbf{E}$. The notation in this paper is the same as that in [16].

As stated in [16], the F-KPP equation related to BBMPE is given by (1.3) with $\mathbf{f}(s) = \mathbf{E}(s^{L+1})$. Traveling wave solutions, that is, solutions satisfying (1.2), do not exist. However, we can consider so-called pulsating traveling waves, that is, solutions $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R} \to [0, 1]$ to (1.3) satisfying

$$\mathbf{u}\left(t + \frac{1}{\nu}, x\right) = \mathbf{u}(t, x - 1), \tag{1.4}$$

as well as the boundary condition

$$\lim_{x\to-\infty} \mathbf{u}(t, x) = 0, \qquad \lim_{x\to+\infty} \mathbf{u}(t, x) = 1,$$

when $\nu > 0$, and

$$\lim_{x\to-\infty} \mathbf{u}(t, x) = 1, \qquad \lim_{x\to+\infty} \mathbf{u}(t, x) = 0,$$

when $\nu < 0$. The quantity ν is called the wave speed. It is known that there is a constant $\nu^* > 0$ (defined below) such that when $|\nu| < \nu^*$ (called the subcritical case) no such solution exists,

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whereas for each $|\nu| \geq \nu^*$ (where $|\nu| > \nu^*$ is called the supercritical case and $|\nu| = \nu^*$ is called the critical case) there exists a unique (up to time-shift) pulsating traveling wave (see Hamel *et al.* [8]).

In [16], we studied the limits of the additive and derivative martingales of BBMPE, and by using these limits we gave a probabilistic proof of the existence of pulsating traveling waves. In this paper, using the relation between BBMPE and the related F-KPP equation, we give probabilistic proofs of the asymptotics and uniqueness of pulsating traveling waves. These extend the results of Kyprianou [13] for classical BBM to BBMPE. However, the methods in Kyprianou [13] do not work for BBMPE. We will therefore adapt ideas from [10]. The non-homogeneous nature of the environment makes the actual arguments much more delicate.

Before stating our main results, we first introduce the minimal speed ν^*. For every $\lambda \in \mathbb{R}$, let $\gamma(\lambda)$ and $\psi(\cdot, \lambda)$ be the principal eigenvalue and the corresponding positive eigenfunction of the periodic problem: for all $x \in \mathbb{R}$,

$$\frac{1}{2}\psi_{xx}(x, \lambda) - \lambda\psi_x(x, \lambda) + \left(\frac{1}{2}\lambda^2 + m\mathbf{g}(x)\right)\psi(x, \lambda) = \gamma(\lambda)\psi(x, \lambda),$$
$$\psi(x+1, \lambda) = \psi(x, \lambda).$$

We normalize $\psi(\cdot, \lambda)$ so that $\int_0^1 \psi(x, \lambda)dx = 1$. Define

$$\nu^* := \min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}, \qquad \lambda^* := \arg\min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}.$$

Then ν^* is the minimal wave speed (see [8]), and the existence of λ^* is proved in [14].

Using the property $\mathbf{u}\left(t + \frac{1}{\nu}, x\right) = \mathbf{u}(t, x-1)$, we can define $\mathbf{u}(-t, x)$ for any $t > 0$. To be more specific, let $\lceil x \rceil$ be the smallest integer greater than or equal to x, and let $\lfloor x \rfloor$ be the integral part of x. When $\nu > 0$, define

$$\mathbf{u}(-t, x) = \mathbf{u}\left(-t + \frac{\lceil \nu t \rceil}{\nu}, x + \lceil \nu t \rceil\right), \quad t > 0, \; x \in \mathbb{R}.$$

When $\nu < 0$, define

$$\mathbf{u}(-t, x) = \mathbf{u}\left(-t + \frac{\lfloor \nu t \rfloor}{\nu}, x + \lfloor \nu t \rfloor\right), \quad t > 0, \; x \in \mathbb{R}.$$

Then $\mathbf{u}(t, x)$ satisfies the F-KPP equation (1.3) and (1.4) in $\mathbb{R} \times \mathbb{R}$.

Our first two main results give the asymptotic behaviors of pulsating traveling waves in the supercritical case of $|\nu| > \nu^*$ and the critical case of $|\nu| = \nu^*$.

Theorem 1.1. *Suppose* $\mathbf{u}(t, x)$ *is a pulsating traveling wave with speed* $\nu > \nu^*$ *and* $\lambda \in (0, \lambda^*)$ *satisfies* $\nu = \frac{\gamma(\lambda)}{\lambda}$. *If* $\mathbf{E}(L \log^+ L) < +\infty$, *then there exists* $\beta > 0$ *such that*

$$1 - \mathbf{u}\left(\frac{y-x}{\nu}, y\right) \sim \beta e^{-\lambda x}\psi(y, \lambda) \quad \textit{uniformly in } y \in [0, 1] \textit{ as } x \to +\infty.$$

Theorem 1.2. *Suppose* $\mathbf{u}(t, x)$ *is a pulsating traveling wave with speed* $\nu = \nu^*$. *If* $\mathbf{E}\big(L(\log^+ L)^2\big) < \infty$, *then there exists* $\beta > 0$ *such that*

$$1 - \mathbf{u}\left(\frac{y-x}{\nu^*}, y\right) \sim \beta x e^{-\lambda^* x}\psi(y, \lambda^*) \quad \textit{uniformly in } y \in [0, 1] \textit{ as } x \to +\infty.$$

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Remark 1.1. By symmetry, we also have the asymptotic behaviors of pulsating traveling waves with negative speed. In the supercritical case of $\nu < -\nu^*$, suppose $\mathbf{u}(t, x)$ is a pulsating traveling wave with speed ν and $\lambda \in (-\lambda^*, 0)$ satisfies $\nu = \frac{\gamma(\lambda)}{\lambda}$. If $\mathbf{E}(L \log^+ L) < +\infty$, then there exists $\beta > 0$ such that

$$1 - \mathbf{u}\left(\frac{y-x}{\nu}, y\right) \sim \beta e^{-\lambda x}\psi(y, \lambda) \quad \text{uniformly in } y \in [0, 1] \text{ as } x \to -\infty.$$

In the critical case, suppose $\mathbf{u}(t, x)$ is a pulsating traveling wave with speed $-\nu^*$. If $\mathbf{E}\big(L(\log^+ L)^2\big) < \infty$, then there exists $\beta > 0$ such that

$$1 - \mathbf{u}\left(\frac{y-x}{-\nu^*}, y\right) \sim \beta |x| e^{\lambda^* x}\psi(y, -\lambda^*) \quad \text{uniformly in } y \in [0, 1] \text{ as } x \to -\infty.$$

For any $\lambda \in \mathbb{R}$, define

$$W_t(\lambda) = e^{-\gamma(\lambda)t} \sum_{u \in N_t} e^{-\lambda X_u(t)} \psi(X_u(t), \lambda) \tag{1.5}$$

and

$$\partial W_t(\lambda) := e^{-\gamma(\lambda)t} \sum_{u \in N_t} e^{-\lambda X_u(t)} \big(\psi(X_u(t), \lambda)(\gamma'(\lambda)t + X_u(t)) - \psi_\lambda(X_u(t), \lambda)\big). \tag{1.6}$$

It follows from [16, Theorem 1.1] that, for any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{(W_t(\lambda))_{t \ge 0}, \mathbb{P}_x\}$ is a martingale, called the additive martingale. The limit $W(\lambda, x) := \lim_{t \uparrow \infty} W_t(\lambda)$ exists $\mathbb{P}_x$-almost surely (a.s.). Moreover, $W(\lambda, x)$ is an $L^1(\mathbb{P}_x)$-limit when $|\lambda| < \lambda^*$ and $\mathbf{E}(L \log^+ L) < \infty$; and $W(\lambda, x) = 0$ $\mathbb{P}_x$-a.s. when $|\lambda| \ge \lambda^*$ or $|\lambda| < \lambda^*$ and $\mathbf{E}(L \log^+ L) = \infty$. It follows from [16, Theorem 1.2] that, for any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{(\partial W_t(\lambda))_{t \ge 0}, \mathbb{P}_x\}$ is a martingale, called the derivative martingale. For all $|\lambda| \ge \lambda^*$, the limit $\partial W(\lambda, x) := \lim_{t \uparrow \infty} \partial W_t(\lambda)$ exists $\mathbb{P}_x$-a.s. Moreover, if $\mathbf{E}\big(L(\log^+ L)^2\big) < \infty$, then $\partial W(\lambda, x) \in (0, \infty)$ when $\lambda = \lambda^*$, and $\partial W(\lambda, x) \in (-\infty, 0)$ when $\lambda = -\lambda^*$. If $|\lambda| > \lambda^*$ or $|\lambda| = \lambda^*$ and $\mathbf{E}\big(L(\log^+ L)^2\big) = \infty$, then $\partial W(\lambda, x) = 0$ $\mathbb{P}_x$-a.s.

Using Theorem 1.1, Theorem 1.2, and [16, Theorem 1.3], we can prove the following result, which gives the existence and uniqueness of pulsating traveling waves.

Theorem 1.3. (*i*) ***Supercritical case.*** *If* $|\nu| > \nu^*$ *and* $\mathbf{E}(L \log^+ L) < \infty$, *then there is a unique (up to time-shift) pulsating traveling wave with speed* ν, *given by*

$$\mathbf{u}(t, x) = \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda)t} W(\lambda, x)\right\}\right),$$

where $|\lambda| \in (0, \lambda^*)$ *is such that* $\nu = \frac{\gamma(\lambda)}{\lambda}$.

(*ii*) ***Critical case.*** *If* $|\nu| = \nu^*$ *and* $\mathbf{E}\big(L(\log^+ L)^2\big) < \infty$, *then there is a unique (up to time-shift) pulsating traveling wave with speed* ν, *given by*

$$\mathbf{u}(t, x) = \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda)t} \partial W(\lambda, x)\right\}\right),$$

where $\lambda = \lambda^*$ *if* $\nu = \nu^*$, *and* $\lambda = -\lambda^*$ *if* $\nu = -\nu^*$.

Theorem 1.1, Theorem 1.2, and the uniqueness in Theorem 1.3 were proved analytically in [7, 9] under slightly different assumptions. The probabilistic representation in Theorem 1.3 is new. For a detailed comparison, see Remark 4.1 at the end of Section 4.

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2. Preliminaries

2.1. Properties of principal eigenvalue and eigenfunction

In this section, we recall some properties of $\gamma(\lambda)$ and $\psi(x, \lambda)$ from [16]. By [16, Lemma 2.1], the function γ is analytic, strictly convex, and even on $\mathbb{R}$. There exists a unique $\lambda^* > 0$ such that

$$\nu^* = \frac{\gamma(\lambda^*)}{\lambda^*} = \min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda} > 0.$$

Furthermore,

$$\lim_{\lambda\to-\infty} \gamma'(\lambda) = -\infty, \qquad \lim_{\lambda\to+\infty} \gamma'(\lambda) = +\infty.$$

By [16, Lemma 2.2], we have $\gamma'(\lambda^*) = \dfrac{\gamma(\lambda^*)}{\lambda^*}$,

$$\gamma'(\lambda) < \frac{\gamma(\lambda)}{\lambda} \quad \text{on } (0, \lambda^*), \quad \text{and} \quad \gamma'(\lambda) > \frac{\gamma(\lambda)}{\lambda} \quad \text{on } (\lambda^*, \infty). \tag{2.1}$$

By [16, Lemma 2.5], we have that $\psi(x, \cdot) \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$, and $\psi_\lambda(x, \lambda)$ satisfies

$$\frac{1}{2}\psi_{\lambda xx}(x, \lambda) - \psi_x(x, \lambda) - \lambda\psi_{\lambda x}(x, \lambda) + \left(\frac{1}{2}\lambda^2 + m\mathbf{g}(x)\right)\psi_\lambda(x, \lambda) + \lambda\psi(x, \lambda)$$
$$= \gamma(\lambda)\psi_\lambda(x, \lambda) + \gamma'(\lambda)\psi(x, \lambda).$$

Define

$$\phi(x, \lambda) := e^{-\lambda x}\psi(x, \lambda), \qquad x \in \mathbb{R}.$$

Then $\phi(x, \lambda)$ satisfies

$$\frac{1}{2}\phi_{xx}(x, \lambda) + m\mathbf{g}(x)\phi(x, \lambda) = \gamma(\lambda)\phi(x, \lambda), \tag{2.2}$$

and $\phi_\lambda(x, \lambda)$ satisfies

$$\frac{1}{2}\phi_{\lambda xx}(x, \lambda) + m\mathbf{g}(x)\phi_\lambda(x, \lambda) = \gamma'(\lambda)\phi(x, \lambda) + \gamma(\lambda)\phi_\lambda(x, \lambda). \tag{2.3}$$

Define

$$h(x) := x - \frac{\psi_\lambda(x, \lambda)}{\psi(x, \lambda)}; \tag{2.4}$$

we also have

$$h(x) = -\frac{\phi_\lambda(x, \lambda)}{\phi(x, \lambda)}.$$

It is easy to see that h' is 1-periodic and continuous, and [16, Lemma 2.10] shows that h' is strictly positive.

2.2. Measure change for Brownian motion

Martingale change of measures for Brownian motion will play an important role in our arguments. In this section, we state the results of [16] about martingale change of measures.

Let $\{B_t, t \geq 0; \Pi_x\}$ be a standard Brownian motion starting from x. Define

$$\Xi_t(\lambda) := e^{-\gamma(\lambda)t - \lambda B_t + m\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\psi(B_t, \lambda);$$

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then by [16, Lemma 2.6], $\{\Xi_t(\lambda), t \geq 0\}$ is a Π_x-martingale. Define a probability measure Π_x^λ by

$$\left.\frac{d\Pi_x^\lambda}{d\Pi_x}\right|_{\mathcal{F}_t^B} = \frac{\Xi_t(\lambda)}{\Xi_0(\lambda)}, \tag{2.5}$$

where $\{\mathcal{F}_t^B : t \geq 0\}$ is the natural filtration of Brownian motion. We have shown in [16] that $\{B_t, \Pi_x^\lambda\}$ is a diffusion with infinitesimal generator

$$(\mathcal{A}f)(x) = \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2} + \left(\frac{\psi_x(x,\lambda)}{\psi(x,\lambda)} - \lambda\right)\frac{\partial f(x)}{\partial x}. \tag{2.6}$$

In the remainder of this paper, we always assume that $\{Y_t, t \geq 0; \Pi_x^\lambda\}$ *is a diffusion with infinitesimal generator* (2.6). It follows from [16, Lemma 2.8] that, for any $x \in \mathbb{R}$,

$$\frac{Y_t}{t} \to -\gamma'(\lambda), \qquad \text{as } t \to \infty, \ \Pi_x^\lambda\text{-a.s.} \tag{2.7}$$

Define

$$M_t := \gamma'(\lambda)t + h(Y_t) - h(Y_0), \qquad t \geq 0.$$

By [16, Lemma 2.12], $\{M_t, t \geq 0; \Pi_x^\lambda\}$ is a martingale. Moreover, there exist two constants $c_2 > c_1 > 0$ such that the quadratic variation $\langle M\rangle_t$ satisfies

$$\langle M\rangle_t = \int_0^t \left(h'(Y_s)\right)^2 ds \in [c_1 t, c_2 t]. \tag{2.8}$$

For any $x \in \mathbb{R}$, define an $\{\mathcal{F}_t^B\}$ stopping time

$$\tau_\lambda^x := \inf\{t \geq 0 : h(B_t) \leq -x - \gamma'(\lambda)t\}. \tag{2.9}$$

Define

$$\Lambda_t^{(x,\lambda)} := e^{-\gamma(\lambda)t - \lambda B_t + m\int_0^t \mathbf{g}(B_s)ds}\psi(B_t,\lambda)\left(x + \gamma'(\lambda)t + h(B_t)\right)\mathbf{1}_{\{\tau_\lambda^x > t\}}; \tag{2.10}$$

then [16, Lemma 2.11] shows that for any $x, y \in \mathbb{R}$ with $y > h^{-1}(-x)$, $\{\Lambda_t^{(x,\lambda)}, t \geq 0\}$ is a Π_y-martingale. For $x, y \in \mathbb{R}$ with $y > h^{-1}(-x)$, define a new probability measure $\Pi_y^{(x,\lambda)}$ by

$$\left.\frac{d\Pi_y^{(x,\lambda)}}{d\Pi_y}\right|_{\mathcal{F}_t^B} = \frac{\Lambda_t^{(x,\lambda)}}{\Lambda_0^{(x,\lambda)}}. \tag{2.11}$$

By [16, Section 2.2], if $\{B_t, t \geq 0; \Pi_y\}$ is a standard Brownian motion starting at y, then $\left\{x + h(y) + M_{T(t)}, t \geq 0; \Pi_y^{(x,\lambda)}\right\}$ is a standard Bessel-3 process starting at $x + h(y)$, where $M_t = \gamma'(\lambda)t + h(B_t) - h(B_0)$ and

$$T(s) = \inf\{t > 0 : \langle M\rangle_t > s\} = \inf\left\{t > 0 : \int_0^t \left(h'(B_r)\right)^2 dr > s\right\}.$$

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3. Proof of Theorem 1.1

Proof of Theorem 1.1. We fix $\nu > \nu^*$ in this proof and so λ is also fixed. We will prove the theorem in five steps. In the first four steps, we assume the number of offspring is 2, that is, $L = 1$. In the last step, we prove the result for general L.

Step 1. Suppose $L = 1$ and thus $m = 1$. Let $\mathbf{w}(t, x) = 1 - \mathbf{u}(t, x)$; then $\mathbf{w}(t, x)$ satisfies

$$\begin{cases} \frac{\partial \mathbf{w}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{w}}{\partial x^2} + \mathbf{g}\cdot\left(\mathbf{w} - \mathbf{w}^2\right), \\ \mathbf{w}\left(t + \frac{1}{\nu}, x\right) = \mathbf{w}(t, x - 1), \end{cases} \tag{3.1}$$

for $t \geq 0$, $x \in \mathbb{R}$. Define

$$\mathbf{w}(-t, x) = \mathbf{w}\left(-t + \frac{\lceil \nu t\rceil}{\nu}, x + \lceil \nu t\rceil\right), \quad \text{for } t > 0,\ x \in \mathbb{R}.$$

By the periodicity of $\mathbf{w}$, we get that $\mathbf{w}(t, x)$ satisfies (3.1). Put

$$\widetilde{\mathbf{w}}(t, x) := \frac{e^{\lambda x - \gamma(\lambda)t}\mathbf{w}(t, x)}{\psi(x, \lambda)}. \tag{3.2}$$

Recall that $\left\{Y_t, \Pi_x^\lambda\right\}$ is a diffusion with infinitesimal generator (2.6). For $t \geq 0$, let $Y_{[0,t]}$ be the restriction of Y to $[0, t]$. Define

$$f\left(Y_{[0,t]}\right) = \widetilde{\mathbf{w}}(-t, Y_t)e^{-\int_0^t \mathbf{g}(Y_s)\mathbf{w}(-s, Y_s)\mathrm{d}s}, \quad t \geq 0. \tag{3.3}$$

In this step we prove that $\left\{\left(f\left(Y_{[0,t]}\right)\right)_{t\geq 0}, \Pi_x^\lambda\right\}$ is a positive martingale.

By the Feynman–Kac formula, we have

$$\mathbf{w}(T, x) = \Pi_x\left[\mathbf{w}(T - t, B_t)e^{\int_0^t \mathbf{g}(B_s)(1-\mathbf{w}(T-s, B_s))\mathrm{d}s}\right], \quad \text{for } T \in \mathbb{R}, t > 0. \tag{3.4}$$

Recall that, since $m = 1$,

$$\left.\frac{\mathrm{d}\Pi_x^\lambda}{\mathrm{d}\Pi_x}\right|_{\mathcal{F}_t^B} = \frac{\Xi_t(\lambda)}{\Xi_0(\lambda)} = \frac{e^{-\gamma(\lambda)t - \lambda B_t + \int_0^t \mathbf{g}(B_s)\mathrm{d}s}\psi(B_t, \lambda)}{e^{-\lambda x}\psi(x, \lambda)}.$$

Therefore,

$$\begin{aligned} \mathbf{w}(T, x) &= \Pi_x^\lambda\left[\frac{\Xi_0(\lambda)}{\Xi_t(\lambda)}\mathbf{w}(T - t, B_t)e^{\int_0^t \mathbf{g}(B_s)(1-\mathbf{w}(T-s, B_s))\mathrm{d}s}\right] \\ &= \Pi_x^\lambda\left[e^{-\lambda x}\psi(x, \lambda)\frac{e^{\lambda B_t + \gamma(\lambda)t - \int_0^t \mathbf{g}(B_s)\mathrm{d}s}\mathbf{w}(T - t, B_t)}{\psi(B_t, \lambda)}e^{\int_0^t \mathbf{g}(B_s)(1-\mathbf{w}(T-s, B_s))\mathrm{d}s}\right] \\ &= \Pi_x^\lambda\left[e^{-\lambda x + \gamma(\lambda)T}\psi(x, \lambda)\frac{e^{\lambda B_t - \gamma(\lambda)(T-t)}\mathbf{w}(T - t, B_t)}{\psi(B_t, \lambda)}e^{-\int_0^t \mathbf{g}(B_s)\mathbf{w}(T-s, B_s)\mathrm{d}s}\right]. \end{aligned}$$

Thus we have

$$\widetilde{\mathbf{w}}(T, x) = \Pi_x^\lambda\left[\widetilde{\mathbf{w}}(T - t, B_t)e^{-\int_0^t \mathbf{g}(B_s)\mathbf{w}(T-s, B_s)\mathrm{d}s}\right].$$

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Note that both $\{B_t, \Pi_x^\lambda\}$ and $\{Y_t, \Pi_x^\lambda\}$ are diffusions with infinitesimal generator $\mathcal{A}$. Thus

$$\widetilde{\mathbf{w}}(T, x) = \Pi_x^\lambda\left[\widetilde{\mathbf{w}}(T-t, Y_t)e^{-\int_0^t \mathbf{g}(Y_s)\mathbf{w}(T-s, Y_s)\mathrm{d}s}\right]. \tag{3.5}$$

It follows from $\nu = \frac{\gamma(\lambda)}{\lambda}$ that

$$\widetilde{\mathbf{w}}\left(t+\frac{1}{\nu}, x+1\right) = \frac{e^{\lambda(x+1)-\gamma(\lambda)\left(t+\frac{1}{\nu}\right)}\mathbf{w}\left(t+\frac{1}{\nu}, x+1\right)}{\psi(x+1, \lambda)} = \frac{e^{\lambda x-\gamma(\lambda)t}\mathbf{w}(t, x)}{\psi(x, \lambda)} = \widetilde{\mathbf{w}}(t, x). \tag{3.6}$$

For $0 < s < t$, we have

$$\begin{aligned}
&\Pi_x^\lambda\left[f\left(Y_{[0,t]}\right)\middle|\mathcal{F}_s\right] = \Pi_x^\lambda\left[\widetilde{\mathbf{w}}(-t, Y_t)e^{-\int_0^t \mathbf{g}(Y_r)\mathbf{w}(-r, Y_r)\mathrm{d}r}\middle|\mathcal{F}_s\right]\\
&= e^{-\int_0^s \mathbf{g}(Y_r)\mathbf{w}(-r, Y_r)\mathrm{d}r}\Pi_{Y_s}^\lambda\left[\widetilde{\mathbf{w}}(-t, Y_{t-s})e^{-\int_0^{t-s} \mathbf{g}(Y_r)\mathbf{w}(-(r+s), Y_r)\mathrm{d}r}\right]\\
&= e^{-\int_0^s \mathbf{g}(Y_r)\mathbf{w}(-r, Y_r)\mathrm{d}r}\Pi_{Y_s}^\lambda\left[\widetilde{\mathbf{w}}\left(-s-(t-s), Y_{t-s}\right)e^{-\int_0^{t-s} \mathbf{g}(Y_r)\mathbf{w}(-s-r, Y_r)\mathrm{d}r}\right]\\
&= e^{-\int_0^s \mathbf{g}(Y_r)\mathbf{w}(-r, Y_r)\mathrm{d}r}\widetilde{\mathbf{w}}(-s, Y_s) = f(Y_{[0,s]}),
\end{aligned}$$

where the penultimate equality follows from (3.5) with $T = -s$. Hence the process $\left\{\left(f\left(Y_{[0,t]}\right)\right)_{t\geq 0}, \Pi_x^\lambda\right\}$ is a positive martingale.

Step 2. Suppose $L = 1$. In this step, we will show that there exists a constant $\beta \geq 0$ such that for any $x \in \mathbb{R}$,

$$\widetilde{\mathbf{w}}(-t, Y_t) \to \beta \quad \text{as } t \to \infty,\ \Pi_x^\lambda\text{-a.s.} \tag{3.7}$$

Moreover, we also prove that for any $x, T \in \mathbb{R}$,

$$\widetilde{\mathbf{w}}(T-t, Y_t) \to \beta \quad \text{as } t \to \infty,\ \Pi_x^\lambda\text{-a.s.} \tag{3.8}$$

It follows from (2.7) and (2.1) that

$$\lim_{s\to\infty} \frac{Y_s + \nu s}{s} = -\gamma'(\lambda) + \frac{\gamma(\lambda)}{\lambda} > 0.$$

Thus $\lim_{s\to\infty}(Y_s + \nu s) = \infty$. Since a positive martingale has a non-negative finite limit, taking logarithms in (3.3) and dividing by $Y_t + \nu t$ gives

$$\limsup_{t\to\infty}\left\{\frac{\ln \widetilde{\mathbf{w}}(-t, Y_t)}{Y_t+\nu t} - \frac{1}{Y_t+\nu t}\int_0^t \mathbf{g}(Y_s)\mathbf{w}(-s, Y_s)\mathrm{d}s\right\} \leq 0 \quad \Pi_x^\lambda\text{-a.s.} \tag{3.9}$$

Put $\|\mathbf{g}\|_\infty = \max_{x\in[0,1]} \mathbf{g}(x)$. Taking $T = t$ in (3.4), we get

$$\begin{aligned}
\mathbf{w}(t, x) &= \Pi_x\left[\mathbf{w}(0, B_t)e^{\int_0^t \mathbf{g}(B_s)(1-\mathbf{w}(t-s, B_s))\mathrm{d}s}\right] \leq \Pi_x\left[\mathbf{w}(0, B_t)e^{\|\mathbf{g}\|_\infty t}\right]\\
&\leq e^{\|\mathbf{g}\|_\infty/\nu}\Pi_0\left[\mathbf{w}(0, B_t+x)\right], \quad t \in \left[0, \frac{1}{\nu}\right].
\end{aligned}$$

Since $\mathbf{w}(0, x) \to 0$ as $x \to \infty$, we have, for any $\epsilon > 0$, $\mathbf{w}(0, x/2) \leq \epsilon/2$ when x is large enough. Since $\Pi_0(B_t + x \leq x/2) = \Pi_0(B_t \leq -x/2)$, we have $\Pi_0(B_t + x \leq x/2) \leq \epsilon/2$ for x large enough. Therefore, for x large enough,

$$\mathbf{w}(t, x) \leq e^{\|\mathbf{g}\|_\infty/\nu}\Pi_0(\mathbf{w}(0, B_t+x)) \leq e^{\|\mathbf{g}\|_\infty/\nu}\epsilon, \quad t \in \left[0, \frac{1}{\nu}\right].$$

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This implies that $\mathbf{w}(t, x) \to 0$ as $x \to \infty$ uniformly in $t \in \left[0, \frac{1}{\nu}\right]$. Combining this with the fact that

$$\mathbf{w}(-t, Y_t) = \mathbf{w}\left(-t + \frac{\lceil \nu t \rceil}{\nu}, Y_t + \lceil \nu t \rceil\right)$$

and $\lim_{t\to\infty} (Y_t + \lceil \nu t \rceil) = \infty$ Π_x^λ-a.s., we get

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{g}(Y_s)\mathbf{w}(-s, Y_s)\mathrm{d}s = 0, \quad \Pi_x^\lambda\text{-a.s.}$$

Therefore, by (3.9),

$$\limsup_{t\to\infty} \left\{ \frac{\ln \widetilde{\mathbf{w}}(-t, Y_t)}{Y_t + \nu t} \right\} \le 0, \quad \Pi_x^\lambda\text{-a.s.}$$

Hence by (3.2), we have

$$\limsup_{t\to\infty} \left\{ \frac{\ln \left[e^{\lambda Y_t + \gamma(\lambda)t}\mathbf{w}(-t, Y_t)/\psi(Y_t, \lambda)\right]}{Y_t + \nu t} \right\} \le 0, \quad \Pi_x^\lambda\text{-a.s.} \tag{3.10}$$

Since $\psi(\cdot, \lambda)$ is positive, continuous, and periodic, we have

$$0 < \inf_{x\in\mathbb{R}} \psi(x, \lambda) \le \sup_{x\in\mathbb{R}} \psi(x, \lambda) < \infty.$$

Combining this with $\nu = \frac{\gamma(\lambda)}{\lambda}$ and (3.10), we get that

$$\limsup_{t\to\infty} \left\{ \frac{\lambda(Y_t + \nu t) + \ln \mathbf{w}(-t, Y_t)}{Y_t + \nu t} \right\} \le 0, \quad \Pi_x^\lambda\text{-a.s.};$$

that is,

$$\limsup_{t\to\infty} \left\{ \frac{\ln \mathbf{w}(-t, Y_t)}{Y_t + \nu t} \right\} \le -\lambda, \quad \Pi_x^\lambda\text{-a.s.}$$

This implies that, for any $\delta > 0$ and Π_x^λ-a.s. all ω, there exists $C(\omega) > 0$ such that

$$\mathbf{w}(-t, Y_t(\omega)) \le C(\omega)e^{-(\lambda-\delta)(Y_t(\omega)+\nu t)}, \quad t \ge 0. \tag{3.11}$$

Therefore,

$$\int_0^\infty \mathbf{g}(Y_s)\mathbf{w}(-s, Y_s)\mathrm{d}s < +\infty, \quad \Pi_x^\lambda\text{-a.s.}$$

Consequently, by (3.3), $\widetilde{\mathbf{w}}(-t, Y_t)$ converges Π_x^λ-a.s. to some limit, say ξ_x.

Next we use a coupling method to prove that ξ_x is a constant Π_x^λ-a.s. Consider $\left\{\left(Y_t^1, Y_t^2\right), t \ge 0; \widetilde{\Pi}_{(x,y)}^\lambda\right\}$ with $\left\{Y_t^1, t \ge 0\right\}$ and $\left\{Y_t^2, t \ge 0\right\}$ being independent, and

$$\{Y_t^1, t \ge 0; \widetilde{\Pi}_{(x,y)}^\lambda\} \overset{d}{=} \{Y_t, t \ge 0; \Pi_x^\lambda\}, \quad \{Y_t^2, t \ge 0; \widetilde{\Pi}_{(x,y)}^\lambda\} \overset{d}{=} \{Y_t, t \ge 0; \Pi_y^\lambda\}.$$

Define

$$M_t^i = h\left(Y_t^i\right) + \gamma'(\lambda)t - h\left(Y_0^i\right), \quad i = 1, 2.$$

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Then $\left\{M_t^1, t \geq 0; \widetilde{\Pi}_{(x,y)}^\lambda\right\}$ and $\left\{M_t^2, t \geq 0; \widetilde{\Pi}_{(x,y)}^\lambda\right\}$ are independent martingales. Hence

$$\langle M^1 - M^2 \rangle_t = \langle M^1 \rangle_t + \langle M^2 \rangle_t \to \infty, \quad \text{as } t \to \infty.$$

By the Dambis–Dubins–Schwarz theorem, we get

$$\liminf_{t\to\infty}\left(M_t^1 - M_t^2\right) = -\infty \text{ and } \limsup_{t\to\infty}\left(M_t^1 - M_t^2\right) = +\infty, \quad \widetilde{\Pi}_{(x,y)}^\lambda\text{-a.s.} \tag{3.12}$$

Since $h(x) = x - \frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}$, we have

$$Y_t^1 - Y_t^2 = M_t^1 - M_t^2 + \frac{\psi_\lambda\left(Y_t^1, \lambda\right)}{\psi\left(Y_t^1, \lambda\right)} - \frac{\psi_\lambda\left(Y_t^2, \lambda\right)}{\psi\left(Y_t^2, \lambda\right)} + h\left(Y_0^1\right) - h\left(Y_0^2\right).$$

Combining (3.12) with the boundedness of $\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}$, we get

$$\liminf_{t\to\infty}\left(Y_t^1 - Y_t^2\right) = -\infty \text{ and } \limsup_{t\to\infty}\left(Y_t^1 - Y_t^2\right) = +\infty, \quad \widetilde{\Pi}_{(x,y)}^\lambda\text{-a.s.}$$

Define $E := \left\{\omega : \exists t_n = t_n(\omega) \to \infty \text{ with } Y_{t_n}^1 = Y_{t_n}^2 \text{ for all } n\right\}$. Then it follows from the display above that

$$\widetilde{\Pi}_{(x,y)}^\lambda(E) = 1. \tag{3.13}$$

If we use $\tilde{\xi}_x$ and $\tilde{\xi}_y$ respectively to denote the limits of $\widetilde{\mathbf{w}}\left(-t, Y_t^1\right)$ and $\widetilde{\mathbf{w}}\left(-t, Y_t^2\right)$ under $\widetilde{\Pi}_{(x,y)}^\lambda$, then (3.13) implies $\widetilde{\Pi}_{(x,y)}^\lambda\left(\tilde{\xi}_x = \tilde{\xi}_y\right) = 1$. Since $\tilde{\xi}_x$ and $\tilde{\xi}_y$ are independent, there is a constant $\beta \geq 0$ such that $\tilde{\xi}_x = \tilde{\xi}_y = \beta$. Since $\tilde{\xi}_x \overset{d}{=} \xi_x$, we have for any $x \in \mathbb{R}$, $\xi_x = \beta$, which means (3.7) holds.

Now we consider

$$f\left(Y_{[0,t]}, T\right) = \widetilde{\mathbf{w}}(T - t, Y_t)e^{-\int_0^t \mathbf{g}(Y_s)\mathbf{w}(T-s, Y_s)\mathrm{d}s}, \quad t \geq 0.$$

The proof in Step 1 also works if $f\left(Y_{[0,t]}\right)$ is replaced by $f(Y_{[0,t]}, T)$. Then, using the same argument as above, there exists another constant β_T such that

$$\widetilde{\mathbf{w}}(T - t, Y_t) \to \beta_T, \quad \Pi_x^\lambda\text{-a.s.}$$

We also have

$$\liminf_{t\to\infty}\left(Y_{t+T}^1 - Y_t^2\right) = -\infty \text{ and } \limsup_{t\to\infty}\left(Y_{t+T}^1 - Y_t^2\right) = +\infty.$$

Hence, if we put $E_T := \left\{\omega : \exists t_n = t_n(\omega) \to \infty \text{ with } Y_{t_n+T}^1 = Y_{t_n}^2 \text{ for all } n\right\}$, then

$$\widetilde{\Pi}_{(x,y)}^\lambda(E_T) = 1. \tag{3.14}$$

Notice that $Y_{t_n+T}^1 = Y_{t_n}^2$ implies $\widetilde{\mathbf{w}}\left(T - (t+T), Y_{t+T}^1\right) = \widetilde{\mathbf{w}}\left(-t, Y_t^2\right)$. Combining this with (3.14), we have $\beta_T = \beta$; that is, for any $x, T \in \mathbb{R}$, (3.8) holds.

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Step 3. In Step 2, we have shown that $\widetilde{\mathbf{w}}(-t, Y_t) \to \beta$ Π_x^λ-a.s., that is, $\widetilde{\mathbf{w}}$ converges along each path. In this step, we prove that

$$\widetilde{\mathbf{w}}(t, x) \to \beta \quad \text{uniformly in } t \in \left[0, \frac{1}{\nu}\right] \text{ as } x \to \infty, \tag{3.15}$$

and β is positive.

First, we prove that $\widetilde{\mathbf{w}}$ is bounded. Using the same notation as in Step 2, we also have, for any $k \in \mathbb{Z}$,

$$\left\langle M^1_{\cdot+\frac{k}{\nu}} - M^2_\cdot \right\rangle_t = \left\langle M^1_{\cdot+\frac{k}{\nu}} \right\rangle_t + \langle M^2_\cdot \rangle_t \to \infty, \quad \text{as } t \to \infty.$$

By the Dambis–Dubins–Schwarz theorem, it holds that

$$\liminf_{t\to\infty}\left(M^1_{t+\frac{k}{\nu}} - M^2_t\right) = -\infty \text{ and } \limsup_{t\to\infty}\left(M^1_{t+\frac{k}{\nu}} - M^2_t\right) = +\infty,$$

so the same conclusion holds for $Y^1_{t+\frac{k}{\nu}} - Y^2_t$. This implies, with $E_k := \Big\{\omega : \exists t_n = t_n(\omega) \to \infty$ with $Y^1_{t_n+\frac{k}{\nu}} + k = Y^2_{t_n}$ for all $n\Big\}$, that

$$\widetilde{\Pi}^\lambda_{(x,y)}(E_k) = 1. \tag{3.16}$$

Consider the event Ω_0 defined by

$$\bigcap_{k\in\mathbb{Z}} E_k \bigcap \left\{\lim_{t\to\infty} \widetilde{\mathbf{w}}\left(-t, Y^i_t\right) = \beta, \ \lim_{t\to\infty} \frac{Y^i_t}{t} = -\gamma'(\lambda), \text{ and } Y^i_t \text{ is continuous for } i = 1, 2\right\}.$$

By (2.7), (3.8), and (3.16), we get $\widetilde{\Pi}^\lambda_{(x,y)}(\Omega_0) = 1$. By (3.6), it suffices to show that $\widetilde{\mathbf{w}}(t, x)$ is bounded in $\left[0, \frac{1}{\nu}\right] \times \mathbb{R}$. Fix $\omega \in \Omega_0$, and consider the continuous curves

$$\mathcal{L}^i_k = \left\{\left(-t + \frac{k}{\nu}, Y^i_t(\omega) + k\right) : t \in \left(\frac{k-1}{\nu}, \frac{k}{\nu}\right)\right\}, \quad k \in \mathbb{N}, \ i = 1, 2.$$

In Step 2 we have shown $\lim_{t\to\infty} (Y^i_t(\omega) + \nu t) = \infty$ for $i = 1, 2$; thus for any x_0 large enough, there exist $k = k(\omega) > j = j(\omega) \in N$ such that

$$x_0 \le Y^1_t(\omega) + k \text{ for any } t \in \left[\frac{k-1}{\nu}, \frac{k}{\nu}\right], \text{ and } x_0 \ge Y^2_t(\omega) + j \text{ for any } t \in \left[\frac{j-1}{\nu}, \frac{j}{\nu}\right].$$

Define

$$\widetilde{\tau}(\omega) = \inf\left\{t \ge j/\nu : Y^1_{t+\frac{k-j}{\nu}}(\omega) + k = Y^2_t(\omega) + j\right\}.$$

By the definition of Ω_0, we know $\widetilde{\tau}(\omega) < +\infty$. Let $\widetilde{\mathcal{L}}$ denote the line segment

$$\left\{\frac{1}{\nu}\right\} \times \left[Y^2_{\frac{j-1}{\nu}}(\omega) + j, Y^1_{\frac{k-1}{\nu}}(\omega) + k\right],$$

and define the curves

$$\widetilde{\mathcal{L}}^1_k = \left\{\left(-t + \frac{k}{\nu}, Y^1_t(\omega) + k\right) : t \in \left[\frac{k-1}{\nu}, \widetilde{\tau}(\omega) + \frac{k-j}{\nu}\right]\right\},$$
$$\widetilde{\mathcal{L}}^2_j = \left\{\left(-t + \frac{j}{\nu}, Y^2_t(\omega) + j\right) : t \in \left[\frac{j-1}{\nu}, \widetilde{\tau}(\omega)\right]\right\}.$$

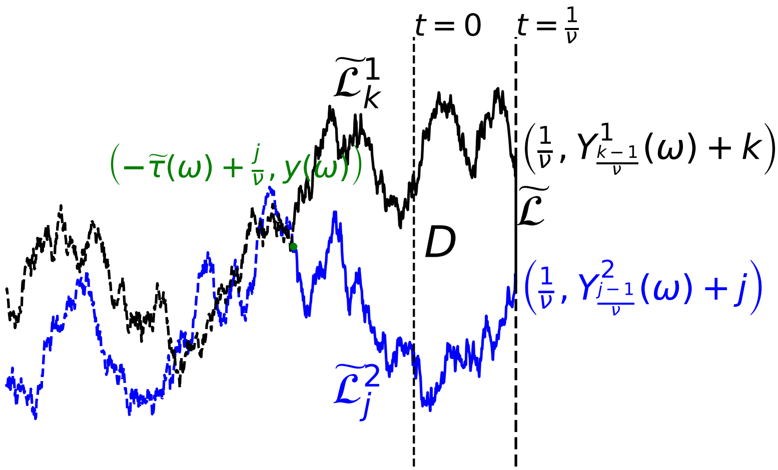

FIGURE 1. Bounded domain D with boundary $\widetilde{\mathcal{L}}\cup\widetilde{\mathcal{L}}_k^1\cup\widetilde{\mathcal{L}}_j^2$.

By the definition of $\widetilde{\tau}(\omega)$, we have

$$\left(-\left(\widetilde{\tau}(\omega)+\frac{k-j}{\nu}\right)+\frac{k}{\nu},\ Y^1_{\widetilde{\tau}(\omega)+\frac{k-j}{\nu}}(\omega)+k\right)=\left(-\widetilde{\tau}(\omega)+\frac{j}{\nu},\ Y^2_{\widetilde{\tau}(\omega)}(\omega)+j\right).$$

Define

$$y(\omega)=Y^1_{\widetilde{\tau}(\omega)+\frac{k-j}{\nu}}(\omega)+k;$$

then $y(\omega)=Y^2_{\widetilde{\tau}(\omega)}(\omega)+j$, and $\widetilde{\mathcal{L}}_k^1$ and $\widetilde{\mathcal{L}}_j^2$ intersect at the point $\left(-\widetilde{\tau}(\omega)+\frac{j}{\nu},\ y(\omega)\right)$. Combining (2.6), (3.5), and the Feynman–Kac formula, we have

$$\frac{\partial\widetilde{\mathbf{w}}}{\partial t}=\frac{1}{2}\frac{\partial^2\widetilde{\mathbf{w}}}{\partial x^2}+\left(\frac{\psi_x(x,\lambda)}{\psi(x,\lambda)}-\lambda\right)\frac{\partial\widetilde{\mathbf{w}}}{\partial x}-\mathbf{g}\mathbf{w}\widetilde{\mathbf{w}}.$$

Let D denote the bounded domain with boundary $\widetilde{\mathcal{L}}\cup\widetilde{\mathcal{L}}_k^1\cup\widetilde{\mathcal{L}}_j^2$ (see Figure 1). By the maximum principle, we have that $\widetilde{\mathbf{w}}$ attains its maximum in $\overline{D}$ on $\widetilde{\mathcal{L}}_k^1\cup\widetilde{\mathcal{L}}_j^2$, where $\overline{D}$ is the closure of D. Hence the maximum of $\widetilde{\mathbf{w}}$ on $\overline{D}$ is less than or equal to $K:=\max_{t\ge 0}\left\{\widetilde{\mathbf{w}}\left(-t,\ Y_t^1(\omega)\right),\ \widetilde{\mathbf{w}}\left(-t,\ Y_t^2(\omega)\right)\right\}$. By the continuity of $\widetilde{\mathbf{w}}$ and since

$$\lim_{t\to\infty}\widetilde{\mathbf{w}}(-t,\ Y_t^i(\omega))=\beta,\qquad i=1,2,$$

we get $K<\infty$. Notice that, fixing $\omega\in\Omega_0$, for any $t_0\in\left[0,\ \frac{1}{\nu}\right]$ and any x_0 large enough, there exist $k=k(\omega,x_0)$, $j=j(\omega,x_0)$ and a bounded domain $D=D(\omega,x_0)$ such that D has boundary $\widetilde{\mathcal{L}}\cup\widetilde{\mathcal{L}}_k^1\cup\widetilde{\mathcal{L}}_j^2$ and $(t_0,x_0)\in\overline{D}$. Thus $\widetilde{\mathbf{w}}(t_0,x_0)\le K$. Combining this with $\lim_{x\to-\infty}\widetilde{\mathbf{w}}(t,x)=0$ and the continuity of $\widetilde{\mathbf{w}}$, we get that $\widetilde{\mathbf{w}}(t,x)$ is bounded in $\left[0,\ \frac{1}{\nu}\right]\times\mathbb{R}$, hence bounded in $\mathbb{R}\times\mathbb{R}$.

Since $\widetilde{\mathbf{w}}(t,x)$ is bounded, by (3.5), (3.8), and the dominated convergence theorem,

$$\widetilde{\mathbf{w}}(t,x)=\Pi_x^\lambda\left[\beta e^{-\int_0^\infty \mathbf{g}(Y_s)\mathbf{w}(t-s,Y_s)\mathrm{d}s}\right]\le\beta.$$

Since $\widetilde{\mathbf{w}}(t,x)>0$, we have $\beta>0$.

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Next we show that

$$\widetilde{\mathbf{w}}(t,x)\to\beta \quad \text{uniformly in } t\in\left[0,\frac{1}{\nu}\right] \text{ as } x\to\infty.$$

It follows from (3.2) that

$$\begin{aligned}\mathbf{w}(t-s,Y_s)&\le e^{-\lambda Y_s+\gamma(\lambda)(t-s)}\psi(Y_t,\lambda)\widetilde{\mathbf{w}}(t-s,Y_s)\\&\le\beta\max_{z\in[0,1]}\psi(z,\lambda)e^{\gamma(\lambda)\frac{1}{\nu}}e^{-\lambda(Y_s+\nu s)}\\&\le C_1e^{-\lambda(Y_s+\nu s)},\quad\forall t\in\left[0,\frac{1}{\nu}\right],\end{aligned}\tag{3.17}$$

where C_1 is a constant depending only on λ. Moreover, we have

$$\mathbf{g}(Y_s)\mathbf{w}(t-s,Y_s)\le Ce^{-\lambda(Y_s+\nu s)},\quad\forall t\in\left[0,\frac{1}{\nu}\right],$$

where $C=C_1\|\mathbf{g}\|_\infty$. For any $y\in[0,1]$ and $n\in\mathbb{N}$, let

$$f_n(y)=\Pi_y^\lambda\left[e^{-\int_0^\infty Ce^{-\lambda(Y_s+n+\nu s)}\mathrm{d}s}\right].$$

Recall that $\{x\}$ is the fractional part of x and $\lfloor x\rfloor$ is the integer part of x. Then by the periodicity of Y_t, we have

$$\widetilde{\mathbf{w}}(t,x)\ge\beta\Pi_x^\lambda\left[e^{-\int_0^\infty Ce^{-\lambda(Y_s+\nu s)}\mathrm{d}s}\right]=\beta\Pi_{\{x\}}^\lambda\left[e^{-\int_0^\infty Ce^{-\lambda(Y_s+\lfloor x\rfloor+\nu s)}\mathrm{d}s}\right]=\beta f_{\lfloor x\rfloor}(\{x\}).$$

Since $\int_0^\infty e^{-\lambda(Y_s+\nu s)}\mathrm{d}s<+\infty$ Π_x^λ-a.s., by the dominated convergence theorem, we get

$$\begin{aligned}\lim_{n\to\infty}f_n(y)&=\Pi_y^\lambda\left[\lim_{n\to\infty}e^{-\int_0^\infty Ce^{-\lambda(Y_s+n+\nu s)}\mathrm{d}s}\right]=\Pi_y^\lambda\left[e^{-\lim_{n\to\infty}\int_0^\infty Ce^{-\lambda(Y_s+n+\nu s)}\mathrm{d}s}\right]\\&=\Pi_y^\lambda\left[e^{-\int_0^\infty\lim_{n\to\infty}Ce^{-\lambda(Y_s+n+\nu s)}\mathrm{d}s}\right]=1.\end{aligned}$$

Notice that $f_n(y)\le f_{n+1}(y)$, using Dini's theorem we get $f_n(y)\to1$ uniformly for $y\in[0,1]$. Combining this with $\beta f_n(y)\le\widetilde{\mathbf{w}}(t,y+n)\le\beta$, we have (3.15).

By (3.2), we get

$$\frac{e^{\lambda x-\gamma(\lambda)t}\mathbf{w}(t,x)}{\psi(x,\lambda)}\to\beta\quad\text{uniformly in } t\in\left[0,\frac{1}{\nu}\right]\text{ as } x\to\infty.\tag{3.18}$$

Step 4. Suppose $L=1$. We will show that

$$\mathbf{w}\left(\frac{y-x}{\nu},y\right)\sim\beta e^{-\lambda x}\psi(y,\lambda)\quad\text{uniformly in } y\in[0,1]\text{ as } x\to+\infty.\tag{3.19}$$

It is equivalent to show that, for any $\epsilon>0$, there exists x_0 such that for any $x>x_0$,

$$\sup_{y\in[0,1]}\left|\frac{e^{\lambda x}\mathbf{w}\left(\frac{y-x}{\nu},y\right)}{\psi(y,\lambda)}-\beta\right|<\epsilon.$$

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By (3.18), for any $\epsilon > 0$, there exists z_0 such that for any $z > z_0$,

$$\sup_{t\in[0,\frac{1}{\nu}]} \left| \frac{e^{\lambda z-\gamma(\lambda)t}\mathbf{w}(t, z)}{\psi(z, \lambda)} - \beta \right| < \epsilon.$$

Put

$$t = \frac{\{y-x\}}{\nu}, \quad z = -\lfloor y-x \rfloor + y.$$

For any $x > z_0 + 1$, we have

$$\sup_{\frac{\{y-x\}}{\nu}\in\left[0,\frac{1}{\nu}\right]} \left| \frac{e^{\lambda(-\lfloor y-x\rfloor+y)-\gamma(\lambda)\frac{\{y-x\}}{\nu}}\mathbf{w}\left(\frac{\{y-x\}}{\nu}, -\lfloor y-x\rfloor + y\right)}{\psi(-\lfloor y-x\rfloor + y, \lambda)} - \beta \right| < \epsilon,$$

that is

$$\sup_{y\in[0,1]} \left| \frac{e^{\lambda x}\mathbf{w}\left(\frac{y-x}{\nu}, y\right)}{\psi(y, \lambda)} - \beta \right| < \epsilon,$$

where we used the periodicity of ψ and the fact that

$$\lambda(-\lfloor y-x\rfloor + y) - \gamma(\lambda)\frac{\{y-x\}}{\nu} = \lambda(-\lfloor y-x\rfloor + y) - \lambda\{y-x\} = \lambda(-(y-x)+y) = \lambda x.$$

Thus (3.19) holds.

Step 5. In Steps 1–4, we have proven the theorem in the case of binary branching. It suffices to prove Steps 1–4 again for general branching mechanism.

For general branching mechanism, $\mathbf{w} = 1 - \mathbf{u}$ satisfies

$$\frac{\partial \mathbf{w}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{w}}{\partial x^2} + \mathbf{g}\cdot(1 - \mathbf{w} - \mathbf{f}(1-\mathbf{w})),$$

where $\mathbf{f}(s) = \mathbf{E}s^{1+L} = \sum_{k=0}^{\infty} p_k s^{1+k}$ and $m = \sum_{k=0}^{\infty} kp_k$. By the Feynman–Kac formula,

$$\mathbf{w}(T, x) = \Pi_x\left[\mathbf{w}(T-t, B_t)e^{\int_0^t \mathbf{g}(B_s)\frac{1-\mathbf{w}-\mathbf{f}(1-\mathbf{w})}{\mathbf{w}}(T-s,B_s)\mathrm{d}s}\right].$$

By the definition in (2.5),

$$\begin{aligned}\mathbf{w}(T, x) &= \Pi_x^\lambda\left[\frac{\Xi_0(\lambda)}{\Xi_t(\lambda)}\mathbf{w}(T-t, B_t)e^{\int_0^t \mathbf{g}(B_s)\frac{1-\mathbf{w}-\mathbf{f}(1-\mathbf{w})}{\mathbf{w}}(T-s,B_s)\mathrm{d}s}\right] \\ &= \Pi_x^\lambda\left[e^{-\lambda x+\gamma(\lambda)T}\psi(x, \lambda)\frac{e^{\lambda B_t-\gamma(\lambda)(T-t)}\mathbf{w}(T-t, B_t)}{\psi(B_t, \lambda)}e^{\int_0^t \mathbf{g}(B_s)\left(\frac{1-\mathbf{w}-\mathbf{f}(1-\mathbf{w})}{\mathbf{w}}(T-s,B_s)-m\right)\mathrm{d}s}\right].\end{aligned}$$

Put

$$A(w) = \begin{cases} m - \frac{1-w-\mathbf{f}(1-w)}{w}, & w\in(0, 1], \\ 0, & w = 0.\end{cases}$$

Since $\left(B_t, \Pi_x^\lambda\right)$ and $\left(Y_t, \Pi_x^\lambda\right)$ have the same law, by (3.2), we have

$$\widetilde{\mathbf{w}}(T, x) = \Pi_x^\lambda\left[\widetilde{\mathbf{w}}(T-t, Y_t)e^{-\int_0^t \mathbf{g}(Y_s)A(\mathbf{w})(T-s,Y_s)\mathrm{d}s}\right].$$

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It follows from [1, Corollary 2, p. 26] that $A(\cdot)$ is non-negative and non-decreasing. Moreover, for any $r, c \in (0, 1)$,

$$\sum_{n=0}^{\infty} A\big(cr^n\big) < +\infty \quad \text{iff} \quad \mathbf{E}\big(L \log^+ L\big) < +\infty. \tag{3.20}$$

Using the argument of Step 1, we get that

$$\left\{\widetilde{\mathbf{w}}(-t, Y_t)e^{-\int_0^t \mathbf{g}(Y_s)A(\mathbf{w})(-s, Y_s)\mathrm{d}s}, \Pi_x^\lambda\right\}_{t\ge 0}$$

is a non-negative martingale. Using the argument at the beginning of Step 2, we get that $\mathbf{w}(t, x) \to 0$ uniformly in $t \in \big[0, \frac{1}{\nu}\big]$ as $x \to \infty$ and $\mathbf{w}(-t, Y_t)$ decays exponentially with rate at least $-\lambda$. We will get that $\widetilde{\mathbf{w}}(-t, Y_t)$ converges Π_x^λ-a.s., if we can show that

$$\int_0^\infty \mathbf{g}(Y_s)A(\mathbf{w})(-s, Y_s)\mathrm{d}s < +\infty, \qquad \Pi_x^\lambda\text{-a.s.}$$

By (3.11) and (3.20), if $\mathbf{E}(L \log^+ L) < \infty$, we have

$$\begin{aligned}\int_0^\infty \mathbf{g}(Y_s)A(\mathbf{w})(-s, Y_s)\mathrm{d}s &\le \|\mathbf{g}\|_\infty \int_0^\infty A\Big(Ce^{-(\lambda-\delta)(Y_s+\nu s)}\Big)\,\mathrm{d}s \\ &\le \|\mathbf{g}\|_\infty \int_0^\infty A\Big(C\big(e^{-(\lambda-\delta)(\nu-\gamma'(\lambda))s}\big)\Big)\,\mathrm{d}s \\ &\le \|\mathbf{g}\|_\infty \sum_{n=0}^{\infty} A\Big(C\big(e^{-(\lambda-\delta)(\nu-\gamma'(\lambda))n}\big)\Big) < +\infty \qquad \Pi_x^\lambda\text{-a.s.,}\end{aligned}$$

where C is a constant depending on ω and may change in value from line to line.

Also by the arguments in Steps 2 and 3, $\widetilde{\mathbf{w}}(t, x)$ satisfies

$$\frac{\partial \widetilde{\mathbf{w}}}{\partial t} = \frac{1}{2}\frac{\partial^2 \widetilde{\mathbf{w}}}{\partial x^2} + \left(\frac{\psi_x(x, \lambda)}{\psi(x, \lambda)} - \lambda\right)\frac{\partial \widetilde{\mathbf{w}}}{\partial x} - \mathbf{g}\cdot A(\mathbf{w})\widetilde{\mathbf{w}},$$

and the maximum principle holds. Hence, $\widetilde{\mathbf{w}}(t, x)$ is bounded in $\big[0, \frac{1}{\nu}\big] \times \mathbb{R}$. Since $A(\cdot)$ is non-decreasing, (3.17) implies

$$A(\mathbf{w})(t-s, Y_s) \le A\Big(C_1 e^{-\lambda(Y_s+\nu s)}\Big).$$

By the periodicity of $\{Y_t\}$, we have

$$\begin{aligned}\widetilde{\mathbf{w}}(t, y+n) &\ge \beta\Pi_{y+n}^\lambda\left[e^{-\int_0^\infty \|\mathbf{g}\|_\infty A(C_1 e^{-\lambda(Y_s+\nu s)})\mathrm{d}s}\right] \\ &= \beta\Pi_y^\lambda\left[e^{-\int_0^\infty \|\mathbf{g}\|_\infty A(C_1 e^{-\lambda(Y_s+n+\nu s)})\mathrm{d}s}\right].\end{aligned}$$

To prove the theorem, it suffices to show

$$\Pi_y^\lambda\left[e^{-\int_0^\infty \|\mathbf{g}\|_\infty A(C_1 e^{-\lambda(Y_s+n+\nu s)})\mathrm{d}s}\right] \to 1, \qquad \text{as } n \to \infty \text{ uniformly for } y \in [0, 1]. \tag{3.21}$$

We know

$$\int_0^\infty \|\mathbf{g}\|_\infty A\Big(C_1 e^{-\lambda(Y_s+n+\nu s)}\Big)\,\mathrm{d}s < +\infty, \qquad \Pi_x^\lambda\text{-a.s.}$$

By an argument similar to that in Step 3, (3.21) follows from $\lim_{u\downarrow 0} A(u) = 0$, the dominated convergence theorem, and Dini's theorem. This completes the proof. □

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4. Proof of Theorem 1.2

In this section, we prove the asymptotic behavior in the critical case.

Proof of Theorem 1.2. We prove the theorem in seven steps. In the first six steps, we consider the case that $L=1$. In the last step we consider general L.

Step 1. Suppose $L=1$. Put $\mathbf{w}(t,x)=1-\mathbf{u}(t,x)$. We first prove that $\mathbf{w}(t,x)$ decays exponentially with rate at least $-\lambda$ uniformly in $t\in\left[0,\frac{1}{\nu}\right]$. Using an argument similar to that of Theorem 1.1, we have that, for $t,x\in\mathbb{R}$, $\mathbf{w}(t,x)$ satisfies

$$\begin{cases}\frac{\partial \mathbf{w}}{\partial t}=\frac{1}{2}\frac{\partial^2\mathbf{w}}{\partial x^2}+\mathbf{g}\cdot\left(\mathbf{w}-\mathbf{w}^2\right),\\ \mathbf{w}\left(t+\frac{1}{\nu^*},x\right)=\mathbf{w}(t,x-1).\end{cases}$$

By the Feynman–Kac formula, we get

$$\mathbf{w}(T,x)=\Pi_x\left[\mathbf{w}(T-t,B_t)e^{\int_0^t \mathbf{g}(B_s)(1-\mathbf{w}(T-s,B_s))\mathrm{d}s}\right],\qquad \text{for } T\in\mathbb{R},\ t>0.$$

For any $\lambda<\lambda^*$, define

$$\widetilde{\mathbf{w}}(t,x)=\frac{e^{\lambda x-\gamma(\lambda)t}\mathbf{w}(t,x)}{\psi(x,\lambda)}. \tag{4.1}$$

Changing measure with $\Xi_t(\lambda)$ and following the same ideas as in Step 1 of the proof of Theorem 1.1, we get that

$$\widetilde{\mathbf{w}}(T,x)=\Pi_x^\lambda\left[\widetilde{\mathbf{w}}(T-t,Y_t)e^{-\int_0^t \mathbf{g}(Y_s)\mathbf{w}(T-s,Y_s)\mathrm{d}s}\right],$$

and

$$\left\{\widetilde{\mathbf{w}}(-t,Y_t)e^{-\int_0^t \mathbf{g}(Y_s)\mathbf{w}(-s,Y_s)\mathrm{d}s},\ \Pi_x^\lambda\right\}_{t\ge 0}$$

is a positive martingale. Therefore, we have

$$\limsup_{t\to\infty}\left\{\frac{\ln\widetilde{\mathbf{w}}(-t,Y_t)}{Y_t+\nu^*t}-\frac{1}{Y_t+\nu^*t}\int_0^t \mathbf{g}(Y_s)\mathbf{w}(-s,Y_s)\mathrm{d}s\right\}\le 0,\qquad \Pi_x^\lambda\text{-a.s.}$$

Notice that for $\lambda<\lambda^*$,

$$\lim_{t\to\infty}\frac{Y_t+\nu^*t}{t}\to-\gamma'(\lambda)+\nu^*>0.$$

Combining this with (4.1), we get

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln\widetilde{\mathbf{w}}(-t,Y_t)}{Y_t+\nu^*t}&=\limsup_{t\to\infty}\frac{\ln\left(e^{\lambda(Y_t+\nu^*t)+(\gamma(\lambda)-\lambda\nu^*)t}\mathbf{w}(-t,Y_t)\right)}{Y_t+\nu^*t}\\&=\limsup_{t\to\infty}\frac{\ln\mathbf{w}(-t,Y_t)}{Y_t+\nu^*t}+\lambda+\frac{\gamma(\lambda)-\lambda\nu^*}{\nu^*-\gamma'(\lambda)}\le 0,\qquad \Pi_x^\lambda\text{-a.s.},\end{aligned}$$

where $\gamma(\lambda)-\lambda\nu^*>0$. Hence $\mathbf{w}(-t,Y_t)$ decays exponentially with rate at least $-\lambda$ Π_x^λ-a.s. This implies

$$\int_0^\infty \mathbf{g}(Y_s)\mathbf{w}(-s,Y_s)\mathrm{d}s<+\infty,\qquad \Pi_x^\lambda\text{-a.s.}$$

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Then $\widetilde{\mathbf{w}}(-t, Y_t)$ converges Π_x^λ-a.s. Using a coupling method similar to that of Step 2 in the proof of Theorem 1.1, we get that the limit of $\widetilde{\mathbf{w}}(-t, Y_t)$ is a constant. Notice that

$$\widetilde{\mathbf{w}}\left(t+\frac{1}{\nu^*}, x+1\right) = \frac{e^{\lambda(x+1)-\gamma(\lambda)\left(t+\frac{1}{\nu^*}\right)}\mathbf{w}\left(t+\frac{1}{\nu^*}, x+1\right)}{\psi(x+1, \lambda)} = e^{\lambda-\frac{\gamma(\lambda)}{\nu^*}}\widetilde{\mathbf{w}}(t, x) < \widetilde{\mathbf{w}}(t, x),$$

where we used $\mathbf{w}\left(t+\frac{1}{\nu^*}, x+1\right) = \mathbf{w}(t, x)$ and $\nu^* < \frac{\gamma(\lambda)}{\lambda}$ for $\lambda < \lambda^*$. Therefore, for any $k \in \mathbb{N}$,

$$\widetilde{\mathbf{w}}\left(-t+\frac{k}{\nu}, Y_t + k\right) \le \widetilde{\mathbf{w}}(-t, Y_t).$$

Using an argument similar to that of Step 3 in the proof of Theorem 1.1, we get that $\widetilde{\mathbf{w}}(t, x)$ is bounded in $\left[0, \frac{1}{\nu^*}\right] \times \mathbb{R}$. Then, by (4.1), $\mathbf{w}(t, x)$ decays exponentially with rate at least $-\lambda$ uniformly in $t \in \left[0, \frac{1}{\nu^*}\right]$.

Step 2. We will show that $\left\{f\left(B_{[0,t]}\right), \Pi_x^{(y,\lambda^*)}\right\}$ is a local martingale, where $f\left(B_{[0,t]}\right)$ is defined in (4.7) below. Recall the definitions of h, τ_λ^x, $\Lambda_t^{(x,\lambda)}$, and $\Pi_y^{(x,\lambda)}$ given in (2.4), (2.9), (2.10), and (2.11) respectively, with $\lambda = \lambda^*$ in (2.4). Fix $y \in \mathbb{R}$. For any (t, x) such that $y - \gamma'(\lambda^*)t + h(x) > 0$, define

$$\widehat{\mathbf{w}}(t, x, y) := \frac{e^{\lambda^* x - \gamma(\lambda^*)t}\mathbf{w}(t, x)}{\psi(x, \lambda^*)(y - \gamma'(\lambda^*)t + h(x))}, \tag{4.2}$$

and for any $z > 0$, define

$$\tau_z := \inf\left\{t \ge 0 : y + \gamma'(\lambda^*)t + h(B_t) \le z\right\}. \tag{4.3}$$

We mention here that τ_z actually depends on y. For any $x \in \mathbb{R}$, we may define

$$\tau_z(x) := \inf\left\{t \ge 0 : x + \gamma'(\lambda^*)t + h(B_t) \le z\right\}.$$

Then τ_z is shorthand for $\tau_z(y)$, and for any $x \in \mathbb{R}$, $\tau_z(y - x) = \tau_{z+x}$. Using (4.2), it is easy to show

$$\widehat{\mathbf{w}}\left(t + \frac{1}{\nu^*}, x+1, y\right) = \widehat{\mathbf{w}}(t, x, y). \tag{4.4}$$

We first prove that for any $T \in \mathbb{R}$ and $t > 0$,

$$\widehat{\mathbf{w}}\left(T, x, y+\nu^* T\right) = \Pi_x^{(y,\lambda^*)}\left(\widehat{\mathbf{w}}\left(T - t\wedge\tau_z, B_{t\wedge\tau_z}, y+\nu^* T\right)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_s)\mathbf{w}(T-s, B_s)\mathrm{d}s}\right). \tag{4.5}$$

First note that, by the Feynman–Kac formula and the optional stopping theorem,

$$\mathbf{w}(T, x) = \Pi_x\left(\mathbf{w}(T - t\wedge\tau_z, B_{t\wedge\tau_z})e^{\int_0^{t\wedge\tau_z}\mathbf{g}(B_s)(1-\mathbf{w}(T-s,B_s))\mathrm{d}s}\right), \quad T \in \mathbb{R},\ t > 0. \tag{4.6}$$

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Noticing that $\Lambda^{(y,\lambda^*)}_{t\wedge\tau_z} > 0$ and $\nu^* = \gamma'(\lambda^*)$, a direct calculation shows that for $x > h^{-1}(y)$,

$$\begin{aligned}
&\Pi^{(y,\lambda^*)}_x\Big(\psi(x,\lambda^*)(y+h(x))e^{-\lambda^* x+\gamma(\lambda^*)T} \\
&\quad\times \widehat{\mathbf{w}}\big(T-t\wedge\tau_z, B_{t\wedge\tau_z}, y+\nu^* T\big)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_s)\mathbf{w}(T-s,B_s)\mathrm{d}s}\Big) \\
=&\Pi^{(y,\lambda^*)}_x\left(\frac{\Lambda^{(y,\lambda^*)}_0}{\Lambda^{(y,\lambda^*)}_{t\wedge\tau_z}}\mathbf{w}\big(T-t\wedge\tau_z, B_{t\wedge\tau_z}\big)e^{\int_0^{t\wedge\tau_z}\mathbf{g}(B_s)(1-\mathbf{w}(T-s,B_s))\mathrm{d}s}\right) \\
=&\Pi_x\left(\frac{\Lambda^{(y,\lambda^*)}_0}{\Lambda^{(y,\lambda^*)}_{t\wedge\tau_z}}\mathbf{w}\big(T-t\wedge\tau_z, B_{t\wedge\tau_z}\big)e^{\int_0^{t\wedge\tau_z}\mathbf{g}(B_s)(1-\mathbf{w}(T-s,B_s))\mathrm{d}s}\frac{\Lambda^{(y,\lambda^*)}_{t\wedge\tau_z}}{\Lambda^{(y,\lambda^*)}_0}\mathbf{1}_{\left\{\Lambda^{(y,\lambda^*)}_{t\wedge\tau_z}>0\right\}}\right) \\
=&\Pi_x\left(\mathbf{w}\big(T-t\wedge\tau_z, B_{t\wedge\tau_z}\big)e^{\int_0^{t\wedge\tau_z}\mathbf{g}(B_s)(1-\mathbf{w}(T-s,B_s))\mathrm{d}s}\right) = \mathbf{w}(T,x),
\end{aligned}$$

where in the last equality we used (4.6). Then using the definition of $\widehat{\mathbf{w}}$ given in (4.2), we get (4.5).

Notice that $\left\{B_t, t\ge 0; \Pi^{(y,\lambda^*)}_x\right\}$ is not a Brownian motion. By the argument in the last paragraph of Subsection 2.2, we have that $\left\{y+\gamma'(\lambda^*)T(t)+h(B_{T(t)}),\ \Pi^{(y,\lambda^*)}_x\right\}$ is a Bessel-3 process starting from $y+h(x)$. Define

$$f\big(B_{[0,t]}\big) = \widehat{\mathbf{w}}(-t, B_t, y)e^{-\int_0^t \mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s}. \tag{4.7}$$

By the Markov property, for any $0 < s < t$,

$$\Pi^{(y,\lambda^*)}_x\Big[f\big(B_{[0,t\wedge\tau_z]}\big)|\mathcal{F}_s\Big] = f\big(B_{[0,s\wedge\tau_z]}\big).$$

The proof of the display above is given in the appendix; see Lemma A.1. This implies that $\left\{f\big(B_{[0,t\wedge\tau_z]}\big), \Pi^{(y,\lambda^*)}_x\right\}$ is a martingale. Since $\tau_z\to\infty$ as $z\downarrow 0$, $\left\{f\big(B_{[0,t]}\big), \Pi^{(y,\lambda^*)}_x\right\}$ is a local martingale.

Step 3. In this step, we will show that $\widehat{\mathbf{w}}(T-t, B_t, y)$ converges $\Pi^{(y-\nu^*T,\lambda^*)}_x$-a.s. to a constant $\beta\ge 0$ as $t\to\infty$. Non-negative local martingales are non-negative super-martingales and hence must converge. Therefore $f(B_{[0,t]})$, defined by (4.7), converges $\Pi^{(y,\lambda^*)}_x$-a.s. as $t\to\infty$. Notice that $\left\{y+\gamma'(\lambda^*)T(t)+h(B_{T(t)}),\ \Pi^{(y,\lambda^*)}_x\right\}$ is a Bessel-3 process starting from $y+h(x)$. It is known (see, for example, [17, Theorem 3.2]) that the Bessel-3 process grows no slower than $t^{1/2-\epsilon}$ for any $\epsilon>0$. Recall that $T(t)=\inf\{s>0:\langle M\rangle_s>t\}$. Equation (2.8) yields $T(t)\in\left[\frac{t}{c_2}, \frac{t}{c_1}\right]$. Therefore, $y+\gamma'(\lambda^*)t+h(B_t)$ grows no slower than $t^{1/2-\epsilon}$ for any $\epsilon>0$. By (2.4) and the boundedness of $\psi_\lambda(\cdot,\lambda)/\psi(\cdot,\lambda)$, we get that $B_t+\nu^*t = B_t+\gamma'(\lambda^*)t$ grows no slower than $t^{1/2-\epsilon}$. By Step 1, $\mathbf{w}(t,x)$ decays exponentially fast uniformly in $t\in\left[0,\frac{1}{\nu^*}\right]$. Hence we have

$$\int_0^\infty \mathbf{g}(B_t)\mathbf{w}(-t,B_t)\mathrm{d}t < +\infty, \qquad \Pi^{(y,\lambda^*)}_x\text{-a.s.}$$

Therefore, by (4.7), the convergence of $f(B_{[0,t]})$ implies that $\widehat{\mathbf{w}}(-t, B_t, y)$ converges $\Pi^{(y,\lambda^*)}_x$-a.s. to some limit, say ξ_x.

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By (4.5), we have

$$\widehat{\mathbf{w}}(T,x,y)$$
$$=\Pi_x^{(y-\nu^*T,\lambda^*)}\left(\widehat{\mathbf{w}}\big(T-t\wedge\tau_z(y-\nu^*T),B_{t\wedge\tau_z(y-\nu^*T)},y\big)e^{-\int_0^{t\wedge\tau_z(y-\nu^*T)}\mathbf{g}(B_s)\mathbf{w}(T-s,B_s)\mathrm{d}s}\right).$$

By the same method, we can get

$$\left\{\widehat{\mathbf{w}}\big(T-t\wedge\tau_z(y-\nu^*T),B_{t\wedge\tau_z(y-\nu^*T)},y\big)e^{-\int_0^{t\wedge\tau_z(y-\nu^*T)}\mathbf{g}(B_s)\mathbf{w}(T-s,B_s)\mathrm{d}s},\ \Pi_x^{(y-\nu^*T,\lambda^*)}\right\}$$

is a local martingale and $\widehat{\mathbf{w}}(T-t,B_t,y)$ converges $\Pi_x^{(y-\nu^*T,\lambda^*)}$-a.s. to some limit, say ξ_x^T.

Next we use a coupling method to prove that there is a constant $\beta\geq 0$ such that

$$\xi_x^T=\beta,\qquad \Pi_x^{(y-\nu^*T,\lambda^*)}\text{-a.s.}\quad \forall T\geq 0. \tag{4.8}$$

Similarly to Step 2 in the proof of Theorem 1.1, consider a process $\left\{\left(B_t^1,B_t^2\right),t\geq 0;\widetilde{\Pi}_x^{(y,T)}\right\}$ with $\left\{B_t^1,t\geq 0\right\}$ and $\left\{B_t^2,t\geq 0\right\}$ being independent, and

$$\left\{B_t^1,t\geq 0;\widetilde{\Pi}_x^{(y,T)}\right\}\stackrel{d}{=}\left\{B_t,t\geq 0;\Pi_x^{(y,\lambda^*)}\right\},$$
$$\left\{B_t^2,t\geq 0;\widetilde{\Pi}_x^{(y,T)}\right\}\stackrel{d}{=}\left\{B_t,t\geq 0;\Pi_x^{(y-\nu^*T,\lambda^*)}\right\}.$$

Define random curves

$$\mathcal{L}_k^1=\left\{\left(-t+\frac{k}{\nu^*},B_t^1+k\right):t\geq\frac{k-1}{\nu^*}\right\},\qquad k\in\mathbb{N},$$

and

$$\mathcal{L}_k^2=\left\{\left(T-t+\frac{k}{\nu^*},B_t^2+k\right):t\geq\frac{k-1}{\nu^*}+T\right\},\qquad k\in\mathbb{N}.$$

Notice that all the curves start from the line $\left\{\frac{1}{\nu^*}\right\}\times\mathbb{R}$, and for each $i=1,2$, if $\mathcal{L}_1^i$ is given, we can get all the curves $\mathcal{L}_k^i$ by translation. Using the fact that $y+h(B_{T(t)})+\gamma'(\lambda^*)T(t)$ is a Bessel-3 process, $\gamma'(\lambda^*)=\nu^*$, and $|h(x)-x|$ is bounded, we have

$$\lim_{t\to\infty}B_t+\nu^*t=\infty\quad \Pi_x^{(y,\lambda^*)}\text{-a.s.}$$

Now we show that for $\widetilde{\Pi}_x^{(y,T)}$-a.s. all ω, it holds that for any $k\in\mathbb{N}$, $\mathcal{L}_k^1$ and $\mathcal{L}_{k+1}^1$ intersect each other. It is also equivalent to show that for any $k\in\mathbb{N}$, $\mathcal{L}_k^1$ and $\mathcal{L}_{k+1}^1$ intersect each other $\widetilde{\Pi}_x^{(y,T)}$-a.s. If there exists $t\geq\frac{k-1}{\nu^*}$ such that $B^1_{t+\frac{1}{\nu^*}}+1=B_t^1$, then

$$\left(-t+\frac{k}{\nu^*},B_t^1+k\right)=\left(-\left(t+\frac{1}{\nu^*}\right)+\frac{k+1}{\nu^*},B^1_{t+\frac{1}{\nu^*}}+k+1\right),$$

which implies $\mathcal{L}_k^1$ and $\mathcal{L}_{k+1}^1$ intersect each other. Notice that

$$B^1_{t+\frac{1}{\nu^*}}+1=B_t^1\Longleftrightarrow B^1_{t+\frac{1}{\nu^*}}+\nu^*t+1=B_t^1+\nu^*t$$
$$\Longleftrightarrow h\left(B^1_{t+\frac{1}{\nu^*}}\right)+\nu^*\left(t+\frac{1}{\nu^*}\right)=h\left(B_t^1\right)+\nu^*t$$
$$\Longleftrightarrow \widehat{R}_{\langle M^1\rangle_{t+\frac{1}{\nu^*}}}=\widehat{R}_{\langle M^1\rangle_t},$$

where $\widehat{R}_t := y + h\big(B^1_{T(t)}\big) + \nu^* T(t)$ is a standard Bessel-3 process starting at $y + h(x)$. By (2.8), we have

$$\langle M^1 \rangle_{t+\frac{1}{\nu^*}} - \langle M^1 \rangle_t \in \left[\frac{c_1}{\nu^*}, \frac{c_2}{\nu^*}\right],$$

where $\langle M^1 \rangle_t = \int_0^t \big(h'\big(B^1_s\big)\big)^2 \mathrm{d}s$. Put

$$l(t) = \widehat{R}_{\langle M^1 \rangle_{t+\frac{1}{\nu^*}}} - \widehat{R}_{\langle M^1 \rangle_t};$$

then $l(t)$ is continuous $\widetilde{\Pi}_x^{(y,T)}$-a.s. Since $\widetilde{\Pi}_x^{(y,T)}\big(\lim_{t\to\infty} \widehat{R}_t = \infty\big) = 1$, we have

$$\widetilde{\Pi}_x^{(y,T)} \text{ (for any } T > 0, \exists\, t > T \text{ s.t. } l(t) > 0) = 1.$$

To prove $\mathcal{L}^1_k$ and $\mathcal{L}^1_{k+1}$ intersect $\widetilde{\Pi}_x^{(y,T)}$-a.s., it suffices to show that

$$\widetilde{\Pi}_x^{(y,T)} \text{ (for any } T > 0, \exists\, t > T \text{ s.t. } l(t) < 0) = 1.$$

Notice that

$$l(t) \le \max_{s \in \left[\frac{c_1}{\nu^*}, \frac{c_2}{\nu^*}\right]} \Big(\widehat{R}_{s+\langle M^1 \rangle_t} - \widehat{R}_{\langle M^1 \rangle_t}\Big).$$

For simplicity, put $b_1 = \frac{c_1}{\nu^*}$ and $b_2 = \frac{c_2}{\nu^*}$. It suffices to show that

$$\widetilde{\Pi}_x^{(y,T)} \left(\text{for any } T > 0, \exists\, t > T \text{ s.t. } \max_{s\in[b_1,b_2]} (\widehat{R}_{t+s} - \widehat{R}_t) < 0\right) = 1. \tag{4.9}$$

A classical result shows $\widehat{R}_t$ satisfies

$$\mathrm{d}\widehat{R}_t = \mathrm{d}\widehat{B}_t + \frac{1}{\widehat{R}_t}\mathrm{d}t, \tag{4.10}$$

where $\Big(\widehat{B}_t; \widetilde{\Pi}_x^{(y,T)}\Big)$ is a standard Brownian motion. By (4.10), we have

$$\begin{aligned}
&\widetilde{\Pi}_x^{(y,T)} \left(\max_{s\in[b_1,b_2]} (\widehat{B}_{t+s} - \widehat{B}_t) < -1\right) \\
\ge &\widetilde{\Pi}_x^{(y,T)} \left(\widehat{B}_{t+b_1} - \widehat{B}_t < -2,\ \max_{s\in[b_1,b_2]} \widehat{B}_{t+s} - \widehat{B}_{t+b_1} < 1\right) \\
= &\Pi_0(B_{b_1} < -2) \cdot \Pi_0 \left(\max_{s\in[0,b_2-b_1]} B_s < 1\right) \ge C > 0,
\end{aligned}$$

where the constant C does not depend on t. Hence

$$\sum_{j=0}^{\infty} \widetilde{\Pi}_x^{(y,T)} \left(\max_{s\in[b_1,b_2]} (\widehat{B}_{jb_2+s} - \widehat{B}_{jb_2}) < -1\right) = +\infty.$$

By the second Borel–Cantelli lemma and the independent increments property of the Brownian motion, we have

$$\widetilde{\Pi}_x^{(y,T)} \left(\max_{s\in[b_1,b_2]} (\widehat{B}_{jb_2+s} - \widehat{B}_{jb_2}) < -1,\ \text{i.o.}\right) = 1. \tag{4.11}$$

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By (4.10), it holds that

$$\widehat{R}_{t+s} - \widehat{R}_t = \widehat{B}_{t+s} - \widehat{B}_t + \int_t^{t+s} \frac{1}{\widehat{R}_r} \mathrm{d}r.$$

If $\widehat{R}_{t+r} > b_2$ for $r \in [0, b_2]$ and $\widehat{B}_{t+s} - \widehat{B}_t < -1$ for $s \le b_2$, then

$$\widehat{R}_{t+s} - \widehat{R}_t < -1 + \frac{s}{b_2} \le 0. \tag{4.12}$$

Since the Bessel-3 process is transient, we have

$$\widetilde{\Pi}_x^{(y,T)}\big(\exists\, T > 0, \text{ s.t. for any } t > T, \widehat{R}_t > b_2\big) = 1. \tag{4.13}$$

Combining (4.11), (4.12), and (4.13), we get (4.9). So $\widetilde{\Pi}_x^{(y,T)}$-a.s., it holds that for any $k \in \mathbb{N}$, $\mathcal{L}_k^1$ and $\mathcal{L}_{k+1}^1$ intersect each other. Using the same method, we can also prove that $\widetilde{\Pi}_x^{(y,T)}$-a.s., for any $k, j \in \mathbb{N}$, $\mathcal{L}_k^1$ and $\mathcal{L}_{k+j}^1$ intersect each other.

Consider

$$\Omega_0 = \bigcap_{k,j\in\mathbb{N}} \left\{\mathcal{L}_k^1, \mathcal{L}_{k+j}^1 \text{ intersect}\right\} \bigcap \left\{\lim_{t\to\infty} \widehat{\mathbf{w}}\big(-t, B_t^1, y\big), \ \lim_{t\to\infty} \widehat{\mathbf{w}}\big(T - t, B_t^2, y\big) \text{ exists}\right\}$$
$$\bigcap \left\{\lim_{t\to\infty} B_t^i + v^* t = +\infty, \text{ and } B_t^i \text{ is continuous for } i = 1, 2\right\};$$

then $\widetilde{\Pi}_x^{(y,T)}(\Omega_0) = 1$. For any $\omega \in \Omega_0$ and $j \in \mathbb{N}$, we know $\mathcal{L}_j^2$ starts from the point $\left(\frac{1}{v^*}, B^2_{\frac{j-1}{v^*}+T} + j\right)$ and there exists $k \in \mathbb{N}$ such that

$$B^1_{\frac{k-1}{v^*}}(\omega) + k \le B^2_{\frac{j-1}{v^*}+T}(\omega) + j \le B^1_{\frac{k}{v^*}}(\omega) + k + 1.$$

Hence the starting point of $\mathcal{L}_j^2(\omega)$ is between the starting points of $\mathcal{L}_k^1(\omega)$ and $\mathcal{L}_{k+1}^1(\omega)$. Since $\mathcal{L}_k^1(\omega)$ and $\mathcal{L}_{k+1}^1(\omega)$ intersect, we have that $\mathcal{L}_j^2(\omega)$ must intersect either $\mathcal{L}_k^1(\omega)$ or $\mathcal{L}_{k+1}^1(\omega)$. We use $(s_1(\omega), x_1(\omega))$ to denote the intersection point. By (4.4), there exist $t_1^1(\omega)$, $t_1^2(\omega)$ satisfying

$$\widehat{\mathbf{w}}\left(-t_1^1(\omega), B^1_{t_1^1}(\omega), y\right) = \widehat{\mathbf{w}}(s_1(\omega), x_1(\omega), y) = \widehat{\mathbf{w}}\left(T - t_1^2(\omega), B^2_{t_1^2}(\omega), y\right).$$

Since j is arbitrary, we can find $\{t_n^i(\omega) : n \in \mathbb{N},\ i = 1, 2\}$ by induction such that

$$\widehat{\mathbf{w}}\left(-t_n^1(\omega), B^1_{t_n^1}(\omega), y\right) = \widehat{\mathbf{w}}\left(T - t_n^2(\omega), B^2_{t_n^2}(\omega), y\right)$$

and satisfying

$$\lim_{n\to\infty} t_n^i(\omega) = \infty \qquad \text{for } i = 1, 2.$$

Therefore, we have

$$\widetilde{\Pi}_x^{(y,T)}\left(\lim_{t\to\infty} \widehat{\mathbf{w}}\big(-t, B_t^1, y\big) = \lim_{t\to\infty} \widehat{\mathbf{w}}\big(T - t, B_t^2, y\big)\right) = 1.$$

By the independence of $\left\{B_t^1, t \ge 0\right\}$ and $\left\{B_t^2, t \ge 0\right\}$, we get that the limits must be the same. So there is a constant $\beta \ge 0$ such that

$$\xi_x = \beta, \quad \Pi_x^{(y,\lambda^*)}\text{-a.s.,} \quad \text{and} \quad \xi_x^T = \beta, \quad \Pi_x^{(y-v^*T,\lambda^*)}\text{-a.s.}$$

Thus (4.8) is valid.

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Step 4. In this step we prove $\beta > 0$ by contradiction. If $\beta = 0$, then $\widehat{\mathbf{w}}(-t, B_t, y) \to 0$ as $t \to \infty$. Hence the positive local martingale

$$\widehat{\mathbf{w}}(-t, B_t, y)e^{-\int_0^t \mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s} \to 0, \qquad \text{as } t \to \infty,\ \Pi_x^{(y,\lambda^*)}\text{-a.s.} \tag{4.14}$$

In Step 2 we proved that $\left\{f\left(B_{[0,t\wedge\tau_z]}\right), \Pi_x^{(y,\lambda^*)}\right\}$ is a martingale. Thus if $y + h(x) > z > 0$, we have

$$\begin{aligned}\widehat{\mathbf{w}}(0, x, y) &= \Pi_x^{(y,\lambda^*)}\left(\widehat{\mathbf{w}}(-t\wedge\tau_z, B_{t\wedge\tau_z}, y)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s}\right)\\ &= \Pi_x^{(y,\lambda^*)}\left(\widehat{\mathbf{w}}(-\tau_z, B_{\tau_z}, y)e^{-\int_0^{\tau_z}\mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s}\mathbf{1}_{\{\tau_z<t\}}\right)\\ &\quad + \Pi_x^{(y,\lambda^*)}\left(\widehat{\mathbf{w}}(-t, B_t, y)e^{-\int_0^{t}\mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s}\mathbf{1}_{\{\tau_z\geq t\}}\right).\end{aligned}$$

Letting $t \to \infty$ and using (4.14), we have

$$\widehat{\mathbf{w}}(0, x, y) = \Pi_x^{(y,\lambda^*)}\left(\widehat{\mathbf{w}}\left(-\tau_z, B_{\tau_z}, y\right)e^{-\int_0^{\tau_z}\mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s}\mathbf{1}_{\{\tau_z<\infty\}}\right).$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}\widehat{\mathbf{w}}(0, x, y) &= \Pi_x^{(y,\lambda^*)}\left(\frac{e^{\lambda^* B_{\tau_z}+\gamma(\lambda^*)\tau_z}\mathbf{w}\left(-\tau_z, B_{\tau_z}\right)}{\psi\left(B_{\tau_z}, \lambda^*\right)\left(y+\gamma'(\lambda^*)\tau_z + h\left(B_{\tau_z}\right)\right)}e^{-\int_0^{\tau_z}\mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s}\mathbf{1}_{\{\tau_z<\infty\}}\right)\\ &= \Pi_x^{(y,\lambda^*)}\left(\frac{e^{\lambda^* B_{\tau_z}+\gamma(\lambda^*)\tau_z}\mathbf{w}\left(-\tau_z, B_{\tau_z}\right)}{\psi\left(B_{\tau_z}, \lambda^*\right)z}e^{-\int_0^{\tau_z}\mathbf{g}(B_s)\mathbf{w}(-s,B_s)\mathrm{d}s}\mathbf{1}_{\{\tau_z<\infty\}}\right).\end{aligned}$$

Since $\left\{y + \gamma'(\lambda^*)T(t) + h(B_{T(t)}),\ \Pi_x^{(y,\lambda^*)}\right\}$ is a Bessel-3 process starting from $y + h(x)$, we have (see, for example, Karatzas and Shreve [11, p. 162, Problem 3.23])

$$\Pi_x^{(y,\lambda^*)}(\tau_z < \infty) = \Pi_x^{(y,\lambda^*)}\left(\inf_{t\geq 0}\left\{y + \gamma'(\lambda^*)T(t) + h\left(B_{T(t)}\right)\right\} \leq z\right) = \frac{z}{y + h(x)}. \tag{4.15}$$

By (4.2), we have

$$\begin{aligned}\frac{e^{\lambda^* x}\mathbf{w}(0, x)}{\psi(x, \lambda^*)(y + h(x))} &= \widehat{\mathbf{w}}(0, x, y) \leq \Pi_x^{(y,\lambda^*)}\left(\frac{e^{\lambda^* B_{\tau_z}+\gamma(\lambda^*)\tau_z}\mathbf{w}\left(-\tau_z, B_{\tau_z}\right)}{\psi\left(B_{\tau_z}, \lambda^*\right)z}\mathbf{1}_{\{\tau_z<\infty\}}\right)\\ &= \Pi_x^{(y,\lambda^*)}\left(\frac{e^{\lambda^*\left(z-y+\psi_\lambda\left(B_{\tau_z},\lambda^*\right)/\psi\left(B_{\tau_z},\lambda^*\right)\right)}\mathbf{w}\left(-\{\tau_z\}, B_{\tau_z} + \nu^*\left[\tau_z\right]\right)}{\psi\left(B_{\tau_z}, \lambda^*\right)z}\mathbf{1}_{\{\tau_z<\infty\}}\right)\\ &\leq C_1\Pi_x^{(y,\lambda^*)}(\tau_z < \infty) \leq C_1\frac{z}{y + h(x)},\end{aligned}$$

where we used the facts that ψ_λ/ψ is bounded and that $\mathbf{w}(t, x)$ is bounded in $(t, x) \in \left[-\frac{1}{\nu^*}, 0\right] \times [-C_2, C_2]$. Here C_1, C_2 are constants depending only on y, z, λ^*. Hence we get that for $x > h^{-1}(z - y)$, $e^{\lambda^* x}\mathbf{w}(0, x)/\psi(x, \lambda^*)$ is bounded. Combining this with the fact that $e^{\lambda^* x}\mathbf{w}(0, x)/\psi(x, \lambda^*) \to 0$ as $x \to -\infty$, we have that $e^{\lambda^* x}\mathbf{w}(0, x)/\psi(x, \lambda^*)$ is bounded on $\mathbb{R}$. Similarly, we can prove that $e^{\lambda^* x-\gamma(\lambda^*)t}\mathbf{w}(t, x)/\psi(x, \lambda^*)$ is bounded on $\left[0, \frac{1}{\nu^*}\right] \times \mathbb{R}$.

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By Step 1 of the proof of Theorem 1.1, we know that

$$\left\{ \frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\psi(Y_t,\, \lambda^*)} e^{-\int_0^t \mathbf{g}(Y_s)\mathbf{w}(-s,Y_s)\mathrm{d}s},\ t \geq 0;\ \Pi_x^{\lambda^*} \right\}$$

is a martingale and satisfies

$$\frac{e^{\lambda x}\mathbf{w}(0,\, x)}{\psi(x,\, \lambda^*)} = \Pi_x^{\lambda^*}\left[\frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\psi(Y_t,\, \lambda^*)} e^{-\int_0^t \mathbf{g}(Y_s)\mathbf{w}(-s,Y_s)\mathrm{d}s} \right], \tag{4.16}$$

where $\left\{Y_t,\, \Pi_x^{\lambda^*}\right\}$ is a diffusion with infinitesimal generator

$$(\mathcal{A}f)(x) = \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2} + \left(\frac{\psi_x(x,\, \lambda^*)}{\psi(x,\, \lambda^*)} - \lambda^* \right)\frac{\partial f(x)}{\partial x}.$$

So

$$\left\{ \frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\psi(Y_t,\, \lambda^*)},\ \Pi_x^{\lambda^*} \right\}$$

is a positive submartingale that is bounded and hence must converge. Using an argument similar to that of Step 2 in the proof of Theorem 1.1, we have that the limit of

$$\frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\psi(Y_t,\, \lambda^*)}$$

is a constant. Since

$$\mathbf{w}(-t,\, Y_t) \leq 1,\ \liminf_{t\to\infty}(Y_t + \nu^* t) = -\infty \text{ and } \inf_{x\in\mathbb{R}} \psi(x,\, \lambda^*) > 0,$$

we have

$$\liminf_{t\to\infty} \frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\psi(Y_t,\, \lambda^*)} \leq \frac{\liminf_{t\to\infty} e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\inf_{x\in\mathbb{R}} \psi(x,\, \lambda^*)} = 0.$$

So the constant must be 0. By (4.16), we have

$$\frac{e^{\lambda x}\mathbf{w}(0,\, x)}{\psi(x,\, \lambda^*)} \leq \Pi_x^{\lambda^*}\left[\frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\psi(Y_t,\, \lambda^*)} \right].$$

Letting $t \to \infty$ and noticing that

$$\frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,\, Y_t)}{\psi(Y_t,\, \lambda^*)} \to 0, \qquad \Pi_x^{\lambda^*}\text{-a.s.},$$

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we have

$$\frac{e^{\lambda x}\mathbf{w}(0,x)}{\psi(x,\lambda^*)} \le \lim_{t\to\infty} \Pi_x^{\lambda^*}\left[\frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,Y_t)}{\psi(Y_t,\lambda^*)}\right] = \Pi_x^{\lambda^*}\left[\lim_{t\to\infty}\frac{e^{\lambda^* Y_t+\gamma(\lambda^*)t}\mathbf{w}(-t,Y_t)}{\psi(Y_t,\lambda^*)}\right] = 0,$$

where we used the bounded dominated convergence theorem. This implies $\mathbf{w}(0,x)\equiv 0$, which contradicts the definition of pulsating traveling waves. Therefore, we have $\beta>0$.

Step 5. In this step, we show that

$$\widehat{\mathbf{w}}(t,x,y)\to\beta \quad \text{uniformly in } t\in\left[0,\frac{1}{\nu^*}\right] \text{ as } x\to\infty. \tag{4.17}$$

Fix $y\in\mathbb{R}$. First we will show that $\widehat{\mathbf{w}}(t,x,y)$ is bounded for $(t,x)\in\left[0,\frac{1}{\nu^*}\right]\times[h^{-1}(z-y+\nu^*t),\infty)$, which implies that (t,x) satisfies $y-\nu^*t+h(x)\ge z>0$. Recall that $\phi:=\phi(x,\lambda^*)=e^{-\lambda^*x}\psi(x,\lambda^*)$. By (4.2), we can rewrite $\widehat{\mathbf{w}}$ as

$$\widehat{\mathbf{w}}(t,x,y)=\frac{e^{-\gamma(\lambda^*)t}\mathbf{w}(t,x)}{\phi(x,\lambda^*)\,(y-\gamma'(\lambda^*)t)-\phi_\lambda(x,\lambda^*)}.$$

By (2.2) and (2.3), a direct calculation yields

$$\frac{\partial\widehat{\mathbf{w}}}{\partial t}=\frac{1}{2}\frac{\partial^2\widehat{\mathbf{w}}}{\partial x^2}+\frac{(y-\gamma'(\lambda^*))\phi_x-\phi_{\lambda x}}{(y-\gamma'(\lambda^*))\phi-\phi_\lambda}\frac{\partial\widehat{\mathbf{w}}}{\partial x}-\mathbf{g}\mathbf{w}\widehat{\mathbf{w}}.$$

Similarly to Step 3 in the proof of Theorem 1.1, fix $\omega\in\Omega_0$; then for any x_0 large enough, there exist $k,j\in\mathbb{N}$ such that (t,x_0) is located between the curves $\mathcal{L}_j^1(\omega)$ and $\mathcal{L}_k^1(\omega)$ for any $t\in\left[0,\frac{1}{\nu^*}\right]$. Since $\mathcal{L}_j^1(\omega)$ and $\mathcal{L}_k^1(\omega)$ must intersect each other, it follows from the maximum principle that $\widehat{\mathbf{w}}(t,x_0,y)$ is bounded by the maximum on boundary $\mathcal{L}_j^1(\omega)$ and $\mathcal{L}_k^1(\omega)$. We know that $\widehat{\mathbf{w}}(t,x,y)$ along $(t,x)\in\mathcal{L}_k^1(\omega)$ converges as $t\to\infty$, and then $\widehat{\mathbf{w}}(t,x,y)$ is bounded on $\bigcup_{j\ge1}\mathcal{L}_j^1(\omega)$. Since $\widehat{\mathbf{w}}(t,x,y)$ is continuous and $\widehat{\mathbf{w}}(t,h^{-1}(z-y+\nu^*t),y)$ is bounded, we have that $\widehat{\mathbf{w}}(t,x_0,y)$ is bounded when x_0 is small. Therefore, $\widehat{\mathbf{w}}(t,x,y)$ is bounded in $\left[0,\frac{1}{\nu^*}\right]\times[h^{-1}(z-y+\nu^*t),\infty)$. We denote the bound by K_z.

Next, we will show (4.17). Recall that

$$\widehat{\mathbf{w}}(T,x,y) = \Pi_x^{(y-\nu^*T,\lambda^*)}\left(\widehat{\mathbf{w}}\big(T-t\wedge\tau_z(y-\nu^*T),B_{t\wedge\tau_z(y-\nu^*T)},y\big)e^{-\int_0^{t\wedge\tau_z(y-\nu^*T)}\mathbf{g}(B_s)\mathbf{w}(T-s,B_s)\mathrm{d}s}\right).$$

Letting $t\to\infty$, we have for $T\in\left[0,\frac{1}{\nu^*}\right]$

$$\widehat{\mathbf{w}}(T,x,y)\le\Pi_x^{(y-\nu^*T,\lambda^*)}\left(\beta\mathbf{1}_{\{\tau_z(y-\nu^*T)=\infty\}}+K_z\mathbf{1}_{\{\tau_z(y-\nu^*T)<\infty\}}\right) =\beta\left(1-\frac{z}{h(x)+y-\nu^*T}\right)+K_z\frac{z}{h(x)+y-\nu^*T},$$

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where we used (4.15). Therefore, we have

$$\limsup_{x\to\infty} \widehat{\mathbf{w}}(T, x, y) \le \beta \quad \text{uniformly for } T \in \left[0, \frac{1}{\nu^*}\right]. \tag{4.18}$$

On the other hand, we have the following estimate:

$$\widehat{\mathbf{w}}(T, x, y) \ge \Pi_x^{\left(y-\nu^*T,\lambda^*\right)} \left(\beta \mathbf{1}_{\{\tau_z(y-\nu^*T)=\infty\}} e^{-\int_0^\infty \mathbf{g}(B_t)\mathbf{w}(T-t,B_t)\mathrm{d}t}\right).$$

By (4.2), we get

$$\begin{aligned}\mathbf{g}(B_t)\mathbf{w}(T-t, B_t) &\le \|\mathbf{g}\|_\infty z K_z \max_{x\in[0,1]} \psi(x, \lambda^*) e^{-\lambda^* B_t+\gamma(\lambda^*)(T-t)}\\ &\le Ce^{-\lambda^*(y-\nu^*T+h(B_t)+\nu^*t)} = Ce^{-\lambda^* \widehat{R}^T_{\langle M\rangle_t}},\end{aligned}$$

where $\widehat{R}^T_t = y-\nu^*T+h(B_{T(t)})+\nu^*T(t)$ is a Bessel-3 process starting at $y-\nu^*T+h(x)$, $T(t)=\inf\{s>0:\langle M\rangle_s>t\}$, and the constant C does not depend on T. Thus we have

$$\begin{aligned}\widehat{\mathbf{w}}(T, x, y) &\ge \Pi_x^{\left(y-\nu^*T,\lambda^*\right)}\left(\beta \mathbf{1}_{\{\tau_z(y-\nu^*T)=\infty\}} e^{-\int_0^\infty Ce^{-\lambda^*\widehat{R}^T_{\langle M\rangle_t}}\mathrm{d}t}\right)\\ &\ge \Pi_x^{\left(y-\nu^*T,\lambda^*\right)}\left(\beta \mathbf{1}_{\{\tau_{z+\nu^*T}=\infty\}} e^{-\int_0^\infty Ce^{-\lambda^*\widehat{R}^T_t}\mathrm{d}t}\right),\end{aligned}$$

where $\tau_z(y-\nu^*T)=\tau_{z+\nu^*T}$. Define the stopping time

$$\sigma_x(\widehat{R}^T) := \inf\left\{t\ge 0 : \widehat{R}^T_t = y-\nu^*T+h(x)\right\}$$

and the function

$$f(x) := \Pi_x^{\left(y-\nu^*T,\lambda^*\right)}\left(\beta \mathbf{1}_{\{\tau_{z+\nu^*T}=\infty\}} e^{-\int_0^\infty Ce^{-\lambda^*\widehat{R}^T_t}\mathrm{d}t}\right).$$

By the Markov property, for $x_1<x_2$ we have

$$f(x_1) = \Pi_{x_1}^{\left(y-\nu^*T,\lambda^*\right)}\left[\mathbf{1}_{\{\sigma_{x_2}(\widehat{R}^T)<\tau_{z+\nu^*T}\}} e^{-\int_0^{\sigma_{x_2}(\widehat{R}^T)} Ce^{-\lambda^*\widehat{R}^T_t}\mathrm{d}t} f(x_2)\right] \le f(x_2);$$

that is, $f(x)$ is increasing. Put

$$f(\infty) := \lim_{x\to\infty} f(x).$$

Since $\beta>0$ and $\int_0^\infty e^{-\lambda^*\widehat{R}^T_t}\mathrm{d}t<\infty$, $\Pi_x^{\left(y-\nu^*T,\lambda^*\right)}$-a.s., we have $f(x)>0$. Moreover,

$$\begin{aligned}\beta f(x) &= \lim_{n\to\infty} \Pi_x^{\left(y-\nu^*T,\lambda^*\right)}\left[\beta \mathbf{1}_{\{\sigma_{x+n}(\widehat{R}^T)<\tau_{z+\nu^*T}\}} e^{-\int_0^{\sigma_{x+n}(\widehat{R}^T)} Ce^{-\lambda^*\widehat{R}^T_t}\mathrm{d}t} f(x+n)\right]\\ &= \Pi_x^{\left(y-\nu^*T,\lambda^*\right)}\left[\lim_{n\to\infty} \beta \mathbf{1}_{\{\sigma_{x+n}(\widehat{R}^T)<\tau_{z+\nu^*T}\}} e^{-\int_0^{\sigma_{x+n}(\widehat{R}^T)} Ce^{-\lambda^*\widehat{R}^T_t}\mathrm{d}t}\right] \lim_{n\to\infty} f(x+n)\\ &= \Pi_x^{\left(y-\nu^*T,\lambda^*\right)}\left[\beta \mathbf{1}_{\{\tau_{z+\nu^*T}=\infty\}} e^{-\int_0^\infty Ce^{-\lambda^*\widehat{R}^T_t}\mathrm{d}t}\right] f(\infty) = f(x)f(\infty).\end{aligned}$$

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Therefore, $f(\infty) = \beta$. Combining this with $\widehat{\mathbf{w}}(T, x, y) \geq f(x)$, we have

$$\liminf_{x\to\infty} \widehat{\mathbf{w}}(T, x, y) \geq \beta \quad \text{uniformly for } T \in \left[0, \frac{1}{\nu^*}\right]. \tag{4.19}$$

By (4.18) and (4.19), we get (4.17).

Step 6. Since $h(x) = x - \psi_\lambda(x, \lambda^*)/\psi(x, \lambda^*)$ and ψ_λ/ψ is bounded, we have

$$\lim_{x\to\infty} \widehat{\mathbf{w}}(t, x, y) = \lim_{x\to\infty} \frac{e^{\lambda^* x - \gamma(\lambda^*)t}\mathbf{w}(t, x)}{\psi(x, \lambda^*)x}.$$

Thus

$$\frac{e^{\lambda^* z - \gamma(\lambda^*)t}\mathbf{w}(t, z)}{\psi(z, \lambda^*)z} \to \beta \quad \text{uniformly in } t \in \left[0, \frac{1}{\nu^*}\right] \text{ as } z \to \infty.$$

Using an argument similar to that in Step 3 of the proof of Theorem 1.1, we have

$$\frac{e^{\lambda^* x}\mathbf{w}\left(\frac{y-x}{\nu^*}, y\right)}{\psi(y, \lambda^*)(-\lfloor y - x\rfloor + y)} \to \beta \quad \text{uniformly in } y \in [0, 1] \text{ as } x \to \infty.$$

By $|-\lfloor y - x\rfloor + y - x| \leq 1$, we have

$$\frac{e^{\lambda^* x}\mathbf{w}\left(\frac{y-x}{\nu^*}, y\right)}{x\psi(y, \lambda^*)} \to \beta \quad \text{uniformly in } y \in [0, 1] \text{ as } x \to \infty.$$

Step 7. For general branching mechanism, $\mathbf{w} = 1 - \mathbf{u}$ satisfies

$$\frac{\partial \mathbf{w}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{w}}{\partial x^2} + \mathbf{g} \cdot (1 - \mathbf{w} - \mathbf{f}(1 - \mathbf{w})).$$

Recall that $A(w) = m - \frac{1-w-\mathbf{f}(1-w)}{w}$. As in Step 3, it suffices to show that

$$\int_0^\infty \mathbf{g}(B_t)\mathbf{w}(-t, B_t)\mathrm{d}t < \infty, \qquad \Pi_x^{(y,\lambda^*)}\text{-a.s.}$$

Since $B_t + \nu^* t$ behaves like $\sqrt{t}$ and $\mathbf{w}(t, x)$ decays exponentially fast, we have

$$\int_0^\infty A\left(e^{-c\sqrt{t}}\right)\mathrm{d}t < \infty \text{ for some } c > 0 \Longrightarrow \int_0^\infty \mathbf{g}(B_t)A(\mathbf{w})(-t, B_t)\mathrm{d}t < \infty.$$

Set $s = e^{-c\sqrt{t}}$; then

$$\int_0^\infty A\left(e^{-c\sqrt{t}}\right)\mathrm{d}t < \infty \Longleftrightarrow \int_0^1 A(s)\frac{|\log s|}{s}\mathrm{d}s < \infty.$$

By [5, Theorem 2], it holds that for $a > 1$,

$$\int_0^1 A(s)\frac{|\log s|^a}{s}\mathrm{d}s < \infty \Longleftrightarrow \mathbf{E}\left(L(\log^+ L)^{1+a}\right) < \infty. \tag{4.20}$$

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Actually the proof of [5, Theorem 2] also works for $a = 1$. So if $\mathbf{E}\big(L(\log^+ L)^2\big) < \infty$, there exists $\beta > 0$ such that

$$1 - \mathbf{u}\left(\frac{y-x}{\nu^*}, y\right) \sim \beta x e^{-\lambda^* x} \psi(y, \lambda^*) \quad \text{uniformly in } y \in [0, 1] \text{ as } x \to +\infty.$$

This completes the proof. □

Remark 4.1. The asymptotic behavior of pulsating traveling waves was studied analytically by Hamel [7], who considered the following more general equation:

$$u_t - \nabla \cdot (A(z)\nabla u) + q(z) \cdot \nabla u = f(z, u), \quad z \in \overline{\Omega},$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain, and $A(z)$, $q(z)$, $f(z, u)$ are periodic in some sense. In our case,

$$\Omega = \mathbb{R},\ A(z) \equiv \frac{1}{2},\ q(z) \equiv 0,\ \text{and}\ f(z, u) = \mathbf{g}(z)(1 - u - \mathbf{f}(1-u)).$$

The main result of the paper, [7, Theorem 1.3], is similar to our Theorems 1.1 and 1.2, with $\phi(t, x)$ in [7] corresponding to $1 - \mathbf{u}\big(\frac{t+x}{\nu}, x\big)$ in this paper, and $p^-(x, y)$, $p^+(x, y)$ corresponding to 0, 1 respectively.

The result in [7, Theorem 1.3] was proved under the assumptions [7, (1.4), (1.7), and (1.8)]. In our setup, [7, (1.4)] is equivalent to $\gamma(0) > 0$, which is true under our assumptions. The condition [7, (1.8)] is equivalent to

$$f(x, s) = \mathbf{g}(x)(1 - s - \mathbf{f}(1-s)) \le m\mathbf{g}(x)s \quad \text{for } s \in [0, 1],$$

which is true for any generating function $\mathbf{f}$. The condition [7, (1.7)] says that there exist $\alpha > 0$ and $\gamma > 0$ such that the map $(x, s) \mapsto \mathbf{g}(x)(\mathbf{f}'(1-s) - 1)$ belongs to $C^{0,\alpha}(\mathbb{R} \times [0, \gamma])$, which is equivalent to $g \in C^{0,\alpha}(\mathbb{R})$ and $\mathbf{f}'(1-s) \in C^{0,\alpha}([0, \gamma])$. We claim that

$$\mathbf{f}'(1-s) \in C^{\alpha}([0, \gamma]) \Longrightarrow \forall p \ge 1,\ \mathbf{E}(L(\log^+ L)^p) < \infty.$$

Thus the condition $\mathbf{f}'(1-s) \in C^{0,\alpha}([0, \gamma])$ is stronger than the condition $\mathbf{E}(L \log^+ L) < \infty$ in the supercritical case and $\mathbf{E}\big(L(\log^+ L)^2\big) < \infty$ in the critical case. Now we prove the claim. Notice that

$$A(w) = m - \frac{1 - w - \mathbf{f}(1-w)}{w} = m + 1 - \frac{1 - f(1-w)}{w} = m + 1 - f'(1 - \theta w),$$

where $\theta \in [0, 1]$ and the last equality follows from the mean value theorem. Since $f'(1) = m + 1$ and $\mathbf{f}'(1-s) \in C^{\alpha}([0, \gamma])$, we have

$$A(w) = |m + 1 - f'(1 - \theta w)| \le C(\theta w)^{\alpha} \le C w^{\alpha}, \quad \forall w \le \gamma,$$

for some constant C. Therefore, for any constant $c > 0$,

$$\int_0^{\infty} A\Big(e^{-ct^{1/p}}\Big) \mathrm{d}t \le \int_0^{\infty} C e^{-c\alpha t^{1/p}} \mathrm{d}t < \infty.$$

Using the substitution $s = e^{-ct^{1/p}}$, we get

$$\int_0^1 A(s) \frac{|\log s|^{p-1}}{s} \mathrm{d}s < \infty.$$

By (4.20), this implies $\mathbf{E}(L(\log^+ L)^p) < \infty$.

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5. Proof of Theorem 1.3

The uniqueness of the pulsating traveling wave was proved analytically in [9, Theorem 1.1]. In this section, we will use probabilistic methods to prove the uniqueness in the supercritical case $|\nu| > \nu^*$ and critical case $|\nu| = \nu^*$.

5.1. Martingales on stopping lines

First, we introduce the space of Galton–Watson trees. Let $\mathbb{T}$ be the space of Galton–Watson trees. A Galton–Watson tree $\tau \in \mathbb{T}$ is a point in the space of possible Ulam–Harris labels

$$\Omega = \emptyset \cup \bigcup_{n \in \mathbb{N}} (\mathbb{N})^n,$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$, such that

(i) $\emptyset \in \tau$ (the ancestor);

(ii) if $u, v \in \Omega$, then $uv \in \tau$ implies $u \in \tau$;

(iii) for all $u \in \tau$, there exists $A_u \in \{0, 1, 2, \ldots\}$ such that for $j \in \mathbb{N}$, $j \in \tau$ if and only if $1 \leq j \leq 1 + A_u$.

(Here $1 + A_u$ is the number of offspring of u, and A_u has the same distribution as L.)

Each particle $u \in \tau$ has a mark $(\eta_u, B_u) \in \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R})$, where η_u is the lifetime of u and B_u is the motion of u relative to its birth position. Then the birth time of u can be written as $b_u = \sum_{v<u} \eta_v$, the death time of u is $d_u = \sum_{v \leq u} \eta_v$, and the position of u at time t is given by $X_u(t) = \sum_{v<u} B_v(\eta_v) + B_u(t - b_u)$, where $v < u$ denotes that u is a descendant of v.

Now, on the space–time half-plane $\{(y, t) : y \in \mathbb{R}, t \in \mathbb{R}^+\}$, consider the barrier $\Gamma^{(x,\nu)}$ described by the line $y + \nu t = x$ for $x > 0$ and $\nu \geq \nu^*$. When a particle hits this barrier, it is stopped immediately. Let $C(x, \nu)$ denote the random collection of particles stopped at the barrier, which is known as a stopping line.

By [16, Theorem 1.1], $W_t(\lambda^*) \to 0$ $\mathbb{P}_x$-a.s., so we have

$$e^{-\lambda^*(m_t + \nu^* t)} \min_{x \in [0,1]} \psi(x, \lambda^*) \leq W_t(\lambda^*) \to 0,$$

where $m_t = \min\{X_u(t) : u \in N_t\}$. This yields

$$\lim_{t \to \infty} (m_t + \nu^* t) = +\infty. \tag{5.1}$$

Therefore, all lines of descent from the ancestor will hit $\Gamma^{(x,\nu)}$ with probability one for all $x > x_0$, where x_0 is the position of the ancestor at time $t = 0$. Similarly to the argument in [13] for BBM, we have $\lim_{x\to\infty} \inf\{|u| : u \in C(x, \nu)\} = \infty$, where $|u|$ is the generation of the particle u.

For any $u \in C(x, \nu)$, let σ_u denote the time at which the particle u hits the barrier $\Gamma^{(x,\nu)}$. Let $\mathcal{F}_{C(x,\nu)}$ be the σ-field generated by

$$\left\{ \begin{array}{l} (w, A_w, \eta_w, \{B_w(s) : s \in [0, \eta_w]\} : \exists u \in C(x, \nu),\ s.t.\ w < u) \text{ and} \\ (u, \{B_u(s) : s \in [0, \sigma_u - b_u]\} : u \in C(x, \nu)) \end{array} \right\}.$$

Using traveling wave solutions of the KPP equation, Chauvin [4] exhibited an intrinsic class of martingales. An argument similar to the one used in [4] gives the analogous martingales for BBMPE.

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Theorem 5.1. *Suppose that $\nu \geq \nu^*$ and that $\mathbf{u}(t, x)$ is a pulsating traveling wave with speed ν. Define*

$$M_x(\nu) := \prod_{u \in C(x,\nu)} \mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big). \tag{5.2}$$

Then, for any $z \in \mathbb{R}$, $\{M_x(\nu) : x \geq z\}$ is a $\mathbb{P}_z$-martingale with respect to $\{\mathcal{F}_{C(x,\nu)}, x \geq z\}$, has expectation u(0, z), and converges $\mathbb{P}_z$-a.s. and in $L^1(\mathbb{P}_z)$.

We prove this theorem via several lemmas.

Lemma 5.1. *Consider BBMPE starting from x and with branching rate function* $\mathbf{g}$*. Let σ be the first fission time and let $1 + A$ denote the number of offspring of the initial particle. Let* $\mathbf{f}(s) = \mathbf{E}s^{1+A}$ *and* $\mathbf{P}(A = k) = p_k$*. Then*

$$\mathbb{P}_x\big(\mathbf{1}_{\{\sigma>t\}}\mathbf{u}(-t, B_t) + \mathbf{1}_{\{\sigma\leq t\}}\mathbf{u}^{1+A}(-\sigma, B_\sigma)\big) = \mathbf{u}(0, x).$$

Proof. Note that

$$\mathbb{P}_x(\sigma > t \mid \{B_s : s \leq t\}) = e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}.$$

Put

$$f(t, B_t) = e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\mathbf{u}(-t, B_t) + \int_0^t \mathbf{g}(B_s)e^{-\int_0^s \mathbf{g}(B_r)\mathrm{d}r} \sum_k p_k \mathbf{u}^{k+1}(-s, B_s)\mathrm{d}s.$$

A standard computation using Itô's formula shows that

$$\begin{aligned}
&\mathrm{d}f(t, B_t)\\
&= e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\frac{\partial \mathbf{u}(-t, B_t)}{\partial x}\mathrm{d}B_t + \frac{1}{2}e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\frac{\partial^2 \mathbf{u}(-t, B_t)}{\partial x^2}\mathrm{d}t\\
&\quad - \mathbf{g}(B_t)e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\mathbf{u}(-t, B_t)\mathrm{d}t - e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\frac{\partial \mathbf{u}(-t, B_t)}{\partial t}\mathrm{d}t\\
&\quad + \mathbf{g}(B_t)e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\sum_k p_k \mathbf{u}^{k+1}(-t, B_t)\mathrm{d}t\\
&= e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\frac{\partial \mathbf{u}(-t, B_t)}{\partial x}\mathrm{d}B_t + e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\times\\
&\quad \left(\frac{1}{2}\frac{\partial^2 \mathbf{u}(-t, B_t)}{\partial x^2} - \mathbf{g}(B_t)\mathbf{u}(-t, B_t) - \frac{\partial \mathbf{u}(-t, B_t)}{\partial t} + \mathbf{g}(B_t)\mathbf{f}(\mathbf{u}(-t, B_t))\right)\mathrm{d}t\\
&= e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\frac{\partial \mathbf{u}(-t, B_t)}{\partial x}\mathrm{d}B_t.
\end{aligned}$$

Hence $f(t, B_t)$ is a martingale and

$$\Pi_x f(t, B_t) = \Pi_x f(0, B_0) = \mathbf{u}(0, x).$$

Note that

$$\begin{aligned}
\mathbb{P}_x(\mathbf{1}_{\{\sigma>t\}}\mathbf{u}(-t, B_t)) &= \mathbb{P}_x\big(\mathbb{P}_x\left[\mathbf{1}_{\{\sigma>t\}}\mathbf{u}(-t, B_t) \mid \{B_s : s \leq t\}\right]\big)\\
&= \mathbb{P}_x\big(\mathbf{u}(-t, B_t)\mathbb{P}_x\left[\mathbf{1}_{\{\sigma>t\}} \mid \{B_s : s \leq t\}\right]\big)\\
&= \Pi_x\left(e^{-\int_0^t \mathbf{g}(B_s)\mathrm{d}s}\mathbf{u}(-t, B_t)\right).
\end{aligned}$$

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Similarly, we have

$$\mathbb{P}_x\Big(\mathbf{1}_{\{\sigma\le t\}}\mathbf{u}^{A+1}(-\sigma, B_\sigma)\Big) = \mathbb{P}_x\Big(\mathbb{P}_x\Big[\mathbf{1}_{\{\sigma\le t\}}\mathbf{u}^{A+1}(-\sigma, B_\sigma) \mid \{B_s : s\le t\}\Big]\Big)$$
$$= \Pi_x\left(\int_0^t \mathbf{g}(B_s)e^{-\int_0^s \mathbf{g}(B_r)\mathrm{d}r}\sum_k p_k\mathbf{u}^{k+1}(-s, B_s)\mathrm{d}s\right).$$

Therefore,

$$\mathbb{P}_x\Big(\mathbf{1}_{\{\sigma>t\}}\mathbf{u}(-t, B_t) + \mathbf{1}_{\{\sigma\le t\}}\mathbf{u}^{1+A}(-\sigma, B_\sigma)\Big) = \Pi_x f(t, B_t) = \mathbf{u}(0, x).$$

$\square$

Recall that $\Omega = \emptyset \cup \bigcup_{n\in\mathbb{N}}(\mathbb{N})^n$ and that, for $u\in\Omega$, σ_u denotes the time at which u hits the barrier $\Gamma^{(x,\nu)}$. For a fixed stopping line $C(x, \nu)$, we define

$$\begin{aligned}
L_\tau &:= \{u\in\Omega : b_u \le \sigma_u < d_u\} = C(x, \nu),\\
D_\tau &:= \{u\in\Omega :\ \exists v\in\Omega,\ v<u,\ v\ne u,\ v\in L_\tau\},\\
A_\tau^{(n)} &:= \{u\in\Omega :\ |u| = n,\ u\notin D_\tau,\ u\notin L_\tau\}.
\end{aligned}$$

In other words, D_τ is the set of strict descendants of the stopping line and $A_\tau^{(n)}$ is the set of the nth-generation particles u such that neither u nor the ancestors of u hit the barrier $\Gamma^{(x,\nu)}$. Let

$$\mathcal{H}_n = \sigma\left(\{(u, A_u, \eta_u, \{B_u : u\in[0, \eta_u]\})\} : |u|\le n-1\right).$$

Recall that $M_x(\nu)$ is defined by (5.2).

Lemma 5.2. *For any $z<x$,*

$$\mathbb{E}_z(M_x(\nu)) = \mathbf{u}(0, z).$$

Proof. For $n\in\mathbb{N}$, similarly to [4], we introduce the following approximation of $M_x(\nu)$:

$$M^{(n)} = \prod_{u\in L_\tau, |u|\le n}\mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big)\prod_{u\in A_\tau^{(n)}}\mathbf{u}^{1+A_u}\big(-d_u, X_u(d_u)\big).$$

It is easy to see that $M^{(n)}\in\mathcal{H}_{n+1}$, $n\ge 0$. We first prove that $\{M^{(n)}, n\ge 0\}$ is an $\mathcal{H}_{n+1}$-martingale:

$$\mathbb{E}_x\big(M^{(n+1)}\mid\mathcal{H}_{n+1}\big) = \prod_{u\in L_\tau, |u|\le n}\mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big)$$
$$\times\,\mathbb{E}_x\left(\prod_{u\in L_\tau, |u|=n+1}\mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big)\prod_{u\in A_\tau^{(n+1)}}\mathbf{u}^{1+A_u}\big(-d_u, X_u(d_u)\big)\mid\mathcal{H}_{n+1}\right). \tag{5.3}$$

Note that

$$\{u : u\in L_\tau,\ |u| = n+1\}\cup A_\tau^{(n+1)} = \{u : |u| = n+1,\ u\notin D_\tau\}.$$

Consider any particle u such that $|u| = n+1$ and $u\notin D_\tau$. If u is in L_τ, it occurs in the second product in (5.3); if not, it occurs in the third one. For any particle u, define

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$\mathbf{v}_u(t, x) = \mathbf{u}(t - b_u, x)$. Then given b_u, $\mathbf{v}_u(t, x)$ satisfies (1.3) and (1.4). By the Markov property and the branching property, we have

$$\begin{aligned}
&\mathbb{E}_x\Bigg(\prod_{u\in L_\tau, |u|=n+1} \mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big) \prod_{u\in A_\tau^{(n+1)}} \mathbf{u}^{1+A_u}\big(-d_u, X_u(d_u)\big) \mid \mathcal{H}_{n+1}\Bigg) \\
&= \prod_{|u|=n+1, u\notin D_\tau} \mathbb{E}_x\Big(\mathbf{1}_{\{\sigma_u<d_u\}}\mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big) + \mathbf{1}_{\left\{\sigma_u\geq d_u\right\}}\mathbf{u}^{1+A_u}\big(-d_u, X_u(d_u)\big) \mid \mathcal{H}_{n+1}\Big) \\
&= \prod_{|u|=n+1, u\notin D_\tau} \mathbb{E}_{X_u(b_u)}\Big(\mathbf{1}_{\{\sigma_u<d_u\}}\mathbf{u}(-\sigma_u, X_u(\sigma_u - b_u)) + \\
&\qquad\qquad \mathbf{1}_{\left\{\sigma_u\geq d_u\right\}}\mathbf{u}^{1+A_u}(-d_u, X_u(d_u - b_u))\Big) \\
&= \prod_{|u|=n+1, u\notin D_\tau} \mathbb{E}_{X_u(b_u)}\Big(\mathbf{1}_{\{\sigma_u-b_u<d_u-b_u\}}\mathbf{v}_u(-(\sigma_u - b_u), X_u(\sigma_u - b_u)) + \\
&\qquad\qquad \mathbf{1}_{\{\sigma_u-b_u\geq d_u-b_u\}}\mathbf{v}_u^{1+A_u}\big(-(d_u - b_u), X_u(d_u - b_u)\big)\Big) \\
&= \prod_{|u|=n+1, u\notin D_\tau} \mathbf{v}_u(0, X_u(b_u)).
\end{aligned}$$

The last equality follows from Lemma 5.1; the only difference is that we substitute the random time $\sigma_u - b_u$ for the deterministic time t in Lemma 5.1. Putting together the offspring of the same particle, it becomes

$$\begin{aligned}
\prod_{|u|=n+1, u\notin D_\tau} \mathbf{v}_u(0, X_u(b_u)) &= \prod_{|u|=n+1, u\notin D_\tau} \mathbf{u}\big(-b_u, X_u(b_u)\big) \\
&= \prod_{|u|=n, u\notin D_\tau, u\notin L_\tau} \mathbf{u}^{1+A_u}\big(-d_u, X_u(d_u)\big) = \prod_{u\in A_\tau^{(n)}} \mathbf{u}^{1+A_u}\big(-d_u, X_u(d_u)\big).
\end{aligned}$$

This shows that $M^{(n)}$ is an $\mathcal{H}_{n+1}$-martingale.

Note that (5.1) implies that all lines of descent from the ancestor will hit $\Gamma^{(x,\nu)}$ $\mathbb{P}_z$-a.s. This yields $A_\tau^{(n)} \to \emptyset$ as $n \to \infty$, and thus $M^{(n)} \to M_x(\nu)$ $\mathbb{P}_z$-a.s. Since $0 \leq \mathbf{u}(t, x) \leq 1$, $M^{(n)}$ is bounded. This yields the L^1-convergence of $M^{(n)}$ to $M_x(\nu)$. Therefore,

$$\mathbb{E}_z(M_x(\nu)) = \mathbb{E}_z\big(M^{(0)}\big) = \mathbf{u}(0, z).$$

This completes the proof of this lemma. □

Now we turn to the proof of Theorem 5.1.

Proof of Theorem 5.1. Fix $\nu \geq \nu^*$. To distinguish the times when a particle hits different barriers, we let σ_u^x denote the time when u hits the barrier $\Gamma^{(x,\nu)}$. For $y > x$,

$$M_y(\nu) = \prod_{u\in C(y,\nu)} \mathbf{u}\big(-\sigma_u^y, X_u\big(\sigma_u^y\big)\big) = \prod_{w\in C(x,\nu)} \prod_{u\in C(y,\nu), u>w} \mathbf{u}\big(-\sigma_u^y, X_u\big(\sigma_u^y\big)\big),$$

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where $u > w$ means that u is a descendant of w. Therefore, by the special Markov property of $\{Z_t, t \geq 0\}$, we have that

$$\begin{aligned}\mathbb{E}_x\big(M_y(\nu) \mid \mathcal{F}_{C(x,\nu)}\big) &= \prod_{w\in C(x,\nu)} \mathbb{E}_x\left(\prod_{u\in C(y,\nu),u>w} \mathbf{u}\big(-\sigma_u^y, X_u\big(\sigma_u^y\big)\big) \,\Big|\, \mathcal{F}_{C(x,\nu)}\right)\\ &= \prod_{w\in C(x,\nu)} \mathbb{E}_{X_w(\sigma_w^x)} \prod_{u\in C(y,\nu),u>w} \mathbf{u}\big(-\sigma_u^y, X_u\big(\sigma_u^y - \sigma_w^x\big)\big)\\ &= \prod_{w\in C(x,\nu)} \mathbf{u}\big(-\sigma_w^x, X_u\big(\sigma_w^x\big)\big) = M_x(\nu),\end{aligned}$$

where the second-to-last equality follows from Lemma 5.2 and an argument similar to that in the proof of Lemma 5.2, by defining $\mathbf{v}(t, x) = \mathbf{u}(t - \sigma_w^x, x)$. The proof is complete. □

5.2. Uniqueness in the supercritical and critical cases

In this section, we give a probabilistic proof of the uniqueness of the pulsating traveling wave with speed $|\nu| \geq \nu^*$.

Theorem 1.1 implies that for $\nu > \nu^*$,

$$-\log \mathbf{u}\left(\frac{y-x}{\nu}, y\right) \sim \beta e^{-\lambda x}\psi(y, \lambda) \quad \text{uniformly in } y \in [0, 1] \text{ as } x \to +\infty. \tag{5.4}$$

Theorem 1.2 implies that

$$-\log \mathbf{u}\left(\frac{y-x}{\nu^*}, y\right) \sim \beta x e^{-\lambda^* x}\psi(y, \lambda^*) \quad \text{uniformly in } y \in [0, 1] \text{ as } x \to +\infty. \tag{5.5}$$

Recall that the additive martingale $\{W_t(\lambda)_{t\geq 0}, \mathbb{P}_x\}$ is defined by (1.5). By [16, Theorem 1.1], for any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, the limit $W(\lambda, x) := \lim_{t\uparrow\infty} W_t(\lambda)$ exists $\mathbb{P}_x$-a.s. and $W(\lambda, x)$ is an $L^1(\mathbb{P}_x)$-limit when $|\lambda| < \lambda^*$ and $\mathbf{E}(L \log^+ L) < \infty$. Recall that $C(x, \nu)$ was defined at the beginning of Section 5.1. In the spirit of [13], we define

$$W_{C(x,\nu)}(\lambda) := \sum_{u\in C(x,\nu)} e^{-\lambda X_u(\sigma_u)-\gamma(\lambda)\sigma_u}\psi(X_u(\sigma_u), \lambda),$$

where $\nu = \gamma(\lambda)/\lambda$. Using arguments similar to those of [13, Theorem 8], we can obtain the following result, whose proof is omitted.

Proposition 5.1. *For any $z \in \mathbb{R}$, $\{W_{C(x,\nu)}(\lambda) : x \geq z\}$ is a $\mathbb{P}_z$-martingale with respect to the filtration $\{\mathcal{F}_{C(x,\nu)} : x \geq z\}$, and, as $x \to \infty$, $W_{C(x,\nu)}(\lambda)$ converges a.s. and in $L^1(\mathbb{P}_z)$ to $W(\lambda, z)$ when $|\lambda| \in [0, \lambda^*)$ and* $\mathbf{E}(L \log^+ L) < \infty$.

Let $Y_x = \sum_{u\in C(x,\nu)} \delta_{\{X_u(\sigma_u)\}}$, where $\{x\}$ is the fractional part of x. Then Y_x is a point measure on $[0, 1]$. Notice that

$$\begin{aligned}W_{C(x,\nu)}(\lambda) &= \sum_{u\in C(x,\nu)} e^{-\lambda(X_u(\sigma_u)+\nu\sigma_u)}\psi(X_u(\sigma_u), \lambda)\\ &= \sum_{u\in C(x,\nu)} e^{-\lambda x}\psi(X_u(\sigma_u), \lambda) = e^{-\lambda x}\langle Y_x, \psi\rangle.\end{aligned}$$

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Thus by Proposition 5.1, we have

$$e^{-\lambda x}\langle Y_x, \psi\rangle \overset{\mathbb{P}_z\text{-a.s.}}{\longrightarrow} W(\lambda, z), \quad \text{as } x \to \infty.$$

Theorem 5.2. *Suppose* $|\nu| > \nu^*$ *and* $\mathbf{E}(L \log^+ L) < \infty$. *If* $\mathbf{u}(t, x)$ *is a pulsating traveling wave with speed* ν, *then there exists* $\beta > 0$ *such that*

$$\mathbf{u}(t, x) = \mathbb{E}_x \exp\left\{-\beta e^{\gamma(\lambda)t} W(\lambda, x)\right\}, \tag{5.6}$$

where $|\lambda| \in (0, \lambda^*)$ *is such that* $\nu = \frac{\gamma(\lambda)}{\lambda}$.

Proof. We assume that $\lambda \geq 0$. The case $\lambda < 0$ can be analyzed by symmetry. By Theorem 5.1, $M_x(\nu)$ is a $\mathbb{P}_z$-martingale with respect to $\{\mathcal{F}_{C(x,\nu)} : x \geq z\}$ with expectation $\mathbf{u}(0, z)$ and converges a.s. and in $L^1(\mathbb{P}_z)$, where

$$M_x(\nu) = \exp\left\{\sum_{u\in C(x,\nu)} \log \mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big)\right\}.$$

So there exists a non-negative random variable Y such that

$$-\sum_{u\in C(x,\nu)} \log \mathbf{u}\big(-\sigma_u, X_u(\sigma_u)\big) \overset{\mathbb{P}_z\text{-a.s.}}{\longrightarrow} Y, \quad \text{as } x \to \infty.$$

Note that

$$X_u(\sigma_u) = \{X_u(\sigma_u)\} + \lfloor X_u(\sigma_u)\rfloor,$$
$$-\sigma_u = \frac{X_u(\sigma_u) - x}{\nu} = \frac{\{X_u(\sigma_u)\} - x}{\nu} + \frac{\lfloor X_u(\sigma_u)\rfloor}{\nu}.$$

The previous convergence can be written as

$$\left\langle Y_x,\ -\log \mathbf{u}\left(\frac{\cdot - x}{\nu}, \cdot\right)\right\rangle \overset{\mathbb{P}_z\text{-a.s.}}{\longrightarrow} Y, \quad \text{as } x \to \infty.$$

By (5.4),

$$\lim_{x\to\infty}\left\langle Y_x, -\log \mathbf{u}\left(\frac{\cdot - x}{\nu}, \cdot\right)\right\rangle = \beta \lim_{x\to\infty}\left\langle e^{-\lambda x} Y_x, \psi(\cdot, \lambda)\right\rangle,$$

and thus $Y = \beta W(\lambda, z)$. By the dominated convergence theorem,

$$\mathbf{u}(0, z) = \lim_{x\to\infty} \mathbb{E}_z e^{\langle Y_x, \log \mathbf{u}(\frac{\cdot - x}{\nu}, \cdot)\rangle} = \mathbb{E}_z \lim_{x\to\infty} e^{\langle Y_x, \log \mathbf{u}(\frac{\cdot - x}{\nu}, \cdot)\rangle}$$
$$= \mathbb{E}_z e^{-Y} = \mathbb{E}_z e^{-\beta W(\lambda, z)}.$$

Theorem 1.3(i) of [16] shows that $\mathbb{E}_x \exp\{-\beta e^{\gamma(\lambda)t} W(\lambda, x)\}$, as a function of (t, x), is a solution of the following initial value problem:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g}\cdot(\mathbf{f}(\mathbf{u}) - \mathbf{u}), \quad \mathbf{u}(0, x) = \mathbb{E}_x e^{-\beta W(\lambda, x)}.$$

Therefore, $\mathbf{u}(t, x)$ and $\mathbb{E}_x \exp\{-\beta e^{\gamma(\lambda)t} W(\lambda, x)\}$ are solutions of the above initial value problem. The uniqueness of solutions of initial value problems implies that (5.6) holds. □

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Now we consider the critical case. Recall that on the space–time half-plane $\{(y, t) : y \in \mathbb{R},\ t \in \mathbb{R}^+\}$, the barrier $\Gamma^{(x,\nu)}$ is described by the line $y + \nu t = x$ for $x > 0$, and $C(x, \nu)$ is the random collection of particles stopped at the barrier. Define the barrier $\mathbf{\Gamma}^{(-x,\lambda)}$ described by $y = h^{-1}(-x - \gamma'(\lambda)t)$, and define $\mathbf{C}(-x, \lambda)$ to be the random collection of particles hitting this barrier.

Define $\widetilde{C}(z, \nu^*)$ to be the set of particles that stopped at the barrier $\Gamma^{(z,\nu^*)}$ before meeting the barrier $\mathbf{\Gamma}^{(-x,\lambda^*)}$. Fix $x > 0$ and $y > h^{-1}(-x)$. Define

$$Y_z^{(-x,\lambda^*)} = \sum_{u \in \widetilde{C}(z,\nu^*)} \delta_{\{X_u(\sigma_u)\}}, \qquad z \geq y.$$

Consider the sequence $\left\{V^x_{\widetilde{C}(z,\nu^*)},\, z \geq y\right\}$, where

$$\begin{aligned} V^x_{\widetilde{C}(z,\nu^*)} :=& \sum_{u \in \widetilde{C}(z,\nu^*)} e^{-\gamma(\lambda^*)\sigma_u - \lambda^* X_u(\sigma_u)} \psi(X_u(\sigma_u), \lambda^*) \times \\ & \left(x + \gamma'(\lambda^*)\sigma_u + X_u(\sigma_u) - \frac{\psi_\lambda(X_u(\sigma_u), \lambda^*)}{\psi(X_u(\sigma_u), \lambda^*)} \right) \\ =& \sum_{u \in \widetilde{C}(z,\nu^*)} e^{-\lambda^* z} \psi(X_u(\sigma_u), \lambda^*) \left(x + z - \frac{\psi_\lambda(X_u(\sigma_u), \lambda^*)}{\psi(X_u(\sigma_u), \lambda^*)} \right) \\ =& \left\langle Y_z^{(-x,\lambda^*)},\ e^{-\lambda^* z} \psi(\cdot, \lambda^*) \left(x + z - \frac{\psi_\lambda(\cdot, \lambda^*)}{\psi(\cdot, \lambda^*)} \right) \right\rangle, \qquad z \geq y. \end{aligned}$$

Using similar arguments as in [13, Theorem 15], we have following proposition.

Proposition 5.2. *Let* $\left\{\mathcal{F}_{\widetilde{C}(z,\nu^*)} : z \geq y\right\}$ *be the natural filtration describing everything in the truncated branching tree up to the barrier* $\Gamma^{(z,\nu^*)}$. *If* $\mathbf{E}\big(L(\log^+ L)^2\big) < +\infty$, *then* $\left\{V^x_{\widetilde{C}(z,\nu^*)},\, z \geq y\right\}$ *is a* $\mathbb{P}_y$*-martingale with respect to* $\left\{\mathcal{F}_{\widetilde{C}(z,\nu^*)},\, z \geq y\right\}$, *and* $V^x_{\widetilde{C}(z,\nu^*)}$ *converges* $\mathbb{P}_y$*-a.s. and in* $L^1(\mathbb{P}_y)$ *to* $V^x(\lambda^*)$ *as* $z \to \infty$.

Proof. For $t > 0$, let

$$\widetilde{C}_t(z, \nu^*) = \left\{u \in \widetilde{C}(z, \nu^*) : \sigma_u \leq t\right\}$$

and

$$\widetilde{A}_t(z, \nu^*) = \left\{u \in \widetilde{N}_t^x : v \notin \widetilde{C}_t(z, \nu^*),\ \forall v \leq u\right\}.$$

Define

$$\begin{aligned} & V^x_{t \wedge \widetilde{C}(z,\nu^*)} \\ & := \sum_{u \in \widetilde{A}_t(z,\nu^*)} e^{-\gamma(\lambda^*)t - \lambda^* X_u(t)} \psi\big(X_u(t), \lambda^*\big) \left(x + \gamma'(\lambda^*)t + X_u(t) - \frac{\psi_\lambda\big(X_u(t), \lambda^*\big)}{\psi\big(X_u(t), \lambda^*\big)} \right) \\ & \quad + \sum_{u \in \widetilde{C}_t(z,\nu^*)} e^{-\lambda^* z} \psi(X_u(\sigma_u), \lambda^*) \left(x + z - \frac{\psi_\lambda(X_u(\sigma_u), \lambda^*)}{\psi(X_u(\sigma_u), \lambda^*)} \right). \end{aligned}$$

A straightforward calculation, similar to the proof of [16, Lemma 2.16], shows that

$$\mathbb{E}_y\Big(V_t^x(\lambda^*) | \mathcal{F}_{\widetilde{C}(z,\nu^*)}\Big) = V^x_{t \wedge \widetilde{C}(z,\nu^*)}. \tag{5.7}$$

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Since $\lim_{t\uparrow\infty}\left|\widetilde{A}_t(z,\nu^*)\right|=0$ and $\lim_{t\uparrow\infty}\widetilde{C}_t(z,\nu^*)=\widetilde{C}(z,\nu^*)$, letting $t\to\infty$ in (5.7), we have

$$\lim_{t\uparrow\infty}\mathbb{E}_y\Big(V_t^x(\lambda^*)|\mathcal{F}_{\widetilde{C}(z,\nu^*)}\Big)=V^x_{\widetilde{C}(z,\nu^*)}.$$

By [16, Theorem 4.2], $V_t^x(\lambda^*)$ converges to $V^x(\lambda^*)$ in $L^1(\mathbb{P}_y)$ as $t\to\infty$. Thus $\mathbb{E}_y\big(V_t^x(\lambda^*)|\mathcal{F}_{\widetilde{C}(z,\nu^*)}\big)$ converges to $\mathbb{E}_y\big(V^x(\lambda^*)|\mathcal{F}_{\widetilde{C}(z,\nu^*)}\big)$ in $L^1(\mathbb{P}_y)$. So

$$\mathbb{E}_y\Big(V^x(\lambda^*)|\mathcal{F}_{\widetilde{C}(z,\nu^*)}\Big)=V^x_{\widetilde{C}(z,\nu^*)}.$$

Letting $z\to\infty$ in (5.7), we get that $\mathbb{E}_y\big(V_t^x(\lambda^*)|\mathcal{F}_\infty\big)=V_t^x(\lambda^*)$, where $\mathcal{F}_\infty=\sigma\big(\cup_{z\geq y}\mathcal{F}_{\widetilde{C}(z,\nu^*)}\big)$. This implies that $V^x(\lambda^*)$ is $\mathcal{F}_\infty$-measurable. Hence

$$V^x_{\widetilde{C}(z,\nu^*)}=\mathbb{E}_y\Big(V^x(\lambda^*)|\mathcal{F}_{\widetilde{C}(z,\nu^*)}\Big)\overset{L^1(\mathbb{P}_y)/a.s.}{\longrightarrow}\mathbb{E}_y\Big(V^x(\lambda^*)\Big|\mathcal{F}_\infty\Big)=V^x(\lambda^*).$$

This completes the proof. □

Recall that the derivative martingale $\{\partial W_t(\lambda)_{t\geq 0},\mathbb{P}_x\}$ is defined by (1.6). By [16, Theorem 1.2], for any $|\lambda|\geq\lambda^*$ and $x\in\mathbb{R}$, the limit $\partial W(\lambda,x):=\lim_{t\uparrow\infty}\partial W_t(\lambda)$ exists $\mathbb{P}_x$-a.s., and $\partial W(\lambda,x)\in(0,\infty)$ when $\lambda=\lambda^*$ and $\mathbf{E}\big(L(\log^+L)^2\big)<\infty$.

Theorem 5.3. *Suppose* $\mathbf{E}\big(L(\log^+L)^2\big)<\infty$. *If* $\mathbf{u}(t,x)$ *is a pulsating traveling wave with speed* ν^**, then there exists* $\beta>0$ *such that*

$$\mathbf{u}(t,x)=\mathbb{E}_x\exp\big\{-\beta e^{\gamma(\lambda^*)t}\partial W(\lambda^*,x)\big\}. \tag{5.8}$$

Proof. From Proposition 5.2, we have

$$\lim_{z\to\infty}V^x_{\widetilde{C}(z,\nu^*)}=\lim_{z\to\infty}\left\langle Y_z^{(-x,\lambda^*)},\ e^{-\lambda^*z}\psi(\cdot,\lambda^*)\left(x+z-\frac{\psi_\lambda(\cdot,\lambda^*)}{\psi(\cdot,\lambda^*)}\right)\right\rangle=V^x(\lambda^*),\quad \mathbb{P}_y\text{-a.s.}$$

Notice that for fixed $x\geq 0$,

$$\left(x+z-\frac{\psi_\lambda(w,\lambda^*)}{\psi(w,\lambda^*)}\right)/z\to 1\quad \text{as } z\to\infty, \text{ uniformly in } w\in[0,1].$$

Therefore,

$$\lim_{z\to\infty}\left\langle Y_z^{(-x,\lambda^*)},\ ze^{-\lambda^*z}\psi(\cdot,\lambda^*)\right\rangle=V^x(\lambda^*),\quad \mathbb{P}_y\text{-a.s.}$$

Let $\gamma^{(-x,\lambda^*)}$ be the event that the BBMPE remains entirely to the right of $\mathbf{\Gamma}^{(-x,\lambda^*)}$. By (5.1) and $\nu^*=\gamma'(\lambda^*)$, we have $\inf_{t\geq 0}\{m_t+\gamma'(\lambda^*)t\}>-\infty$ $\mathbb{P}_y$-a.s. By the definition of $\mathbf{\Gamma}^{(-x,\lambda^*)}$, $\gamma^{(-x,\lambda^*)}=\{\forall\, t\geq 0, m_t\geq h^{-1}(-x-\gamma'(\lambda^*)t)\}$. Therefore

$$\mathbb{P}_y\big(\gamma^{(-x,\lambda^*)}\big)\geq\mathbb{P}_y\left(\inf_{t\geq 0}\{m_t+\gamma'(\lambda^*)t\}>-x+\max_{z\in[0,1]}\frac{\psi_\lambda(z,\lambda)}{\psi(z,\lambda)}\right)\uparrow 1\quad \text{as } x\to\infty,$$

which implies that

$$\lim_{x\to\infty}\mathbb{P}_y\big(\gamma^{(-x,\lambda^*)}\big)=1. \tag{5.9}$$

Note that on the event $\gamma^{(-x,\lambda^*)}$, $V^x(\lambda^*)=\partial W(\lambda^*,y)$ $\mathbb{P}_y$-a.s. and $Y_z^{(-x,\lambda^*)}=Y_z$, where $Y_z=\sum_{u\in C(z,\nu^*)}\delta_{\{X_u(\sigma_u)\}}$. Thus it follows that under $\mathbb{P}_y$,

$$\lim_{z\to\infty}\big\langle Y_z,\ ze^{-\lambda^*z}\psi(\cdot,\lambda^*)\big\rangle=\partial W(\lambda^*,y)\quad \text{on } \gamma^{(-x,\lambda^*)}.$$

By (5.9),

$$\lim_{z\to\infty}\left\langle Y_z,\ ze^{-\lambda^* z}\psi(\cdot,\lambda^*)\right\rangle = \partial W(\lambda^*, y)\quad \mathbb{P}_y\text{-a.s.}$$

So by the dominated convergence theorem and the asymptotic behavior (5.5),

$$\begin{aligned}\mathbf{u}(0,y) &= \lim_{z\to\infty}\mathbb{E}_y e^{\left\langle Y_z,\ \log \mathbf{u}\left(\frac{\cdot-z}{\nu^*},\cdot\right)\right\rangle} = \mathbb{E}_y \lim_{z\to\infty} e^{-\left\langle Y_z,\ -\log \mathbf{u}\left(\frac{\cdot-z}{\nu^*},\cdot\right)\right\rangle}\\ &= \mathbb{E}_y \lim_{z\to\infty} e^{-\langle Y_z,\ \beta z e^{-\lambda z}\psi(\cdot,\lambda^*)\rangle} = \mathbb{E}_y e^{-\beta\partial W(\lambda^*, y)}.\end{aligned}$$

Theorem 1.3(ii) of [16] shows that $\mathbb{E}_x \exp\left\{-\beta e^{\gamma(\lambda^*)t}\partial W(\lambda^*, x)\right\}$, as a function of (t, x), is a solution of the following initial value problem:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g}\cdot(\mathbf{f}(\mathbf{u})-\mathbf{u}),\qquad \mathbf{u}(0,x) = \mathbb{E}_x e^{-\beta\partial W(\lambda^*, x)}.$$

Therefore, $\mathbf{u}(t, x)$ and $\mathbb{E}_x \exp\left\{-\beta e^{\gamma(\lambda^*)t}\partial W(\lambda^*, x)\right\}$ are solutions of the above initial value problem. The uniqueness of solutions of initial value problems implies (5.8) holds. □

Proof of Theorem 1.3. Combining Theorem 5.2, Theorem 5.3, and [16, Theorem 1.3], we have Theorem 1.3. □

A. Appendix

Lemma A.1. *Let* $\{f(B_{[0,t]}), t \geq 0\}$ *be defined by* (4.7). *For any* $0 < s < t$,

$$\Pi_x^{(y,\lambda^*)}\left[f(B_{[0,t\wedge\tau_z]})|\mathcal{F}_s\right] = f(B_{[0,s\wedge\tau_z]}),$$

where τ_z *is defined by* (4.3),

Proof. For any $s > 0$, we use θ_s to denote the shift operator. First note that

$$\begin{aligned}&\Pi_x^{(y,\lambda^*)}\big(f(B_{[0,t\wedge\tau_z]})|\mathcal{F}_s\big)\\ &= \Pi_x^{(y,\lambda^*)}\left[\widehat{\mathbf{w}}(-t\wedge\tau_z, B_{t\wedge\tau_z}, y)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s\geq\tau_z}\middle|\mathcal{F}_s\right]\\ &\quad + \Pi_x^{(y,\lambda^*)}\left[\widehat{\mathbf{w}}(-t\wedge\tau_z, B_{t\wedge\tau_z}, y)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s<\tau_z}\middle|\mathcal{F}_s\right]\\ &=: I + II.\end{aligned}\tag{A.1}$$

For I, we have

$$I = \widehat{\mathbf{w}}\big(-\tau_z, B_{\tau_z}, y\big)e^{-\int_0^{\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s\geq\tau_z}.\tag{A.2}$$

For II, we will prove that

$$II = \widehat{\mathbf{w}}(-s, B_s, y)e^{-\int_0^{s}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s<\tau_z},\tag{A.3}$$

which is equivalent to

$$\begin{aligned}&\Pi_x\left(\frac{\Lambda_{t\wedge\tau_z}^{(y,\lambda^*)}}{\Lambda_0^{(y,\lambda^*)}}\widehat{\mathbf{w}}(-t\wedge\tau_z, B_{t\wedge\tau_z}, y)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{\{\Lambda_{t\wedge\tau_z}^{(y,\lambda^*)}>0\}}\mathbf{1}_{s<\tau_z}\middle|\mathcal{F}_s\right)\\ &= \frac{\Lambda_s^{(y,\lambda^*)}}{\Lambda_0^{(y,\lambda^*)}}\widehat{\mathbf{w}}(-s, B_s, y)e^{-\int_0^{s}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s<\tau_z}.\end{aligned}\tag{A.4}$$

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Recall that

$$\tau_z = \tau_z(y) = \inf\{t \geq 0 : y + \gamma'(\lambda^*)t + h(B_t) \leq z\},$$

and $\tau_z(y + \nu^* s) = \tau_{z-\nu^* s}$. For $0 < s < t$, we have on $\{s < \tau_z\}$

$$t \wedge \tau_z = s + (t - s) \wedge \tau_{z-\nu^* s} \circ \theta_s$$

and

$$B_{t\wedge\tau_z} = B_{(t-s)\wedge\tau_{z-\nu^* s}} \circ \theta_s.$$

Using the Markov property of $\{B_t, t \geq 0\}$ and the fact that $\left\{\Lambda^{(y,\lambda^*)}_{t\wedge\tau_z} > 0\right\}$, we have the following:

$$\begin{aligned}
&\text{left side of (A.4)}\\
&= \Pi_x\Big(\big(\Lambda_0^{(y,\lambda^*)}\big)^{-1} e^{-\gamma(\lambda^*)t\wedge\tau_z - \lambda^* B_{t\wedge\tau_z} + \int_0^{t\wedge\tau_z} \mathbf{g}(B_r)\mathrm{d}r}\psi(B_{t\wedge\tau_z}, \lambda^*)\\
&\qquad \times (y + \gamma'(\lambda^*)t\wedge\tau_z + h(B_{t\wedge\tau_z}))\widehat{\mathbf{w}}(-t\wedge\tau_z, B_{t\wedge\tau_z}, y)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s<\tau_z}\Big|\mathcal{F}_s\Big)\\
&= \mathbf{1}_{s<\tau_z}\big(\Lambda_0^{(y,\lambda^*)}\big)^{-1} e^{-\gamma(\lambda^*)s + \int_0^s \mathbf{g}(B_r)(1-\mathbf{w}(-r,B_r))\mathrm{d}r}\\
&\quad \times \Pi_x\Big[e^{-\gamma(\lambda^*)((t-s)\wedge\tau_{z-\nu^* s}) - \lambda^* B_{(t-s)\wedge\tau_{z-\nu^* s}} + \int_0^{(t-s)\wedge\tau_{z-\nu^* s}}\mathbf{g}(B_r)(1-\mathbf{w}(-s-r,B_r))\mathrm{d}r}\\
&\quad \times \psi\big(B_{(t-s)\wedge\tau_{z-\nu^* s}}, \lambda^*\big)\big(y + \gamma'(\lambda^*)\big(s + (t-s)\wedge\tau_{z-\nu^* s}\big) + h\big(B_{(t-s)\wedge\tau_{z-\nu^* s}}\big)\big)\\
&\quad \times \widehat{\mathbf{w}}\big(-s - (t-s)\wedge\tau_{z-\nu^* s}, B_{(t-s)\wedge\tau_{z-\nu^* s}}, y\big)\circ\theta_s\Big|\mathcal{F}_s\Big]\\
&= \mathbf{1}_{s<\tau_z}\big(\Lambda_0^{(y,\lambda^*)}\big)^{-1} e^{-\gamma(\lambda^*)s + \int_0^s \mathbf{g}(B_r)(1-\mathbf{w}(-r,B_r))\mathrm{d}r}\\
&\quad \times \Pi_{B_s}\Big[e^{-\gamma(\lambda^*)((t-s)\wedge\tau_{z-\nu^* s}) - \lambda^* B_{(t-s)\wedge\tau_{z-\nu^* s}} + \int_0^{(t-s)\wedge\tau_{z-\nu^* s}}\mathbf{g}(B_r)(1-\mathbf{w}(-s-r,B_r))\mathrm{d}r}\\
&\quad \times \psi\big(B_{(t-s)\wedge\tau_{z-\nu^* s}}, \lambda^*\big)\big(y + \gamma'(\lambda^*)s + \gamma'(\lambda^*)((t-s)\wedge\tau_{z-\nu^* s}) + h\big(B_{(t-s)\wedge\tau_{z-\nu^* s}}\big)\big)\\
&\quad \times \widehat{\mathbf{w}}\big(-s - (t-s)\wedge\tau_{z-\nu^* s}, B_{(t-s)\wedge\tau_{z-\nu^* s}}, y\big)\Big].
\end{aligned}$$

By (2.11), we have the following:

$$\begin{aligned}
&\text{left side of (A.4)}\\
&= \Pi_{B_s}^{(y+\nu^* s,\lambda^*)}\left[e^{-\int_0^{(t-s)\wedge\tau_{z-\nu^* s}}\mathbf{g}(B_r)\mathbf{w}(-s-r,B_r)\mathrm{d}r}\widehat{\mathbf{w}}\big(-s-(t-s)\wedge\tau_{z-\nu^* s}, B_{(t-s)\wedge\tau_{z-\nu^* s}}, y\big)\right]\\
&\quad \times \mathbf{1}_{s<\tau_z}\left(\Lambda_0^{(y,\lambda^*)}\right)^{-1} e^{-\gamma(\lambda^*)s + \int_0^s \mathbf{g}(B_r)(1-\mathbf{w}(-r,B_r))\mathrm{d}r}e^{-\lambda^* B_s}\psi(B_s, \lambda^*)(y + \nu^* s + h(B_s))\\
&= \Pi_{B_s}^{(y+\nu^* s,\lambda^*)}\left[e^{-\int_0^{(t-s)\wedge\tau_{z-\nu^* s}}\mathbf{g}(B_r)\mathbf{w}(-s-r,B_r)\mathrm{d}r}\widehat{\mathbf{w}}\big(-s-(t-s)\wedge\tau_{z-\nu^* s}, B_{(t-s)\wedge\tau_{z-\nu^* s}}, y\big)\right]\\
&\quad \times \mathbf{1}_{s<\tau_z}\left(\Lambda_0^{(y,\lambda^*)}\right)^{-1}\Lambda_s^{(y,\lambda^*)}e^{-\int_0^s \mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\\
&= \Pi_{B_s}^{(y+\nu^* s,\lambda^*)}\left[e^{-\int_0^{(t-s)\wedge\tau_{z-\nu^* s}}\mathbf{g}(B_r)\mathbf{w}(-s-r,B_r)\mathrm{d}r}\widehat{\mathbf{w}}\big(-s-(t-s)\wedge\tau_{z-\nu^* s}, B_{(t-s)\wedge\tau_{z-\nu^* s}}, y\big)\right]\\
&\quad \times \frac{\Lambda_s^{(y,\lambda^*)}}{\Lambda_0^{(y,\lambda^*)}}e^{-\int_0^{s\wedge\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s<\tau_z}\\
&= \frac{\Lambda_s^{(y,\lambda^*)}}{\Lambda_0^{(y,\lambda^*)}}\widehat{\mathbf{w}}(-s, B_s, y)e^{-\int_0^{s\wedge\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\mathbf{1}_{s<\tau_z} = \text{right side of (A.4)},
\end{aligned}$$

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where in the last equality we used (4.5) with T replaced by $-s$, x replaced by B_s, y replaced by $y+\nu^*s$, t replaced by $t-s$, and z replaced by $z-\nu^*s$. Hence (A.3) holds. Combining (A.1), (A.2), and (A.3), we obtain

$$\Pi_x^{(y,\lambda^*)}\left[\widehat{\mathbf{w}}(-t\wedge\tau, B_{t\wedge\tau_z}, y)e^{-\int_0^{t\wedge\tau_z}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}\Big|\mathcal{F}_s\right]$$
$$=\widehat{\mathbf{w}}(-s\wedge\tau, B_{s\wedge\tau}, y)e^{-\int_0^{s\wedge\tau}\mathbf{g}(B_r)\mathbf{w}(-r,B_r)\mathrm{d}r}=f(B_{[0,s\wedge\tau]}).$$

□

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