Proceedings of the Edinburgh Mathematical Society (1995) **38**, 77–97 ©

FRONTS, DOMAIN WALLS AND PULSES IN A GENERALIZED GINZBURG–LANDAU EQUATION*

by JINQIAO DUAN and PHILIP HOLMES

(Received 17th February 1993)

We discuss the existence and non-existence of front, domain wall and pulse type traveling wave solutions of a Ginzburg–Landau equation with cubic terms containing spatial derivatives and a fifth order term, in both subcritical and supercritical cases. Our results appear to be the first rigorous existence and non-existence proofs for the full equation with all possible terms derived from second order perturbation theory present.

1991 *Mathematics subject classification*: 58F39, 34C27, 34C35, 34C37, 35Q55.

1. Background and definitions

One way to investigate the dynamics of a pattern formation system modeled by a partial differential equation (PDE) of evolution type, with a single space variable, is via the traveling frame reduction. Introducing the traveling coordinate $z=x-ct$ with wave speed c, we get a boundary value problem for a system of ordinary differential equations (ODE). The critical points of these ODEs correspond to the plane waves or the zero amplitude (trivial) wave of the original PDE. The connections, or transitions, between critical points carry important dynamical information. These connections are a subclass (i.e., uniformly translating structures) of the so-called "coherent structures" (e.g. Saarloos and Hohenberg [29, 30]). A heteroclinic connection between the zero amplitude wave and a plane wave is called a front. We call a heteroclinic connection between two different plane waves a heteroclinic domain wall. We call a homoclinic connection from a plane wave to itself a homoclinic domain wall. A homoclinic orbit from the zero amplitude wave to itself is called a pulse. For example, in binary fluid convection (see below), these connections correspond to the "boundary layer" between the coexisting conductive state and periodic, confined convective states.

Several studies have been made of fronts, domain walls and pulses in the classical or cubic Ginzburg–Landau (GL) equation:

$$u_t=a_0u+(1+ib_1)u_{xx}+(a_2+ib_2)|u|^2u. \tag{1.1}$$

See Holmes [24], Sirovich and Newton [34], Landman [27], Bernoff [3], Doelman [11, 12], Schopf and Kramer [31] and references therein, for example. Recently,

*Partially supported by AFOSR 91-0329.

Eckmann and Gallay [16] have proved the existence of heteroclinic domain walls (which they call fronts) connecting stationary plane wave solutions for (1.1) with $b_1 = b_2 = 0$. They appear similar in nature to our solutions shown in Figure 6 (Section 7).

In this paper we discuss the existence of front, domain wall and pulse type solutions for a generalized Ginzburg–Landau equation (Doelman [10]), which is a generic amplitude equation describing the nonlinear evolution of patterns near criticality. Specifically, we consider the equation

$$u_t = a_0 u + (1 + ib_1) u_{xx} + (a_2 + ib_2)|u|^2 u \\ + (a_3 + ib_3)|u|^2 u_x + (a_4 + ib_4) u^2 \bar{u}_x - (1 + ib_5)|u|^4 u, \tag{1.2}$$

where $x \in R^1$, $t > 0$ and the over bar denotes complex conjugate. The coefficients a_j, b_j are real. Observe that the real coefficients of the diffusive and quintic terms can be reduced to ± 1 by suitable rescaling; the former must be positive for well posedness, and the latter negative for global existence of large amplitude solutions. Schopf and Zimmermann [32, 33] also derived (1.1) in the context of binary fluid convection, and proposed the addition of higher order terms in the degenerate case when a_2 changes sign.

The two terms $|u|^2 u_x, u^2 \bar{u}_x$ appear naturally in the asymptotic derivation. Deissler and Brand [9] showed numerically that these two terms can significantly slow down the propagating speed of pulses and also cause the nonsymmetry of pulses. Duan and Holmes [15] proved a sufficient condition, $|b_3 - b_4| < 2$, for global existence of solutions to the Cauchy problem for this equation. This condition appears to be sharp: see Duan, Holmes and Titi [14]. However, we observe that restricted classes of bounded solutions, including traveling waves, may exist outside this parameter range. We remark that the nonlinear Schrodinger (NLS) equation describing the nonlinear evolution of water waves (Hasimoto and Ono [22], Craig et al. [8]) is a special integrable case of the GL equation.

In a special case when $a_3 = a_4 = b_3 = b_4 = 0$, equation (1.2) becomes the so-called quintic GL equation

$$u_t = a_0 u + (1 + ib_1) u_{xx} + (a_2 + ib_2)|u|^2 u - (1 + ib_5)|u|^4 u. \tag{1.3}$$

Thual and Fauve [36], and Fauve and Thual [18] discuss pulses for this equation. Jones, Kapitula and Powell [26] show the existence of certain fronts in the case that b_j are small. Hakim, Jakobsen and Pomeau [21] discuss fronts and pulses (which they call solitary waves) with wave speed $c = 0$. Saarloos and Hohenberg [29] discuss fronts and pulses. Malomed and Nepomnyashchy [28] also discussed, in this special case, fronts (which they call kinks) and pulses (which they call kink-antikink, or soliton-like solutions).

For the general equation (1.2), Saarloos and Hohenberg [30] present a framework for the discussion of front, pulse and domain wall dynamics. Doelman and Eckhaus [13], Theorem 3.4, find some homoclinic domain walls with wave speed $c=0$, via Poincaré maps and Melnikov integrals, extending and correcting the work of Holmes [**24**].

The reduced ODE system is a spatial system since the time variable is absorbed in the traveling frame coordinate. In order to find connections (fronts, pulses and domain walls) in this ODE system, topological and analytical methods are available. A topological quantity, the Conley index for an isolated invariant set, which is a generalization of the Morse index for a nondegenerate fixed point, could be used to find such connections, see Smoller [**35**]. However, it is difficult to construct isolating blocks in our case. Therefore we use an analytical approach, see Guckenheimer and Holmes [**19**, Chapter 4], Holmes [**24**], Wiggins [**37**], Jones, Kapitula and Powell [**26**], Jones, Kopell and Langer [**25**], Doelman and Eckhaus [**13**], and Campbell and Holmes [**5**].

In this paper, after reviewing the derivation of ODEs for uniformly traveling waves and their integrable structure in an appropriate limit in Sections 2 and 3, we prove three main results in Sections 4–6. Theorem 1 gives conditions sufficient for fronts and domain walls to exist in case $a_0<0$ and generalizes the earlier result due to Jones *et al.* [**26**]. Theorem 2 gives conditions sufficient for the existence of domain walls in case $a_0>0$ and generalizes the earlier result due to Doelman and Eckhaus [**13**]. Theorem 3 gives conditions sufficient for the existence of fronts in case $a_0>0$ and Theorems 3 and 4 also give conditions sufficient for non-existence of domain walls and pulses. We conclude and discuss some implications in Section 7.

2. Traveling frame reduction

We first seek the plane waves of the form

$$u=re^{i(kz-wt)}, \tag{2.1}$$

with r, k and ω real and $z=x-ct$. When interpreting the results to follow, we must recall that typically the parameters a_j and b_j are set by the specific application (with a_0 playing the role of a distinguished bifurcation parameter), and that the frequency ω, wavenumber k and wave speed c are adjustable, although only two of them are free, in view of conditions such as (2.3) below. Thus, our task will be to show that plane waves and connections exist for certain values or ranges of values of, say, ω and c.

Substitutions of (2.1) into (1.2) yields

$$r^4+[(b_3-b_4)k-a_2]r^2+k^2-a_0=0, \tag{2.2}$$

$$\omega=-ck+b_1k^2-[b_2+(a_3-a_4)k]r^2+b_5r^4 \tag{2.3}$$

and solving (2.2) for r we get

$$r^2=\frac{a_2-(b_3-b_4)k\pm\sqrt{[(b_3-b_4)^2-4]k^2-2a_2(b_3-b_4)k+a_2^2+4a_0}}{2}. \tag{2.4}$$

https://doi.org/10.1017/S0013091500006210 Published online by Cambridge University Press

We see that, if $|b_3-b_4|\geqq 2$, the amplitude r goes to infinity as k increases. Therefore we frequently assume the global existence condition $|b_3-b_4|<2$. In order for the square root sign to make sense, we assume $a_0>-a_2^2/4$. In this case, the plane waves exist for k in a bounded interval $[k_{\text{lower}}, k_{\text{upper}}]$, where

$$k_{\text{lower}}=\frac{a_2(b_3-b_4)+2\sqrt{-a_0(b_3-b_4)^2+a_2^2+4a_0}}{(b_3-b_4)^2-4}, \tag{2.5}$$

$$k_{\text{upper}}=\frac{a_2(b_3-b_4)-2\sqrt{-a_0(b_3-b_4)^2+a_2^2+4a_0}}{(b_3-b_4)^2-4}. \tag{2.6}$$

When $-a_2^2/4<a_0<0$, the zero amplitude wave is linearly stable, while it is unstable when $a_0>0$. In this paper we initially assume that $a_0<0$, since this seems to be the most interesting case, for in it we have a potential competition between the stable zero amplitude wave ($r=0$) and the large amplitude plane wave ($r=r_0$); see the next section. However, we also discuss the case $a_0>0$. We also assume that $a_2>0$, otherwise the cubic GL equation suffices for physical modeling close to the onset of instability (see for example, Schopf and Zimmermann [32, 33].

We now seek more general, non-periodic traveling waves. Setting $u=Be^{-i\omega t}$ with $B(z)=r(z)e^{i\int^z k(z)dz}$, $z=x-ct$ as above, and inserting equation (1.2), we first get,

$$\begin{aligned}-cB_z=&\,i\omega B+a_0B+(1+ib_1)B_{zz}+(a_2+ib_2)|B|^2B\\&+(a_3+ib_3)|B|^2B_z+(a_4+ib_4)B^2\bar{B}_z-(1+ib_5)|B|^4B.\end{aligned} \tag{2.7}$$

We generally do not have symmetry $c\to -c$ and $z\to -z$, but, if $a_3=a_4=b_3=b_4=0$, i.e., for the quintic GL equation (1.3), this symmetry does hold. After some manipulations, we obtain the three dimensional system

$$r'=(1+b_1^2)s, \tag{2.8}$$

$$\begin{aligned}s'=&\,[-a_0-b_1\omega-b_1ck+(1+b_1^2)k^2]r-cs-(a_2+b_1b_2)r^3\\&-[(a_3+a_4)+b_1(b_3+b_4)]r^2s\\&-[b_1(a_3-a_4)-(b_3-b_4)]r^3k+(1+b_1b_5)r^5,\end{aligned} \tag{2.9}$$

$$\begin{aligned}k'=&\,[b_1c-2(1+b_1^2)k]s/r+a_0b_1-\omega-ck+(a_2b_1-b_2)r^3\\&+[b_1(a_3+a_4)-(b_3+b_4)]rs\\&-[(a_3-a_4)+b_1(b_3-b_4)]r^2k+(b_5-b_1)r^4,\end{aligned} \tag{2.10}$$

where $'=d/d\tau, \tau=z/(1+b_1^2)$ and $s=r'/(1+b_1^2)$. The fact that a complex, second order (four dimensional) ODE reduces to a three dimensional flow follows from translation invariance of the original PDE (1.2). We note that the "wavenumber" k is now a dependent variable which must be solved for, but that the wave speed c and frequency ω retain their character as adjustable parameters.

The (r,s,k) system (2.8)–(2.10) has a singularity at $r=0$. In order to overcome this difficulty we introduce the "blow up" transform or σ-process (Arnold [2]). Letting

$$v=s/r, \tag{2.11}$$

we compute $v'=s'/r-(1+b_1^2)v^2$ and (2.8)–(2.10) becomes the desingularized (r,v,k) system

$$r'=(1+b_1^2)rv, \tag{2.12}$$

$$\begin{aligned} v'= & -a_0-b_1\omega-b_1ck+(1+b_1^2)k^2-cv-(1+b_1^2)v^2 \\ & -(a_2+b_1b_2)r^2-[(a_3+a_4)+b_1(b_3+b_4)]r^2v \\ & -[b_1(a_3-a_4)-(b_3-b_4)]r^2k+(1+b_1b_5)r^4, \end{aligned} \tag{2.13}$$

$$\begin{aligned} k'= & [b_1c-2(1+b_1^2)k]v+a_0b_1-\omega-ck+(a_2b_1-b_2)r^2 \\ & +[b_1(a_3+a_4)-(b_3+b_4)]r^2v \\ & -[(a_3-a_4)+b_1(b_3-b_4)]r^2k+(b_5-b_1)r^4. \end{aligned} \tag{2.14}$$

This system has the invariant plane $r=0$. Thus, by the σ-process, a singular point is transformed to a (singular) invariant plane.

On the invariant plane $r=0$, the (r,v,k) system reduces to

$$v'=-a_0-b_1\omega-b_1ck+(1+b_1^2)k^2-cv-(1+b_1^2)v^2, \tag{2.15}$$

$$k'=[b_1c-2(1+b_1^2)k]v+a_0b_1-\omega-ck. \tag{2.16}$$

This system does not depend on a_3, a_4, b_3, b_4 (nor on b_2 and b_5). As in Jones, Kapitula and Powell [26], we can show that this (v,k) system (2.15)–(2.16) characterizes the behavior of solutions of (r,s,k) system as $r\to 0$. We will need this characterization when looking for fronts connecting the zero amplitude wave and plane waves, and pulses connecting the zero amplitude wave to itself.

3. Integrable structures, symmetry and critical points

In the following, we assume that b_1, b_2, b_5, a_3, a_4 are all small and denote

$$\varepsilon=(b_1,b_2,a_3,a_4,b_5). \tag{3.1}$$

For $\varepsilon=0$, equations (2.12)–(2.14) reduce to

$$r'=rv, \tag{3.2}$$

$$v'=-a_0+k^2-cv-v^2-a_2r^2+(b_3-b_4)r^2k+r^4, \tag{3.3}$$

$$k'=-\omega-(c+2v)k-(b_3+b_4)r^2v. \tag{3.4}$$

If, in addition, $b_3+b_4=0$, we obtain

$$r'=rv, \tag{3.5}$$

$$v'=-a_0+k^2-cv-v^2-a_2r^2+(b_3-b_4)r^2k+r^4, \tag{3.6}$$

$$k'=-\omega-(c+2v)k. \tag{3.7}$$

We remark that if $\omega=0$, the plane $k=0$ is invariant for the second system above. This observation is useful in seeking critical points for the desingularized (r,v,k) system for ε, b_3+b_4 and ω small enough. The zero amplitude wave critical points are $(0,v^{\pm}(\varepsilon,b_3,b_4,\omega,c),k^{\pm}(\varepsilon,b_3,b_4,\omega,c))$ with

$$v^{\pm}(\mathbf{0},0,0,0,c)=\frac{-c\pm\sqrt{c^2-4a_0}}{2},$$

$$k^{\pm}(\mathbf{0},0,0,0,c)=0.$$

By linear analysis, the critical point $(0,v^+,k^+)$ has a one-dimensional unstable manifold, and $(0,v^-,k^-)$ has a one-dimensional stable manifold, when $-a_2^2/4<a_0<0$. The associated eigenvectors are normal to the $\{r=0\}$ plane, for any c. When restricted on the invariant plane $\{r=0\}$, (v^+,k^+) is a sink and (v^-,k^-) is a source, for any c.

For $a_0>0$, this pair of critical points exist if $|c|>2\sqrt{a_0}$. For $c>2\sqrt{a_0}$, $(0,v^+,k^+)$ is a sink and $(0,v^-,k^-)$ has a one-dimensional stable manifold with the associated eigenvector normal to the $\{r=0\}$ plane. For $c<-2\sqrt{a_0}$, $(0,v^-,k^-)$ is a source and $(0,v^+,k^+)$ has a one-dimensional unstable manifold with the associated eigenvector normal to the $\{r=0\}$ plane. When restricted on the invariant plane $\{r=0\}$, (v^+,k^+) is a sink and (v^-,k^-) is a source for $|c|>2\sqrt{a_0}$.

We also have plane wave critical points $(r_0,0,k_0)$ and $(r_1,0,k_1)$, where $r_{0,1}(\varepsilon,b_3,b_4,\omega,c)$ and $k_{0,1}(\varepsilon,b_3,b_4,\omega,c)$ satisfy (2.2) and (2.3). In particular,

$$r_0^2(\mathbf{0},0,0,0,c)=\frac{a_2+\sqrt{a_2^2+4a_0}}{2},$$

$$r_0^2(\mathbf{0},0,0,0,c)=\frac{a_2-\sqrt{a_2^2+4a_0}}{2},$$

$$k_0(\mathbf{0},0,0,0,c)=0,$$
$$k_1(\mathbf{0},0,0,0,c)=0.$$

We refer to r_0 as the large, and r_1 as the small, amplitude plane waves, respectively. Linearizing (3.5)–(3.7) at $(r_0,0,k_0)$ we obtain the eigenvalues

$$\lambda^{\pm}=\frac{-c\pm\sqrt{c^2+4\gamma}}{2},\quad \lambda_3=-c,$$

where

$$\gamma=a_2^2+4a_0+a_2\sqrt{a_2^2+4a_0},$$

and the corresponding eigenvectors $(r_0,\lambda^{\pm},0)$, $(0,0,1)$. By linear analysis in (r,v,k) space, for $-a_2^2/4<a_0<0$, $(r_0,0,k_0)$ has a 2-dimensional stable manifold and a one-dimensional unstable manifold for $c>0$; one-dimensional stable manifold and 2-dimensional unstable manifold for $c<0$; one-dimensional stable manifold, one-dimensional unstable manifold and one-dimensional center manifold for $c=0$. However, when restricted to the plane $\{k=0\}$, $(r_0,0,k_0)$ has both one-dimensional unstable manifold and one-dimensional stable manifold, for any c. At $(r_1,0,k_1)$ the linearized operator has a pair of complex conjugate eigenvalues and one real eigenvalue for any c. More precisely, $(r_1,0,k_1)$ is a stable focus for $c>0$; an unstable focus for $c<0$, and has a pure imaginary pair of eigenvalues and a zero eigenvalue for $c=0$.

For $a_0>0$, there is only one plane wave critical point $(r_0,0,k_0)$ as r_1^2 is negative. However, it has the same linearized structure as in the $-a_2^2/4<a_0<0$ case.

The global structure of the flow of (2.12)–(2.14) for small ε, b_3+b_4 and ω can be deduced from that for $\varepsilon=\mathbf{0}$, $b_3+b_4=0$ and $\omega=0$. If additionally $c=0$, then the two functions

$$E=\frac{1}{2}r^2(v^2+k^2)+\frac{a_0}{2}r^2+\frac{a_2}{4}r^4-\frac{r^6}{6}, \tag{3.8}$$

$$M=r^2k \tag{3.9}$$

are constants of motion, for we compute

$$E'=-(c(v^2+k^2)+\omega k)r^2, \tag{3.10}$$

$$M'=-(\omega+ck)r^2. \tag{3.11}$$

In fact for $b_3,b_4\neq 0$, as Doelman and Eckhaus [13] observe, one can modify the integrals as follows

$$\tilde{E}=E+b_4k+b_4(b_3+b_4)\frac{r^6}{6}, \tag{3.12}$$

$$\tilde{M} = M + (b_3 + b_4)\frac{r^4}{4} \tag{3.13}$$

to obtain

$$\tilde{E}' = -(c(v^2 + k^2) + \omega k)r^2 - b_4(\omega + ck)r^4, \tag{3.14}$$

$$\tilde{M}' = -(\omega + ck)r^2, \tag{3.15}$$

which also vanish when $\omega = c = 0$.

We may summarize the consequences of these local and global results in the phase portraits shown in Figure 1 ($\omega = c = 0$). In Figure 1 we see examples of the three classes of solutions (fronts, domain walls and pulses) "in embryo". See also Figures 2 and 3 for phase portraits when $\omega = 0$ but $c > 0$. The case $c < 0$ follows from that for $c > 0$ via the symmetry

$$(r, v, k, \omega, c, z) \to (r, -v, k, -\omega, -c, -z). \tag{3.16}$$

In the remainder of the paper we will use these integrable structures to investigate the behavior for $\varepsilon, b_3 + b_4, \omega$ small, and for appropriate choices of wave speed c.

4. Fronts and domain walls: $a_0 < 0$

In the $a_0 < 0$ case, there are various possibilities of connections between any two of the zero amplitude wave $r = 0$, the large amplitude wave $r = r_0$, and the small amplitude wave $r = r_1$, as the phase portraits of Figure 1 suggest. Although the small amplitude plane wave is dynamically unstable (see Doelman and Eckhaus [**13**], p. 255]), the existence of connections to and from it does indicate how solutions started near this wave might evolve: see Section 7.

Our first result summarizes much of what we know about connecting orbits and includes earlier results such as those of Jones, Kapitula and Powell [**26**].

Theorem 1. *Let $\varepsilon = (b_1, b_2, a_3, a_4, b_5)$, $b_3 + b_4$ and ω be small enough. Then,*

(1) For $a_0 \in (-\frac{1}{4}a_2^2, -\frac{3}{16}a_2^2)$ and any $c > 0$ there is a heteroclinic domain wall connecting the large amplitude wave ($z \to -\infty$) to the small amplitude wave ($z \to \pm\infty$).

(2) For $a_0 \in (-\frac{3}{16}a_2^2, 0)$ and any $c > 0$ there is a front connecting the zero amplitude wave ($z \to -\infty$) to the small amplitude wave ($z \to \pm\infty$).

(3) For any $a_0 \in (-\frac{1}{4}a_2^2, 0)$ there exists two locally unique functions $\tilde{c} = \tilde{c}(a_0, a_2) \geqq 0$, $\tilde{\tilde{c}} = \tilde{\tilde{c}}(a_0, a_2) \geqq 0$, with

$$\tilde{c}\left(-\frac{3}{16}a_2^2, a_2\right) = \tilde{\tilde{c}}\left(-\frac{3}{16}a_2^2, a_2\right) = 0,$$

such that

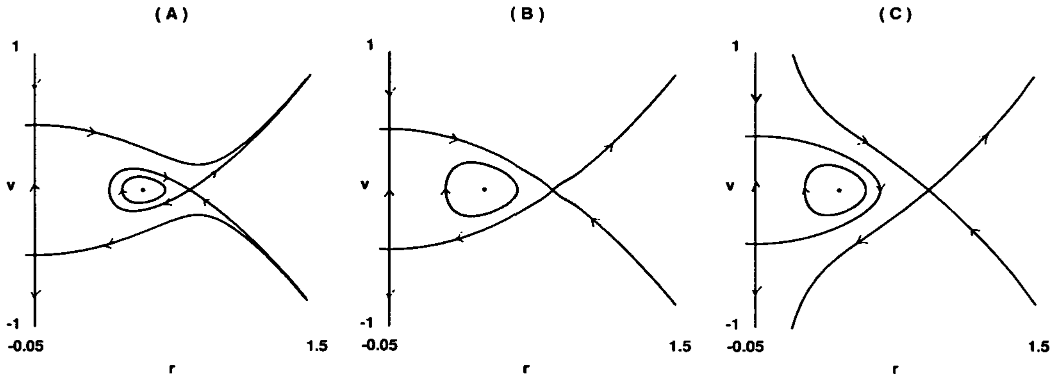

FIGURE 1 Phase portraits for (r, v, k) system (3.5)–(3.7) on the invariant plane $k=0$ when $c=\omega=0$: (A) $a_0 \in (-\frac{1}{4}a_2^2, -\frac{3}{16}a_2^2)$; (B) $a_0 = -\frac{3}{16}a_2^2$; (C) $a_0 \in (-\frac{3}{16}a_2^2, 0)$

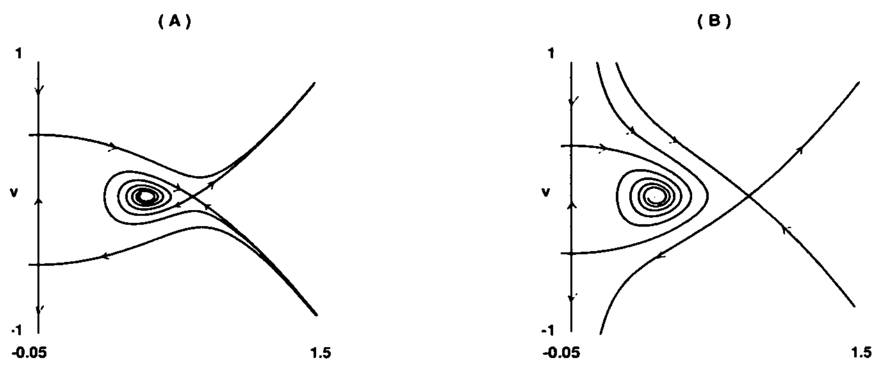

FIGURE 2 Phase portraits for (r, v, k) system (3.5)–(3.7) on the invariant plane $k=0$ when $c>0$ and $\omega=0$: (A) $a_0 \in (-\frac{1}{4}a_2^2, -\frac{3}{16}a_2^2)$; (B) $a_0 \in (-\frac{3}{16}a_2^2 0)$

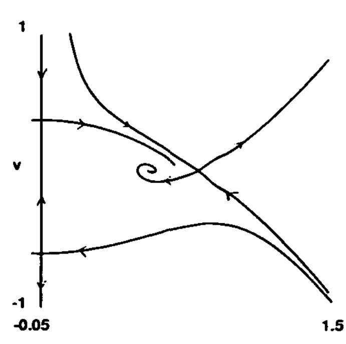

FIGURE 3 $g^s(c_0) > g^u(c_0)$

(A) in case (1) above: if $c=\tilde{c}$, a front connecting the zero amplitude wave ($z\to-\infty$) to the large amplitude plane wave ($z\to+\infty$) also exists; if $c>\tilde{c}$, a front connecting the zero amplitude wave ($z\to-\infty$) to the small amplitude wave ($z\to+\infty$) also exists, and

(B) in case (2) above: if $c=\tilde{\tilde{c}}$, a front connecting the large amplitude plane wave ($z\to-\infty$) to the zero amplitude wave ($z\to+\infty$) also exists; if $c>\tilde{\tilde{c}}$, a heteroclinic domain wall connecting the large amplitude wave ($z\to-\infty$) to the small amplitude wave ($z\to+\infty$) also exists.

(4) Analogous results hold for $c<0$, with the fronts or heteroclinic domain walls running in the opposite z direction.

Proof. Set $\varepsilon=0$, $b_3+b_4=\omega=0$ and consider the flow on the invariant (r,v) plane for $k=0$.

We first assume $a_0\in(-\frac{1}{4}a_2^2,-\frac{3}{16}a_2^2)$. For $c=0$ there is a homoclinic orbit to $r=r_0$ (Figure 1(A)), which, in view of the expression (3.10) for E' ($E'=-cr^2v^2$ for $\omega=k=0$) is broken for all $c\neq 0$, yielding a structurally stable (transversal) saddle $\to$ sink connection from r_0 to r_1 for $c>0$ (Figure 2(A)) and a source $\to$ saddle connection from r_1 to r_0 for $c<0$ (from symmetry (3.16) and Figure 2(A)). This will be used to prove part 1.

We now turn to part 3(A). We employ a modest generalization of Jones, Kapitula and Powell [**26**]. Consider the system

$$r'=rv, \tag{4.1}$$

$$v'=-a_0-cv-v^2-a_2r^2+r^4, \tag{4.2}$$

$$c'=0, \tag{4.3}$$

on $k=0$, in which we have included the parameter c as a trivial component, thus permitting a study of parameterized families of manifolds. Let $W^s=W^s(r_0,0,k_0)$ and $W^u(0,v^+,k^+)$, both being dependent on c. They are each one-dimensional when restricted to the plane $k=0$. Denote the center-stable and center-unstable manifolds as $W^{cs}=\bigcup_c(W^s\times\{c\})$ and $W^{cu}=\bigcup_c(W^u\times\{c\})$, both being 2-dimensional.

Let $S=\{(r,v,c):r=r_1,v>0\}$ denote a cross section to the flow of (4.1)–(4.3). Since $r'>0$ for all $0<r<r_0$ when $v>0$, W^s and W^u intersect S once and only once for each c. Let $g^s(c),g^u(c)$ denote the curves $W^{cs}\cap S$, $W^{cu}\cap S$, respectively, which are (at least) C^1 smooth. From above (see Figure 1(A)), we know $g^s(0)$. We only need to show that there exists a $c_0>0$ such that $g^s(c_0)>g^u(c_0)$ (see Figure 3), which will imply the existence of a $\tilde{c}>0$ for which $g^s(\tilde{c})=g^u(\tilde{c})$, in turn implying existence of a front, i.e., the intersection of W^{cs} and W^{cu} at $\tilde{c}>0$.

Consider a reference plane $v=\alpha(r-r_0)$ with $\alpha<0$, in the (r,v,c) space. On this plane the vector held satisfies

$$\frac{v'}{r'}=-\frac{[c+\alpha(r-r_0)]}{r}+\frac{(r^2-r_1^2)(r+r_0)}{\alpha r}. \tag{4.4}$$

So there exists a $c_1>0$ such that $(v'/r')<\alpha$ for $c>c_1$ and $r\in[r_1,r_0)$. On the other hand, on $v=v^+, 0<r<r_1$, we have $v'=r^4-a_2r^2<0$ and we note that $v^+\to 0$ as $c\to+\infty$. Hence there is a $c_2>0$ such that $v^+<\alpha(r_1-r_0)$. So there is a $c_0>0$ such that $g^s(c_0)>g^u(c_0)$, as claimed. See Figure 3.

For $c>\tilde{c}$, the unstable manifold W^u of $(0,v^+,k^+)$, the upper zero amplitude fixed point, lies below the upper branch of the stable manifold W^s of $(r_0,0,k_1)$ when restricted to the $k=0$ plane, and therefore lies in the domain of attraction of $(r_1,0,k_1)$ which is a sink on this invariant plane. This establishes the existence of structurally stable saddle$\to$sink connections corresponding to fronts when $c>0$ and $c>\tilde{c}$ (Figure 4(A)) (they are source$\to$saddle connections for $c<0$). Part 1, and the $c>\tilde{c}$ case of Part 3(A) of the theorem follow from the fact that $(r_1,0,k_1)$ is a sink in the full (r,v,k) space for $c>0$ (respectively a source for $c<0$) and so these connections are structurally stable and hence survive small perturbations.

To prove the $c=\tilde{c}$ case of Part 3(A), we additionally need to show that W^{cs} and W^{cu} intersect transversally at $\tilde{c}$ in $k=0$. We recall the argument which employs differential forms to show the transversality of stable and unstable manifolds. See Jones, Kopell and Langer [25]. (For the geometric relevance of differential forms, see Arnold [2] or Boothby [4]. Let x,y,z be a coordinate system on R^3 and let W^s and W^u be 2-dimensional (center-)stable and (center-)unstable manifolds, respectively, for a dynamical system. Suppose that W^s and W^u intersect and pick arbitrary orthonormal bases $\{f_1,f_2\},\{g_1,g_2\}$ for the tangent planes of W^s and W^u on their intersection, respectively. When a 2-form, say $dxdy$, acts on a basis, we obtain a number which is given by a determinant with elements from the x or y components of the basis vectors. We associate a 2-form with a tangent plane via the action on an orthonormal basis. For convenience, we denote

$$P_{xy}=dxdy, P^-_{xy}=dxdy(f_1,f_2), P^+_{xy}=dxdy(g_1,g_2);$$

etc. ($-$ for stable, $+$ for unstable). Hence we associate with the tangent planes to W^s and W^u, the two vectors

$$(P^-_{xy},P^-_{xz},P^-_{yz}),(P^+_{xy},P^+_{xz},P^+_{yz}).$$

Note that each of these vectors takes the same value for any orthonormal bases with the same orientation (via the right-hand rule). If these two vectors are independent, then the tangent planes to W^s and W^u are not coincident or parallel, which implies that W^s and W^u intersect transversally.

The same argument is true in R^{2n-1} for n-dimensional (center-)stable and (center)-unstable manifolds. In this case, however, we need to use n-forms, since each basis of the tangent planes to these manifolds has n vectors.

We now complete the proof of the $c=\tilde{c}$ case of Part 3(A).

From (4.1)–(4.3) we compute the evolution equations for 1-forms (i.e., differentials or variations of first order):

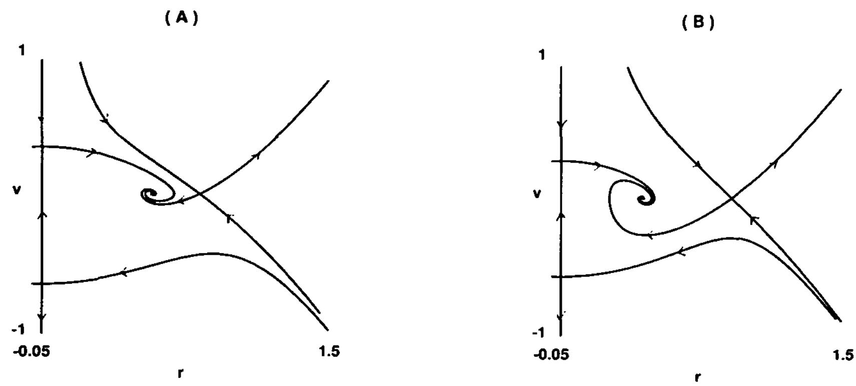

FIGURE 4 Phase portrait for (r, v, k) system (3.5)–(3.7) on the invariant plane $k=0$ when $c>0$ and $\omega=0$: (A) $a_0 \in (-\frac{1}{4}a_2^2, -\frac{3}{16}a_2^2)$ and $c>\tilde{c}$; (B) $a_0 \in (-\frac{3}{16}a_2^2, 0)$ and $c>\tilde{\tilde{c}}$.

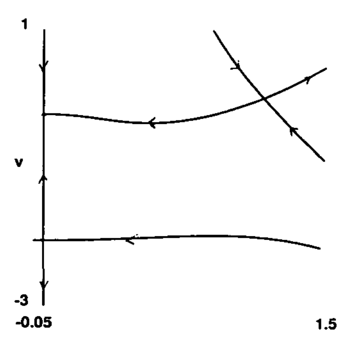

FIGURE 5 The birth of a saddle-sink connection for $a_0>0$ and $c>2\sqrt{a_0}$ large enough

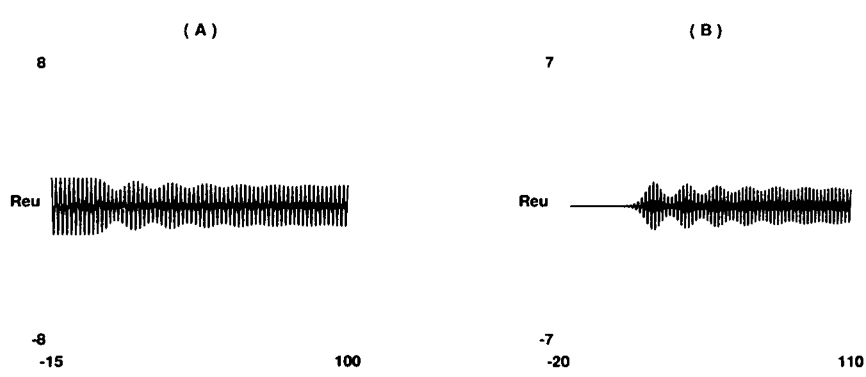

FIGURE 6 Behavior of (the real part of) $u(x,t)$ corresponding to (A) a heteroclinic domain wall; (B) a front. These are associated with Figure 2(A) and 2(B) respectively, and show the dependence of u on x at a fixed time t. Both waves travel to the right, without change of form, as t increases.

$$\delta r' = v\delta r + r\delta v, \tag{4.5}$$

$$\delta v' = (4r^3 - 2a_2 r)\delta r - (c+2v)\delta v - v\delta c, \tag{4.6}$$

$$\delta c' = 0. \tag{4.7}$$

We can also get the evolution equations for 2-forms $P_{rv} = \delta r\delta v$, etc. We only need the following one.

$$P'_{rv} = -(c+v)P_{rv} - vP_{rc}. \tag{4.8}$$

Note that

$$\Lambda_1^{\pm} = (rv, -a_0 - cv - v^2 - a_2 r^2 + r^4, 0),$$

the vector along orbits of (4.1)–(4.3), is a tangent vector for both W^{cs} and W^{cu} and let

$$\Lambda_2^{\pm} = (\delta r^{\pm}, \delta v^{\pm}, 1)$$

respectively, denote the second basis vectors for tangent planes of W^{cs} and W^{cu}. They are clearly linearly independent and so constitute a basis for each tangent plane. Using the Gram–Schmidt process we get an orthonormal basis $\{\hat{\Lambda}_1^{\pm}, \hat{\Lambda}_2^{\pm}\}$ and we let $N > 0$ be the normalization factor. Therefore we have

$$P_{rc}^{\pm} = \delta r\delta c(\Lambda_1^{\pm}, \Lambda_2^{\pm}) = N\begin{vmatrix} rv & 0 \\ \delta r^{\pm} & 1 \end{vmatrix} = Nrv > 0. \tag{4.9}$$

for $v > 0$. Hence, from (4.8), we have

$$P_{rv}^{\pm\prime} = -(c+v)P_{rv}^{\pm} - Nrv^2. \tag{4.10}$$

Also, W^{cs} and W^{cu} each contain a line of critical points whose tangent vectors are in the c direction. For any plane containing such a line, the 2-form $P_{rv} = 0$, which implies that

$$P_{rv}^{+} \to 0, \; as \; z \to -\infty;$$

$$P_{rv}^{-} \to 0, \; as \; z \to +\infty.$$

Since $P_{rv}^{+} = 0$ at $z = -\infty$, (4.10) shows that initially P_{rv}^{+} becomes negative. Moreover, it must remain strictly negative for all z since $P_{rv}^{+\prime}=-Nrv^2<0$ for all $r,v>0$ on $P_{rv}^{+}=0$ and hence solutions of (4.10) cannot cross $P_{rv}^{+}=0$ in the direction of increasing P_{rv}^{+}. A similar argument applied to P_{rv}^{-} yields the conclusion

$$P_{rv}^{+}<0;\quad P_{rv}^{-}>0.$$

From (4.9) we know that $P_{rc}^{+}=P_{rc}^{-}$ and similarly we compute that $P_{vc}^{+}=P_{vc}^{-}$. Thus the two vectors

$$(P_{rv}^{+},P_{rc}^{+},P_{vc}^{+})$$

and

$$(P_{rv}^{-},P_{rc}^{-},P_{vc}^{-})$$

are independent. This implies that W^{cs} and W^{cu} intersect transversally at $c=\tilde{c}>0$ in $k=0$. By a dimension count, W^{cs} and W^{cu} intersect transversally at $\tilde{c}$ in (r,v,k,c) space. All of the above calculations were done for $\varepsilon=\mathbf{0}$, $b_3+b_4=0$ and $\omega=0$. Finally, for sufficiently small and nonzero ε, b_3+b_4 and ω, we merely note that the intersections are preserved by transversality.

This proves the $c=\tilde{c}$ case of Part 3(A).

Part 2 and 3(B) are proved in an analogous fashion. Part 4 follows from the symmetry $(r,v,k,\omega,c,t)\to(r,-v,k,-\omega,-c,-t)$.

We observe that one can estimate the functions $\tilde{c}$ and $\tilde{\tilde{c}}$ for $a_0\simeq-\frac{3}{16}a_2^2$ by a simple application of the Melnikov perturbation method (Guckenheimer and Holmes [**19**, Chapter 4]) to the system

$$r'=rv, \tag{4.11}$$

$$v'=\frac{3}{16}a_2^2-cv-v^2-a_2r^2+r^4+\alpha. \tag{4.12}$$

Here $a_0=-\frac{3}{16}a_2^2-\alpha$. For $c=\alpha=0$ the integral E becomes

$$\hat{E}=\frac{1}{2}r^2v^2-\frac{r^2}{6}\left(\frac{3}{4}a_2-r^2\right)^2, \tag{4.13}$$

and there are saddle-saddle conections from $(0,v^+,0)$ to $(r_0,0,0)$ and $(r_0,0,0)$ to $(0,v^-,0)$ (Figure 1(B)). The preservation of these connections for $a_0<-\frac{3}{16}a_2^2$ and $a_0>-\frac{3}{16}a_2^2$ respectively correspond to the critical $\tilde{c}$ and $\tilde{\tilde{c}}$, above which the fronts or heteroclinic domain walls of part (3) exist. To estimate $\tilde{c}$ and $\tilde{\tilde{c}}$ we compute

$$\int_{-\infty}^{\infty}\hat{E}'dz=\int_{-\infty}^{\infty}(-cr^2v^2+\alpha r^2v)dz, \tag{4.14}$$

which must be integrated along the appropriate unperturbed homoclinic orbit. Since both orbits lie on the level set $E=0$, from (4.13) we have, for the upper orbit,

$$v=\frac{1}{\sqrt{3}}\left(\frac{3}{4}a_2-r^2\right), \tag{4.15}$$

and using this in (4.14) and changing the integration variable via (4.11), we compute

$$\begin{aligned}\int_{-\infty}^{\infty}\hat{E}'dz&=\int_{0}^{\sqrt{3a_2/4}}\left(-\frac{cr}{\sqrt{3}}\left(\frac{3}{4}a_2-r^2\right)+\alpha r\right)dr\\&=\frac{3a_2}{8}\left(\alpha-\frac{\sqrt{3}}{8}a_2c\right).\end{aligned} \tag{4.16}$$

Similarly, for the lower orbit, we obtain

$$\int_{-\infty}^{\infty}\hat{E}'dz=\frac{3a_2}{8}\left(-\alpha-\frac{\sqrt{3}}{8}a_2c\right). \tag{4.17}$$

From the usual Melnikov bifurcation results, we therefore obtain the estimates:

$$\tilde{c}: c\simeq\frac{8\alpha}{\sqrt{3a_2}}=-\frac{8}{\sqrt{3a_2}}\left(\frac{3}{16}a_2^2+a_0\right). \tag{4.18}$$

$$\tilde{\tilde{c}}: c\simeq-\frac{8\alpha}{\sqrt{3a_2}}=\frac{8}{\sqrt{3a_2}}\left(\frac{3}{16}a_2^2+a_0\right), \tag{4.19}$$

which are good for small c and for $a_0\simeq-\frac{3}{16}a_2^2$. □

5. Fronts and domain walls: $a_0>0$

In the $a_0>0$ case, we have few possibilities of connections since there is only one plane wave critical point, which we will continue to call r_0, as it is a continuation of the large amplitude plane wave for $a_0<0$.

Doelman and Eckhaus [13, Theorem 3.4] proved the existence of homoclinic connections connecting the unique plane wave to itself, with traveling speed $c=0$. In the terminology used in this paper, those connections are homoclinic domain walls. They used the approximate Poincaré map and Melnikov integrals as in Holmes [24]. Their proof can be generalized to the case of non-zero traveling speed c. We therefore have the following result.

Theorem 2. *Let $\varepsilon=(b_1,b_2,a_3,a_4,b_5)$ and ω be small enough. If $a_0>0$, then for coefficients in a subset of $(b_1,b_2,b_3,b_4,a_3,a_4,b_5)$-space, there exists a homoclinic domain wall connecting the unique plane wave ($z\to-\infty$) to itself ($z\to+\infty$), with appropriate choice of frequency ω and wave speed c.*

We now prove the following theorem on the existence of fronts, and nonexistence of domain walls in case $a_0>0$.

Theorem 3. *Let $\varepsilon=(b_1,b_2,a_3,a_4,b_5)$, b_3+b_4 and ω be small enough, and assume $a_0>0$. Then for $c>2\sqrt{a_0}$ and large enough, there is a front connecting the unique plane wave ($z\to-\infty$) to the zero amplitude wave ($z\to+\infty$). For $c<-2\sqrt{a_0}$ and $|c|$ large enough, there is a front running in the opposite direction. Moreover, for any $|c|>2\sqrt{a_0}$ no homoclinic domain walls exist.*

Proof. Consider (3.5)–(3.7) on the invariant plane $k=0$ for $\omega=0$. We only need to prove the $c>2\sqrt{a_0}$ case since the $c<-2\sqrt{a_0}$ case can be obtained from the symmetry (3.16). Since the unstable manifold $W^u(r_0,0,k_0)$ is tangent to the eigenvector $(r_0,\lambda^+,0)$ near $(r_0,0,k_0)$, its lower branch enters the $v<0$ half-space (see Figure 5). Along this branch r is initially decreasing as $r'=rv<0$. Moreover, this branch cannot enter the $v>0$ half-space since on $v=0$,

$$\begin{aligned} v' &= -a_0-a_2r^2+r^4 \\ &= (r^2-r_0^2)\left(r^2-\frac{a_2-\sqrt{a_2^2+4a_0}}{2}\right)<0. \end{aligned} \tag{5.1}$$

Hence the lower branch continues in the direction of decreasing r. This branch cannot cross the half-line $\{v=v^+, r>\sqrt{a_2}\}$ as on this half-line

$$v'=r^2(r^2-a_2)>0. \tag{5.2}$$

However, it may cross the half-line $\{v=v^+, r<\sqrt{a_2}\}$. If it does so, we can choose c large enough such that

$$\begin{aligned} v' &= -a_0-cv-v^2-a_2r^2+r^4 \\ &= -(v-v^+)(v-v^-)+r^2(r^2-a_2)>0. \end{aligned} \tag{5.3}$$

Here we use the fact that $v^+\to 0$ and $v^-\to-\infty$ as $c\to+\infty$, and that $r^2(r^2-a_2)$ has minimum $-(a_2^2/4)$. For such c, the lower branch moves upward and eventually enters the domain of attraction of the sink $(0,v^+,k^+)$ and thus we get a structurally stable saddle-sink connection which survives the small perturbation.

Since the lower branch of $W^u(r_0,0,r_0)$ is always going in the direction of decreasing r, it cannot come back to r_0 again. Similarly we can show that the upper branch is always going in the direction of increasing r. Thus no homoclinic domain walls are possible. □

6. Pulses: $a_0<0$ and $a_0>0$

It seems considerably more difficult to find conditions sufficient for the existence of pulses, although we can rule out their existence in certain parameter ranges by showing that necessary conditions are not met. We only consider non-trivial pulses (r not identically zero). Recall that a pulse involves connection (=identification) of the one-dimensional unstable manifold $W^u(0,v^+,k^+)$ and the stable manifold $W^s(0,v^-,k^-)$ when $-a_2^2/4<a_0<0$. While such a connection exists on $k=0$ for $c=\omega=0$, $a_0\in(-\frac{3}{16}a_2^2,0)$ and $\boldsymbol{\varepsilon}=\mathbf{0}$, $b_3+b_4=0$ (Figure 1(B)), simple estimates show that no such connection can exist for $c,\omega\neq 0$ and $\boldsymbol{\varepsilon}, b_3+b_4$ sufficiently small. In fact we have:

Theorem 4. *Suppose that either $a_0\in(-\frac{1}{4}a_2^2,0)$ and $(\omega,c)\neq(0,0)$, or $a_0>0$ and $|c|>2\sqrt{a_0}$. Then for $\boldsymbol{\varepsilon}=(b_1,b_2,a_3,a_4,b_5)$, b_3+b_4 and ω sufficiently small, no pulses exist.*

Proof. As in the proof above, we initially set $\boldsymbol{\varepsilon}=\mathbf{0}$, $b_3+b_4=0$ and then appeal to structural stability and transversality. First consider the case $\omega=0$ for which the plane $k=0$ remains invariant. Without loss of generality, let $c>0$ (for $c<0$ use the symmetry (3.16)). Since the zero amplitude wave fixed points $(0,v^+,0),(0,v^-,0)$, with

$$v^{\pm}=\frac{-c\pm\sqrt{c^2-4a_0}}{2}, \tag{6.1}$$

lie in this plane, so must any pulse connection. However, for $a_0\in(-\frac{1}{4}a_2^2,-\frac{3}{16}a_2^2)$, the unstable manifold of $(0,v^+,0)$ cannot enter the $v<0$ half space where $(0,v^-,0)$ lies, due to the connection orbit of Theorem 1, part 1 (see also Figure 2(A)) and the fact that on $\{v=0,0<r<r_1\}$, the derivative satisfies

$$v'=-a_0-a_2r^2+r^4=(r^2-r_0^2)(r^2-r_1^2)>0.$$

Moreover, in the case $a_0\in[-\frac{3}{16}a_2^2,0)$, since $E'=-cr^2v^2\leqq 0$, the interior of the compact component of any level set of

$$E=\frac{1}{2}r^2v^2+\frac{a_0}{2}r^2+\frac{a_2}{4}r^4-\frac{r^6}{6}, \tag{6.2}$$

is positively invariant (see Figure 2(B)). Since $(0,v^+,0)$ lies inside the level set containing the points $r=0$, $v=\pm\sqrt{-a_0}$ and $(0,v^-,0)$ lies outside it, no orbit can pass from the former to the latter.

Now suppose $\omega>0$ and $c\leqq 0$. The fixed points $(0,v^+,k^+)$, $(0,v^-,k^-)$ now lie in the left ($k<0$) and right ($k>0$) halves of the (k,v) plane, for we have $k^{\pm}=-\omega/(c+2v^{\pm})$. Consider the unstable manifold $W^u(0,v^+,k^+)$, which enters the half space $H_-=\{(r,v,k)\,|\,k<0<r\}$. On any orbit, including ones in this manifold, we have

$$M'=-(\omega+ck)r^2, \tag{6.3}$$

which is strictly negative in H_-. Thus the orbit crosses the level sets $M=r^2k=constant$ in the direction of decreasing M, implying that such an orbit can never leave H_- and enter the positive half-space $H_+=\{(r,v,k)\,|\,k,r>0\}$ in which the local stable manifold $W^s(0,v^-,k^-)$ lies. Thus no pulse exists.

For $\omega>0$ and $c\geqq 0$ a similar argument shows that the stable manifold $W^s(0,v^-,k^-)$ is confined to H_+ and so cannot connect to the unstable manifold $W^u(0,v^+,k^+)$. The case $\omega<0$ is treated similarly.

For $a_0>0$ and $c>2\sqrt{a_2}$, $(0,v^+,k^+)$ is a sink in (r,v,k) space, while the two-dimensional unstable manifold of $(0,v^-,k^-)$ is restricted to the invariant plane $r=0$. Thus no (non-trivial) pulses exist. The $c<2\sqrt{a_0}$ case is similar. □

Hakim, Jakobsen and Pomeau [**21**] conjectured the existence of pulses of non-zero traveling speed c for the quintic equation (1.3) for $a_0<0$. Our result above shows that such connections in fact cannot exist for b_1,b_2,b_3 small. Deissler and Brand [**9**] found some pulses or "one-particle" solutions for equation (1.2) numerically. Setting $a_3=b_3=a_4=b_4=0$ (no nonlinear terms with derivatives), they found a symmetric pulse which became asymmetric and whose propagation speed dropped significantly when $a_3=b_3=a_4=b_4\neq 0$. Since the symmetry $c\to -c$ and $z\to -z$ does not hold in general for the ODEs (2.12)–(2.14), such pulses indeed cannot be symmetric. Deissler and Brand's [**9**] parameter values a_j,b_j are all $O(1)$ and therefore lie outside the range in which our nonexistence result, Theorem 4, applies. They also took $a_2>0$. We have not been able to prove the existence of pulses under conditions comparable to those chosen by Deissler and Brand [**9**]. Our result also shows that for $a_0>0$ and $a_2>0$ pulses may exist only if $|c|\leqq 2\sqrt{a_0}$ and/or $\varepsilon=(b_1,b_2,a_3,a_4,b_5)$, b_3+b_4 and ω are sufficiently large. In fact, van Saarloos and Hohenberg [**30**, Section III.C] found an exact pulse solution of the quintic equation (1.3) for $a_0>0$ and $c=0$, which is a generalization of Hocking and Stewartson's [**23**] pulse solution for the cubic equation (1.1). The Hocking and Stewartson pulse solution is also called a breather since it is additionally periodic in time (also see Holmes [**24**] or Landman [**27**]).

7. Discussion

In Theorem 1 we give conditions for the existence of fronts connecting the zero amplitude wave to the small amplitude wave, and heteroclinic domain walls connecting the large amplitude wave to the small amplitude wave. Since the small amplitude plane wave is typically unstable, such connections may seem of little interest.

However, they do provide a simple class of solutions starting near the unstable wave and approaching a stable wave as $t \to \infty$. (Here we implicitly use a suitable weighted Sobolev norm such as

$$\|f\|_b = \int_{-\infty}^{\infty} |f|^2 b(x)dx. \tag{7.1}$$

where $b(x)$ is a weighting function which decays sufficiently fast as $|z| \to \infty$). For $c > 0$, the right going waves of either type show how the (dynamically stable) zero or large amplitude wave "swallows" the small amplitude wave as t increases. See Figure 6.

Note that our approach is rather different from those of Collet and Eckmann [6, 7], and Eckmann and Wayne [17], who use either the contraction principle or the center manifold theorem to obtain the existence of propagating fronts for the Swift–Hohenberg equation.

The differential form techniques used to prove transversality and hence persistence of heteroclinic orbits in Theorem 1 were used earlier by Jones et al. [26] to obtain a similar result for equation (1.2) in the case $c_3 = b_3 = a_4 = b_4 = b_5 = 0$, but our Theorems 1 and 2 appear to be the first for the full problem with all terms included to second order in the asymptotic derivation as in Doelman [10]. We are unaware of any results analogous to Theorems 3 and 4; in particular our non-existence conditions seem new.

A very important question about the long time dynamics of a pattern formation equation of PDE type, is the dynamical behavior of plane wave, front, pulse and domain wall type solutions such as those found in this paper. In particular, their stability or instability as solutions of the PDE is of great interest (Weinstein [38]). Further work concerning the stability of such solutions is in progress.

Acknowledgement. We would like to thank Arjen Doelman for helpful discussions. The phase portraits were computed using the software package DsTool developed by Guckenheimer et al. [20].

REFERENCES

1. V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, Berlin, 2nd ed., 1988).

2. V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, Berlin, 1978).

3. A. J. Bernoff, Slowly varying fully nonlinear wavetrains in the Ginzburg–Landau equation, *Phys. D* **23** (1988), 363–381.

4. W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry* (Academic Press, New York, 2nd ed., 1986).

5. S. Campbell and P. Holmes, Bifurcation from $O(2)$ symmetric heteroclinic cycles with three interacting modes, *Nonlinearity* **4** (1991), 697–726.

6. P. Collet and J.-P. Eckmann, The existence of dendritic fronts, *Comm. Math. Phys.* **107** (1986), 39–92.

7. P. Collet and J.-P. Eckmann, *Instabilities and Fronts in Extended Systems* (Princeton University Press, 1990).

8. W. Craig, C. Sulem and P. L. Sulem, Nonlinear modulation of gravity waves: a rigorous approach, *Nonlinearity* **5** (1992), 497–522.

9. R. J. Deissler and H. R. Brand, The effect of nonlinear gradient terms on localized states near a weakly inverted bifurcation, *Phys. Lett.* **146** (1990), 252–255.

10. A. Doelman, *On the nonlinear evolution of patterns (modulation equations and their solutions)* (Ph.D. Thesis, University of Utrecht, the Netherlands, 1990).

11. A. Doelman, Slow time-periodic solutions of the Ginzburg–Landau equation, *Phys. D* **40** (1989), 156–172.

12. A. Doelman, Traveling waves in the complex Ginzburg–Landau equation, *J. Nonlinear Sci.* **3** (1993), 225–266.

13. A. Doelman and W. Eckhaus, Periodic and quasiperiodic solutions of degenerate modulation equations, *Phys. D* **53** (1991), 249–266.

14. J. Duan, P. Holmes and E. S. Titi, Global existence theory for a generalized Ginzburg–Landau equation, *Nonlinearity* **5** (1992), 1303–1314.

15. J. Duan and P. Holmes, On the Cauchy problem for a generalized Ginzburg–Landau equation, in *Nonlinear Anal.: Theory, Methods and Applications* **22** (1994), 1033–1040.

16. J.-P. Eckmann and Th. Galay, Front solutions for the Ginzburg–Landau equation, *Comm. Math. Phys.* **152** (1993), 221–248.

17. J.-P. Eckmann and C. E. Wayne, Propagating fronts and the center manifold theorem, *Comm. Math. Phys.* **136** (1991), 285-307.

18. S. Fauve and O. Thual, Solitary waves generated by subcritical instabilities in dissipative systems, *Phys. Rev. Lett.* **64** (1990), 282–284.

19. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, 1983 (3rd printing, 1990)).

20. J. Guckenheimer, M. R. Myers, F. J. Wicklin and P. A. Worfolk, *dstool: Dynamical System Toolkit with Interactive Graphic Interface, User's Manual* (Center for Applied Mathematics, Cornell University, USA, 1991).

21. V. Hakim, P. Jakobsen and Y. Pomeau, Fronts vs. solitary waves in nonequilibrium systems, *Europhys. Lett.* **11** (1990), 19–24.

22. H. Hasimoto and H. Ono, Nonlinear modulation of gravity waves. *J Phys. Soc. Japan* **33** (1972), 805–811.

23. L. M. Hocking and K. Stewartson, On the nonlinear response of a marginally unstable plane parallel flow to a two-dimensional disturbance. *Proc. R. Soc. London A* **326** (1972), 289–313.

24. P. Holmes, Spatial structure of time-periodic solutions of the Ginzburg–Landau equation, *Phys. D* **23** (1986), 84-90.

25. C. K. R. T. Jones, N. Kopell and R. Langer, Construction of the Fitzhugh–Nagumo pulse using differential forms, in *Patterns and dynamics in reactive media*, G. Aris, D. Aronson and H. Swinney (eds.), IMA volumes in mathematics and its applications **37**, Springer-Verlag, Berlin, 1991, 101-116.

26. C. K. R. T. Jones, T. Kapitula and J. Powell, Nearly real fronts in a Ginzburg–Landau equation, *Proc. Roy. Soc. Edinburgh A* **116** (1990), 193–206.

27. M. J. Landman, Solutions of the Ginzburg–Landau equation of interest in shear flow transition. *Stud. Appl. Math.* **76** (1987), 187-237.

28. B. A. Malomed and A. A. Nepomnyashchy, Kinks and solitons in the generalized Ginzburg–Landau equations, *Phys. Rev. A* **42** (1990), 6009–6014.

29. W. Van Saarloos and P. C. Hohenberg, Pulses and fronts in the complex Ginzburg–Landau equation near a subcritical bifurcation, *Phys. Rev. Lett.* **64** (1990), 749–752.

30. W. Van Saarloos and P. C. Hohenberg, Fronts, pulses, sources and sinks in generalized complex Ginzburg–Landau equations, *Phys. D* **56** (1992), 303–367.

31. W. Schopf and L. Kramer, Small amplitude periodic and chaotic solutions of the complex Ginzburg–Landau equation for a subcritical bifurcation, *Phys. Rev. Lett.* **66** (1991), 2316–2319.

32 W. Schopf, and W. Zimmermann, Multicritical behavior in binary fluid convection, *Europhys. Lett.* **8** (1989), 41–46.

33. W. Schopf and W. Zimmermann, Convection in binary fluids: amplitude equations, codimension-2 bifurcation and thermal fluctuations, *Phys. Rev. E* **47** (1993), 1739–1764.

34. L. Sirovich and P. K. Newton, Periodic solutions of the Ginzburg–Landau equation, *Phys. D* **21** (1986), 115–125.

35. J. Smoller, *Shock Waves and Reaction–Diffusion Equations* (Springer-Verlag, Berlin, 1983).

36. O. Thual and S. Fauve, Localized structures generated by subcritical instabilities, *J. Phys. France* **49** (1988), 1829–1833.

37. S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer-Verlag, Berlin, 1990).

38. M. I. Weinstein, Modulational stability of ground states of nonlinear Schrodinger equations, *SIAM J. Math. Anal.* **16** (1985), 472–490.

Applied Mechanics 104–44
California Institute of Technology
Pasadena, California 91125
U.S.A.

Department of Mechanical
and Aerospace Engineering
and
Program in Applied and
Computational Mathematics
Princeton University
Princeton, New Jersey 08544
U.S.A.