J. Fluid Mech. (2023), *vol.* 965, A14, doi:10.1017/jfm.2023.416

Scaling of turbulent velocity structure functions: plausibility constraints

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(Received 19 October 2022; revised 29 March 2023; accepted 11 May 2023)

The nth-order velocity structure function S_n in homogeneous isotropic turbulence is usually represented by $S_n \sim r^{\zeta_n}$, where the spatial separation r lies within the inertial range. The first prediction for ζ_n (i.e. $\zeta_3 = n/3$) was proposed by Kolmogorov (*Dokl. Akad. Nauk SSSR*, vol. 30, 1941) using a dimensional argument. Subsequently, starting with Kolmogorov (*J. Fluid Mech.*, vol. 13, 1962, pp. 82–85), models for the intermittency of the turbulent energy dissipation have predicted values of ζ_n that, except for $n = 3$, differ from $n/3$. In order to assess differences between predictions of ζ_n, we use the Hölder inequality to derive exact relations, denoted plausibility constraints. We first derive the constraint $(p_3 - p_1)\zeta_{2p_2} = (p_3 - p_2)\zeta_{2p_1} + (p_2 - p_1)\zeta_{2p_3}$ between the exponents ζ_{2p}, where $p_1 \leq p_2 \leq p_3$ are any three positive numbers. It is further shown that this relation leads to $\zeta_{2p} = p\zeta_2$. It is also shown that the relation $\zeta_n = n/3$, which complies with $\zeta_{2p} = p\zeta_2$, can be derived from constraints imposed on ζ_n using the Cauchy–Schwarz inequality, a special case of the Hölder inequality. These results show that while the intermittency of ϵ, which is not ignored in the present analysis, is not incompatible with the plausible relation $\zeta_n = n/3$, the prediction $\zeta_n = n/3 + \alpha_n$ is not plausible, unless $\alpha_n = 0$.

Key words: isotropic turbulence, turbulence theory

1. Introduction

There is a general consensus that the following power law (or similarity law) holds in the inertial range of homogeneous isotropic turbulence (HIT), at very, if not infinitely, large Reynolds numbers:

$$\overline{(\delta u)^n} \sim r^{\zeta_n}, \tag{1.1}$$

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where $\delta u = u(x + r, t) - u(x, t)$ is the longitudinal velocity increment between two points, r is the separation between the two points, and t is the time; the overbar represents classical conventional averaging used regularly in the literature for testing small-scale statistics. The power-law form (1.1) was first introduced for $n = 2$ by Kolmogorov (1941*a*) (hereafter referred to as K41) on dimensional grounds and written as

$$\overline{(\delta u)^2} = C_K(\bar{\epsilon} r)^{\zeta_2}, \tag{1.2}$$

where $\zeta_2 = 2/3$, C_K is supposed to be a universal constant (i.e. independent of the Reynolds number and the macro flow structure), and $\bar{\epsilon}$ is the mean turbulent kinetic energy dissipation rate. To arrive at (1.2), often denoted the 2/3-law, $\overline{(\delta u)^2}$ is assumed to depend only on r and $\bar{\epsilon}$ in a scaling range, denoted the inertial range, where the effects of viscosity and the large-scale motion are negligible. This implies that the Reynolds number is infinite or at least very large. However, acting on a remark made by Landau & Lifshitz (1987), Kolmogorov (1962) (hereafter referred to as K62) revisited his earlier analysis, and still based on dimensional grounds, proposed

$$|\overline{(\delta u)^n}| = C_n\,(\bar{\epsilon} r)^{n/3} \left(\frac{L}{r}\right)^{kn(n-3)/2}, \tag{1.3}$$

where k is a universal constant. The exponent $kn(n - 3)/2$ is based on the assumption that the probability density function (p.d.f.) of ϵ is log-normal; however, this assumption has been shown to be incorrect and abandoned. Notice the absolute value in (1.3), which was dropped in the studies that followed. Using the relation $\bar{\epsilon} = C_\epsilon(u_0^3/L)$ (where C_ϵ is a constant), (1.3) is commonly expressed as

$$\overline{(\delta u)^n} = A_n u_0^n \left(\frac{r}{L}\right)^{\zeta_n}, \tag{1.4}$$

with $\zeta_n = n/3 + \alpha_n$, where α_n is a real number; the scaling set (u_0, L) is representative of the large-scale motion (u_0 is often taken as the velocity root mean square (r.m.s.)), and A_n is a positive numerical constant that may depend on the flow macrostructure but is Reynolds-number-independent (see K62). The power law (1.4) (or (1.2) for that matter) is yet to be derived from the Navier–Stokes equations. Frisch (1995) used this expression for $n = 2p$ (where p is a positive integer) and applied the Hölder inequality to it to derive his convexity inequality (see relation (2.2) below). The length scale L was introduced initially by K62 to account for the intermittency of the turbulent kinetic energy dissipation ϵ (Frisch 1995) caused by the large-scale motion (Landau & Lifshitz 1987).

In the literature, another scaling set based on the Kolmogorov velocity ($v_K = (\nu\bar{\epsilon})^{1/4}$) and Kolmogorov length ($\eta = (\nu^3/\bar{\epsilon})^{1/4}$) is often used for presenting the data, and we can use this to rewrite (1.4). Using a simple change of variables where we introduce v_K and η in (1.4), and noting that for HIT, $\eta/L = C_\epsilon^{-1/4} Re_L^{-3/4}$ and $v_K/u_0 = C_\epsilon^{1/4} Re_L^{-1/4}$ (we use the relation $\bar{\epsilon} = C_\epsilon(u_0^3/L) = v_K^3/\eta$, where $Re_L = u_0 L/\nu$ is the large-scale Reynolds number and C_ϵ is a constant), we obtain the trivial relation

$$A_n u_0^n \left(\frac{r}{L}\right)^{\zeta_n} = A_n v_K^n \frac{Re_L^{(n-3\zeta_n)/4}}{C_\epsilon^{(n+\zeta_n)/4}} \left(\frac{r}{\eta}\right)^{\zeta_n}, \tag{1.5}$$

or, following Buaria & Sreenivasan (2022) (see § IV in the supplemental material of Buaria & Sreenivasan 2022),

$$A_n u_0^n \left(\frac{r}{L}\right)^{\zeta_n} = A_n v_K^n \frac{C_\epsilon^{-\zeta_n}}{15^{(n-3\zeta_n)/4}} Re_\lambda^{(n-3\zeta_n)/2} \left(\frac{r}{\eta}\right)^{\zeta_n}, \tag{1.6}$$

since $Re_L = C_\epsilon\, Re_\lambda^2/15$, where $Re_\lambda = u_0\lambda/\nu$ is the Taylor microscale Reynolds number. It is important for the rest of the analysis to stress that relation (1.5) (or (1.6)) is an exact identity for HIT and, accordingly, both sides of this identity have the exact same range of r, i.e. $\eta \ll r \ll L$. The explicit Reynolds number dependence on the right-hand side of (1.5) and (1.6), which results from the change of variables, warrants a comment. We observe that when $n = 3$, the terms $Re_L^{(n-3\zeta_n)/4}$ and $Re_\lambda^{(n-3\zeta_n)/2}$ disappear, i.e. the explicit Reynolds number dependence vanishes from the right-hand side of (1.5) and (1.6) because of the 4/5-law (Kolmogorov 1941*b*)

$$\overline{(\delta u)^3} = -\tfrac{4}{5}\bar{\epsilon} r; \tag{1.7}$$

that is, $\zeta_3 = 1$. Both (1.5) and (1.6) can now be written as

$$\overline{(\delta u)^3} = A_3 C_\epsilon \bar{\epsilon} r. \tag{1.8}$$

Comparing (1.8) with (1.7) shows that $A_3 = -(4/5)C_\epsilon^{-1}$. Recall that the 4/5-law exists only when the Reynolds number dependence disappears from the Kármán–Howarth (KH) equation (Kármán & Howarth 1938) when expressed in terms of $\overline{(\delta u)^2}$ and $\overline{(\delta u)^3}$. This means that the effects of viscosity and the large-scale motion, referred to as the finite Reynolds number (FRN) effect (Antonia & Burattini 2006; Antonia *et al.* 2019), vanish and the 4/5-law is verified. Further, there is a strong theoretical basis for the independence of the Reynolds number in the asymptotic case, $Re_L \to \infty$. For example, Djenidi, Antonia & Tang (2019) showed that as Re_L increases, the scaling of the KH equation based on (v_K, η) extends to increasingly larger values of r/η, while the scaling based on (u_0, L) extends to increasingly smaller values of r/L. When $Re_L \to \infty$, both scalings hold over a common range of scales (i.e. the inertial range), and the solutions of the KH equation become Reynolds-number-independent. It is difficult to imagine that while the FRN effect vanishes when $n = 3$, it should persist for all other values of n. The removal of such dependence for all n requires $\zeta_n = n/3$, which would lead to the existence of a dual scaling where $\overline{(\delta u)^n}$ scales with both (u_0, L) and (v_K, η) in the inertial range (a plausible derivation of (1.1) based on this dual scaling is given in Appendix A). However, the multifractal approach proposes the following model for ζ_n (Frisch 1995):

$$\zeta_n = \inf_h [ph + 3 - D(h)], \tag{1.9}$$

where h is an arbitrary positive scaling exponent, and $D(h)$ is the corresponding multifractal dimension, assumed to be independent of the way turbulent flows are generated (Frisch 1995). The model (1.9), which is based on the intermittency of the velocity and can be related to the multifractal prediction of ζ_n based on the intermittency of ϵ (Meneveau & Sreenivasan 1991; Frisch 1995), was developed to mimic the behaviour $\zeta_n = n/3 + \alpha_n$ of the empirically determined ζ_n with increasing n, and the constraint $\zeta_3 = 1$.

Clearly, the above discussion illustrates the importance of determining the correct values of ζ_n as they will provide a definitive answer to the question: is K62 a more suitable descriptor of small-scale turbulence than K41? There are at least two difficulties in relation to estimating ζ_n. First, the prospect of determining the power-law form, such (1.1) or (1.4), from the Navier–Stokes equations is rather grim. That is unfortunate because one cannot be strictly sure that a power-law formulation for $\overline{(\delta u)^n}$ is actually consistent with the Navier–Stokes equations. Indeed, one has to recall that the viscosity is never zero in these equations, while both K41 and K62 assume that ν has no influence in the inertial range. This leads to the second difficulty, the empirical estimation of ζ_n from experimental and direct numerical simulation data. Such an approach is hindered by the impossibility of achieving very large Reynolds numbers that would ensure a well-defined inertial range where the conditions $r/\eta \gg 1$ and $r/L \ll 1$ are both satisfied adequately. Nevertheless, it is generally assumed in theories of steady-state small-scale turbulence that the Reynolds number is large enough (but not necessarily infinite) so that the small-scale statistics are Reynolds-number-independent as long as the energy input is balanced by the energy dissipation. The case $\nu = 0$ (inviscid flow) is actually excluded as it raises the issue of a singularity or blow-up (see Frisch (1995), who comments on the potential blow-up at $\nu = 0$); after all, all real flows are viscous. Accordingly, it is in this context that the power-law form (1.4) is assumed to be Reynolds-number-independent and asymptotically correct when the Reynolds number is very large. Further, the experimental and direct numerical simulation data suggest that $\overline{(\delta u)^n}$, at least for $n = 2$, 3 and 4, tends to approach a power-law behaviour of the form (1.1) as the Reynolds number increases.

It is clear that, at least for a foreseeable future, the ability to reach very large Reynolds numbers in HIT is quite remote, posing a challenging task for estimating ζ_n empirically. Nevertheless, one can still attempt to test the plausibility of ζ_n predicted by K41, i.e. $\zeta_n = n/3$, the models based on the intermittency of ϵ (i.e. (1.9) or $\zeta_n = n/3 + \alpha_n$), or indeed any future model for small-scale turbulence in the inertial range, against exact constraints that have to be satisfied if the predicted values of ζ_n are to be deemed plausible (Frisch 1995). One such constraint is given by (1.7) and requires ζ_3 to be equal to 1. In this paper, we follow the approach of Frisch (1995), who proposed a method for assessing the plausibility of predictions for ζ_n. Frisch (1995) applied the Hölder inequality to (1.4), an approach that is independent of any theoretical phenomenology that underpins (1.4) and yields exact mathematical results. Frisch (1995) considered only (1.4) or the left-hand side of (1.5). However, since (1.5) is an identity, the same constraints must be imposed on both sides of this identity. This approach is outlined in § 2 where (1.5) (or equivalently (1.6)) is treated as a simple 'mathematical object' whose theoretical derivation and physical meaning bear no relevance to the analysis of this section. In § 3, we discuss the consequences of the mathematical results of § 2 in relation to the predictions of ζ_n based on K41 and intermittency models. We then provide concluding remarks in § 4.

2. Plausibility of the exponents ζ_n

2.1. *Plausibility constraints on ζ_n*

In this subsection, we follow K62 and assume that (1.4) – or equivalently, the identity (1.5) – holds in the inertial range. The method for determining constraints for the exponents ζ_n follows that of Frisch (1995), who also assumes that (1.4) holds in the inertial range, and is based on determining mathematical constraints with which the power-law exponents must comply. While mathematical constraints, such as those provided by the Cauchy–Schwarz and Hölder inequalities, bear no relation to the Navier–Stokes equations, they nevertheless provide rigorous conditions that must be complied with, and thus can be used as a test for any model that predicts the exponents ζ_n. Indeed, they provide necessary (but not sufficient) conditions for assessing the plausibility of the exponents ζ_n determined empirically or theoretically (Frisch 1995; Falkovich, Gawęędzki & Vergassola 2001; Eling & Oz 2015). Any value of ζ_n that fails this plausibility test must be dismissed. It is thus of interest to apply similar tests to the identities (1.5) and (1.6).

We start by applying the Hölder inequality (Feller 1968) to $\overline{(\delta u)^{2p}}$, and follow Frisch (1995). Thus we obtain

$$\left(\overline{(\delta u)^{2p_2}}\right)^{(p_3-p_1)} \leq \left(\overline{(\delta u)^{2p_1}}\right)^{(p_3-p_2)} \left(\overline{(\delta u)^{2p_3}}\right)^{(p_2-p_1)} \tag{2.1}$$

for any three positive numbers $p_1 \leq p_2 \leq p_3$. When we consider the left-hand side of the identity (1.5), i.e. $\overline{(\delta u)^n} \sim (r/L)^{\zeta_n}$, we obtain the following constraint imposed on ζ_{2p}:

$$(p_3 - p_1)\zeta_{2p_2} \geq (p_3 - p_2)\zeta_{2p_1} + (p_2 - p_1)\zeta_{2p_3}, \tag{2.2}$$

since $r/L \ll 1$ in the inertial range. This inequality, commonly referred to as the convexity inequality, is often used to validate or invalidate the exponents ζ_n obtained empirically or theoretically (Falkovich *et al.* 2001; Eling & Oz 2015). However, when we now consider the right-hand side of (1.5), i.e. $\overline{(\delta u)^n} \sim (r/\eta)^{\zeta_n}$, (2.1) leads to a new constraint on ζ_{2p}:

$$(p_3 - p_1)\zeta_{2p_2} \leq (p_3 - p_2)\zeta_{2p_1} + (p_2 - p_1)\zeta_{2p_3}, \tag{2.3}$$

since $r/\eta \gg 1$ in the inertial range. Considering that the inequalities (2.2) and (2.3) result from (2.1) applied to both sides of the identity (1.5), they are then satisfied simultaneously. This yields the following relation amongst the exponents ζ_{2p}:

$$(p_3 - p_1)\zeta_{2p_2} = (p_3 - p_2)\zeta_{2p_1} + (p_2 - p_1)\zeta_{2p_3}. \tag{2.4}$$

Thus for any three positive numbers $p_1 \leq p_2 \leq p_3$, any plausible exponents ζ_n must obey relation (2.4), which is a plausibility constraint. This constraint is far more restrictive than either (2.2) or (2.3). It is important to recall that (1.5), or equivalently (1.6), and (2.1) and accordingly (2.4) hold only in the inertial range $\eta \ll r \ll L$ when $L/\eta \to \infty$ as $Re_L \to \infty$. Now, noting that for $\zeta_0 = 0$, $\overline{(\delta u)^0} \sim r^{\zeta_0} = 1$, we can see that (2.1) holds also when $p_1 = 0$. Thus taking $p_1 = 0$, $p_2 = 1$ and $p_3 = n$ $(n \geq 2)$ into (2.4) yields

$$\zeta_{2n} = n\zeta_2, \tag{2.5}$$

which relates ζ_{2n} to ζ_2, where the latter remains to be determined. Interestingly, this relation shows that the power-law exponents of even order increase linearly with n. We also notice that $\zeta_n = n/3$ verifies this relation, while $\zeta_n = n/3 + \alpha_n$ does not unless $\alpha_n = 0$. But let us next determine whether a plausible expression for ζ_n can be derived analytically.

2.2. *Determination of plausible ζ_n*

We apply the Cauchy–Schwarz (CS) inequality, which is a particular case of the Hölder inequality (Feller 1968), to $\overline{(\delta u)^n}$. For any arbitrary random variables ϕ and ψ, the CS inequality can be expressed as

$$|\overline{(\phi\psi)}|^2 \leq \overline{\phi^2} \times \overline{\psi^2}. \tag{2.6}$$

If we select $\phi = (\delta u)^n$ ($n \geq 2$) and $\psi = 1$, then (2.6) leads to

$$|\overline{(\delta u)^n}| \leq \overline{(\delta u)^{2n}}^{1/2}. \tag{2.7}$$

Applying this relation to the left- and right-hand sides of (1.5) yields

$$2\zeta_n \geq \zeta_{2n} \tag{2.8}$$

and

$$2\zeta_n \leq \zeta_{2n}, \tag{2.9}$$

respectively. Since these two inequalities are constraints imposed on ζ_n by applying (2.7) to both sides of (1.5), they hold simultaneously, which leads to

$$2\zeta_n = \zeta_{2n}. \tag{2.10}$$

Similarly, selecting now $\phi = (\delta u)^{n-1}$ and $\psi = \delta u$, we have

$$|\overline{(\delta u)^n}| = |\overline{(\delta u)(\delta u)^{n-1}}| \leq \overline{(\delta u)^2}^{1/2}\ \overline{(\delta u)^{2n-2}}^{1/2} \tag{2.11}$$

which yields

$$2\zeta_n \geq \zeta_2 + \zeta_{2n-2} \tag{2.12}$$

if one considers the left-hand side of (1.5), and

$$2\zeta_n \leq \zeta_2 + \zeta_{2n-2} \tag{2.13}$$

if one considers the right-hand side of (1.5). As above, since (2.12) and (2.13) hold simultaneously, we have

$$2\zeta_n = \zeta_2 + \zeta_{2n-2}. \tag{2.14}$$

Using (2.10), we can write (2.14) as

$$2\zeta_n = \zeta_2 + 2\zeta_{n-1}. \tag{2.15}$$

Applying again the CS inequality to $\overline{(\delta u)^{n-1}} = \overline{(\delta u)(\delta u)^{n-2}}$ yields

$$2\zeta_{n-1} = \zeta_2 + \zeta_{2n-4}. \tag{2.16}$$

Combining (2.15) and (2.16), we obtain

$$2\zeta_n = 2\zeta_2 + \zeta_{2n-4}. \tag{2.17}$$

Taking $n = 3$ and noting that $\zeta_3 = 1$ (Kolmogorov 1941*b*), (2.17) leads to

$$\zeta_2 = \frac{2}{3}. \tag{2.18}$$

Finally, combining (2.5), (2.10) and (2.18) yields

$$\zeta_n = \frac{n}{3}. \tag{2.19}$$

This result may not be surprising since the CS inequality is a special case of the Hölder inequality, which led to (2.5) and which is satisfied by $\zeta_n = n/3$. We can, in fact, expect to obtain (2.5) from the CS inequality. This can be shown by applying a recursive process n times to (2.15) as follows:

$$2\zeta_n = \zeta_2 + 2\zeta_{n-1} = \zeta_2 + \zeta_2 + 2\zeta_{n-2} = \cdots = n\zeta_2, \tag{2.20}$$

where we used $\zeta_0 = 0$. Now using (2.10) to replace $2\zeta_n$ by ζ_{2n}, (2.5) is obtained.

It is important to stress that in this section we treat (1.4) as a simple 'mathematical' function, regardless of any phenomenology used to derive it, or its physical meaning. Accordingly, the results of this section are purely mathematical. For example, the results indicate that the relation (1.9) for ζ_n fails to comply with the plausibility constraints. It is only when a physical meaning is attributed to (1.4) that one can draw some conclusions on the phenomenology used to derive the prediction (1.9). In the present case, the results show that the relation (1.9) developed under multifractal arguments is not plausible. On the other hand, the relation $\zeta_n = n/3$ complies with the plausibility test, suggesting that the K41 phenomenology is plausible.

3. Discussion

When one attaches a physical meaning to (1.4) and considers the theoretical phenomenology used to derive it, the result $\zeta_n = n/3$ obtained in § 2 is somewhat unexpected and conflicts with the dominant view that $\zeta_n = n/3 + \alpha_n$. Indeed, the starting point of the determination of ζ_n is the expression $\overline{(\delta u)^n} \sim u_0^n (r/L)^{\zeta_n}$ as obtained via K62 or intermittency models, which predict $\zeta_n = n/3 + \alpha_n$. It is important to re-emphasize that since the analysis in § 2 is purely mathematical, its outcome is independent of the phenomenology used to derive (1.4). We nevertheless recall that K62 attempts to account for the effect of the intermittency of ϵ in the inertial range, and multifractal models have been proposed to explain the so-called anomalous scaling of ζ_n, i.e. the deviation of ζ_n from $n/3$ observed in experimental and numerical data (Anselmet *et al.* 1984; Frisch 1995; Sreenivasan & Antonia 1997; Sreenivasan & Dhruva 1998; Anselmet, Antonia & Danaila 2001; Chen *et al.* 2005; Iyer, Sreenivasan & Yeung 2020); the multifractal prediction (1.9) quantifies this deviation. The anomalous scaling is generally attributed to the intermittency of ϵ in the inertial range. The intermittency in the inertial range is certainly real, as illustrated by the fact that the skewness of δu is not zero, and according to K62 leads to (1.4), i.e. the starting point of the present analysis. In light of this, the finding $\zeta_n = n/3$ would indicate that the intermittency of ϵ does not necessarily invalidate the 1941 theory of K41, which is at odds with the arguments that led to the relation (1.9). We cannot explain why the K62 prediction for ζ_n and the relation (1.9) do not comply with the constraints (2.4). We can only provide a few comments. In the multifractal framework, (1.9) is independent of the way the flow is produced. This requires the Reynolds number to be large enough for the effect of the large-scale motion to be negligible in the inertial range. Unless this requirement is truly satisfied, it is difficult to test the multifractal models and K62 (or K41 for that matter) since both experimental and numerical data will be affected by the Reynolds number effect (Antonia & Burattini 2006; Antonia *et al.* 2019). A key assumption in determining the exponent ζ_n from experimental and numerical data is that one can fit a power-law form, such as (1.1), over a range of scales where the effects of viscosity and the large-scale motion are negligible. This requires a large enough Reynolds number so that both $r/\eta \gg 1$ and $r/L \ll 1$ are satisfied in this range of scales, i.e. the inertial range; an inertial range where $r/L \ll 1$ has yet to be achieved in experiments and direct numerical simulations. If these conditions are satisfied, then the Kármán–Howarth equation (Kármán & Howarth 1938) leads to the 4/5-law or $\overline{(\delta u)^3} = -(4/5)u_0^3 C_\epsilon (r/L)$, as expressed in terms of u_0 and L. This means that only when this law is observed can one expect $\overline{(\delta u)^n}$ to follow (1.4) in the inertial range. Unfortunately, as noted earlier (Antonia *et al.* 2019), this 4/5-law is yet to be observed convincingly in the literature, although the data indicate that it is approached as the Reynolds number increases. Accordingly, the determination of ζ_n from experimental and numerical data is inevitably hindered by the Reynolds number effect.

Let us now assess the implication of the results of § 2 for the ratio S_n defined as

$$S_n(r) = \left| \frac{\overline{(\delta u)^n}}{\overline{(\delta u)^2}^{n/2}} \right|. \tag{3.1}$$

Using (1.5), we then have

$$S_n(r) \sim \left(\frac{r}{L}\right)^{(\zeta_n - n\zeta_2/2)} \sim Re_\lambda^\alpha \left(\frac{r}{\eta}\right)^{(\zeta_n - n\zeta_2/2)}, \tag{3.2}$$

where $\alpha = \frac{3}{2}(n\zeta_2/2 - \zeta_n)$. When $n = 3$ and 4, $-S_3$ and S_4 are the skewness and flatness factor of δu, respectively. We have already established relation (2.20) (i.e. $2\zeta_n = n\zeta_2$). Substituting this relation in (3.2) shows that $S_n(r)$ should be independent of both the Reynolds number and the increment r in the inertial range, leaving S_n as a non-zero constant. This is at variance with K62. Indeed, for example, when $n = 3$, K62 predicts $S_3(r) \sim (r/L)^{-3\mu/2}$, where μ is a small positive number (recall that $r \ll L$), implying that $S_3(r)$, and consequently the intermittency, are functions of r in the inertial range. Let us examine the behaviour of (3.2) in the inertial range. To do that, we arbitrarily set $r = \lambda$ (this value of r should adequately satisfy the inertial range requirement – in fact, any value of r in the inertial range will lead to the same result) in (3.2), and noting that $\lambda/\eta \sim Re_\lambda^{1/2}$, we obtain

$$S_n(r = \lambda) \sim Re_\lambda^{(n\zeta_2/2 - \zeta_n)}. \tag{3.3}$$

Unless $n\zeta_2/2 - \zeta_n = 0$, (3.3) indicates that $S_n(r = \lambda)$ either increases without bound (if $\zeta_n < n\zeta_2/2$) or decreases to zero (if $\zeta_n > n\zeta_2/2$) with increasing Re_λ. However, neither behaviour seems realistic. Let us take $n = 3$. It is difficult to imagine that $S_3(r)$ can increase without bound with Re_λ in the inertial range, while $S_3(r) \to 0$ as $r/L \to 1$. On the other hand, a decrease of $S_3(r)$ as Re_λ increases would indicate an unrealistic weakening of the intermittency with increasing Re_λ in the inertial range. The available data for S_3 examined by Antonia *et al.* (2015) (see figure 9 in their paper) indicate that $S_3(r)$ slowly approaches a constant over the range $\eta < r < L$ as Re_λ increases.

In light of the above, it is worthwhile assessing the behaviour of the moments of the longitudinal velocity gradient, denoted S_{n,u_x}; the subscript u_x stands for the longitudinal velocity gradient $\partial u/\partial x$. These moments can be obtained by applying the limit $r \to 0$ to (3.1). Multifractal arguments show that S_{n,u_x} increases with Re_λ, that is,

$$S_{n,u_x} \sim Re_\lambda^{\xi_n}, \tag{3.4}$$

with $\xi_n > 0$. Independently of the various arguments used to arrive at (3.4), as well as of the different expressions for ξ_n, S_{n,u_x} must be consistent with the Navier–Stokes equations. To assess this consistency, we once again take $n = 3$. Starting with the Kármán–Howarth equation, we can obtain the following expression for S_{3,u_x} when $r \to 0$

(Djenidi *et al.* 2019):

$$- S_{3,u_x} = 2 \times 15^{3/2} \beta_K - \frac{60}{7} \frac{1}{Re_\lambda}, \tag{3.5}$$

where $\beta_K = 2G/Re_\lambda$; here, G is the destruction coefficient of $\bar{\epsilon}$ (Batchelor & Townsend 1947). The second term on the right-hand side of (3.5) is associated with the effect of the large-scale motion. This relation, which must be satisfied in the same way as the 4/5-law is satisfied in the inertial range, confirms the Reynolds number dependence of S_{3,u_x} observed in experimental and numerical data. However, since Antonia *et al.* (2015) showed that β_K approaches a constant as Re_λ increases, the Re_λ dependence of S_{3,u_x} gradually vanishes as the Reynolds number increases, indicating that $-S_{3,u_x}$ cannot grow without bound (for further critical discussion on this issue, see Qian 1994; Antonia *et al.* 2015; Tang *et al.* 2019). A similar observation can be made for the flatness factor S_{4,u_x}, which can be written as (Djenidi *et al.* 2019)

$$S_{4,u_x} = -750 \left[\gamma_{1,K} - 4\gamma_{2,K} \right] - \frac{100}{3} \frac{S_{3,u_x}}{Re_\lambda}, \tag{3.6}$$

where $\gamma_{1,K}$ and $\gamma_{2,K}$ are constants when Re_λ is sufficiently large; the second term on the right-hand side of (3.6) is associated with the effect of the large-scale motion. Expressions (3.5) and (3.6) derived from the Navier–Stokes equations are in conflict with (3.4) unless $\xi_n = 0$.

It is instructive to examine how the multifractal model – or indeed other types of model – leads to (3.4). Essentially, the restriction that r should be in the inertial range is relaxed, and the limit $r \to 0$ is allowed. For example, Nelkin (1990) simply extrapolates the multifractal description of the inertial range down to the dissipative range; Frisch (1995) followed the same approach. This approach (see also, for example, Wyngaard & Tennekes 1970; Frisch, Sulem & Nelkin 1978; Van Atta & Antonia 1980; Monin & Yaglom 2007) is inappropriate since the viscosity, which is a controlling parameter in the dissipative range, is assumed to have no influence in the inertial range. In fact, Nelkin (1990) acknowledges that this ‘brute-force’ extension of the multifractal picture to the very small scales need not be correct. In that respect, Nelkin (1990) remarked that Kraichnan (1990) proposed a model in which nearly exponential tails in the p.d.f. of the velocity derivative exist without recourse to multifractalisms, and for which S_{3,u_x} and S_{4,u_x} are independent of the Reynolds number. One should also point out that using a non-Gaussian p.d.f. of δu with stretched exponential tails, together with a ‘quasi-closure’ scheme, Qian (1998, 2000, 2001) showed that the resulting scaling exponents favoured the K41 theory over the K62 theory.

4. Concluding remarks

We have formulated the mathematical constraints on the exponents ζ_n that have been predicted by K41 and various intermittency models. In order to do this, the Hölder inequality has been applied to the identity (1.5). This method is independent of any phenomenology that underpins (1.4) and yields mathematical conditions, which have been referred to as plausibility constraints. The latter are independent of the types of arguments used previously in developing intermittency models that predict ζ_n. Any plausible prediction for ζ_n must comply with these constraints. Thus any prediction for ζ_n that fails this plausibility test should result in the abandonment of the corresponding model. In that regard, we found that predictions for ζ_n based on K62 or multifractal arguments and the empirically determined values of ζ_n reported in the literature are implausible.

Of course, all models yield the correct value of ζ_n when $n = 3$ (i.e. $\zeta_3 = 1$) since they are developed to satisfy the constraint imposed by the 4/5-law. The relation $\zeta_n = n/3$, as given by K41, satisfies the plausibility test for all n. It should be stressed that the present results do not exclude the phenomenon of intermittency since the analysis is performed on (1.4), whose derivation is based on the phenomenology of the intermittency of small-scale turbulence. Although relation (1.4) has yet to be derived from the equation of motion and thus has yet to be validated, its validity has nevertheless been assumed by supporters of K62. The present analysis assesses the exponents ζ_n only when (1.4) is assumed to exist and hold in the inertial range $\eta \ll r \ll L$ when $Re_L \to \infty$. In that regard, the results provide only a set of mathematical constraints for ζ_n to comply with, and do not offer any direct insight into the small-scale turbulence phenomenology. For example, they show that while the intermittency of ϵ is compatible with the plausible relation $\zeta_n = n/3$, the prediction $\zeta_n = n/3 + \alpha_n$ is not plausible, unless $\alpha_n = 0$.

We have also derived $\zeta_n = n/3$ by applying the Cauchy–Schwarz inequality to (1.5), thus confirming the compliance with the plausibility test. While $\zeta_n = n/3$ may appear controversial since it conflicts with the dominant view that ζ_n is 'anomalous', i.e. it deviates from $\zeta_n = n/3$ (see Benzi & Biferale (2015) for a relatively short review of work done in the last decades), it is nevertheless mathematically correct and therefore cannot be ignored or dismissed, in the same way that the 4/5-law cannot be ignored when models for ζ_n are developed. The present results may raise an 'apparent' paradox, i.e. $\zeta_n = n/3$ even though the phenomenon of intermittency is believed to lead to (1.4) with $\zeta_n \neq n/3$ except for $n = 3$. However, the paradox arises only when intermittency models are introduced to explain 'pseudo-scaling exponents' obtained at finite Reynolds numbers. These latter exponents, if they really exist, are not the exponents that pertain to the inertial range when $Re_\lambda \to \infty$. It should be recalled that the result $\zeta_n = n/3$ emerges from the constraints imposed on (1.4) when $Re_\lambda \to \infty$, the necessary and required condition for both K41 and K62. Further, when $\zeta_n = n/3$, (1.4) simply reflects K41; this does not ignore intermittency since, at least for $n = 2$ and $n = 3$, the Kármán–Howarth equation, which does not ignore intermittency, admits similarity based on either (v_K, η) or (u', L) when $Re_\lambda \to \infty$ (where u' is the velocity r.m.s.). In summary, not only is intermittency not excluded from the present analysis, it is in fact not incompatible with $\zeta_n = n/3$. Accordingly, while one must not question the small-scale intermittency, one cannot nevertheless exclude the possibility that some of the arguments advanced in K62 – which played a major role in guiding subsequent intermittency models – may be flawed, thus leading to an incorrect prediction of ζ_n. For example, a basic assumption of K62 is that the variance of ϵ increases without bound with the Reynolds number, while $\bar{\epsilon}$ remains bounded. This assumption has yet to be validated. Another critical aspect to be considered when assessing any phenomenology proposed to describe small-scale turbulence is that the physical results derived from that phenomenology must be consistent with the Navier–Stokes equations.

The analysis and results reported in the present work apply to values of ζ_n that pertain to a well-established inertial range, i.e. when the effect of the Reynolds number and any influence from the large-scale motion have disappeared. While these results can also be used for assessing the values of ζ_n determined empirically from experimental and numerical simulation data, it should be stressed that the Reynolds number in experiments and numerical simulations is finite. There is strong evidence that these data are affected by the finite Reynolds number effect (Qian 1997, 1999, 2000; Moisy, Tabeling & Willaime 1999; Antonia & Burattini 2006; McComb 2014; McComb *et al.* 2014; Tang *et al.* 2017; Antonia *et al.* 2019). Qian (1997) was first to draw attention to the finite Reynolds number (FRN) effect. As already mentioned above, Qian (1998) used a non-Gaussian model for the p.d.f. of $|\delta u|$ to show that the anomalous scaling observed in experiments is an FRN effect, and that normal scaling is valid in the inertial range when $Re_\lambda \to \infty$. This effect has since been scrutinized fairly comprehensively (e.g. Antonia & Burattini 2006; Tang *et al.* 2017; Antonia *et al.* 2019). The present results vindicate the concerns expressed previously in the literature with regard to the FRN effect on the magnitude of 'pseudo-scaling exponents'. Interestingly, the present results are consistent with Lundgren's derivation (Lundgren 2002) of $\overline{(\delta u)^2} \sim r^{2/3}$, which applied a method of matched asymptotic expansions to the Kármán–Howarth equation (Kármán & Howarth 1938) when Re_λ is infinitely large. His result supports the argument that the departure from K41 or 'anomalous' behaviour observed at finite Reynolds number disappears at infinitely large Reynolds number.

The present results may be perceived to be at odds with those for Burgers turbulence or for a passive scalar advected by HIT. We note that the present power-law exponent ($\zeta_n = n/3$) differs from that in Burgers turbulence i.e. $\zeta_n = n$ for $n < 1$, and $\zeta_n = 1$ for $n > 1$ (Bouchaud, Mézard & Parisi 1995; Frisch 1995; Friedrich *et al.* 2018). However, caution is required when comparing Burgers turbulence and three-dimensional HIT. Indeed, it is well established that the results from the Burgers equation differ from results usually expected for turbulent flow fields. This may not be too surprising since the Burgers equation not only does not include a pressure term but lacks an important property attributed to turbulence: the solutions do not exhibit chaotic features that are sensitive to initial conditions (Bec & Khanin 2007). Further, it is worth quoting Frisch (1995) in connection to the Burgers equation: 'We shall not here open the Pandora's box of Burgers equation how it does (and often does not) relate to the turbulence problem'.

Regarding the passive scalar structure functions, if one assumes that $\overline{(\delta\phi)^n} \sim (r/L)^{\alpha_n}$ holds in the inertial-convective range (Van Atta 1971; Antonia *et al.* 1984) (as far as we are aware, a rigorous derivation of the power-law form for $\overline{(\delta\phi)^n}$ that does not introduce assumptions in the scalar transport equation does not exist), then applying the same analysis as reported in § 2 would lead to a similar result: $\alpha = n/3$. To obtain it, all that is required is to carry out the change of variables $L = C_\epsilon^{1/4}\, Re_L^{3/4}\eta$. This leads to an identity for $\overline{(\delta\phi)^n}$ similar to the identity (1.5). Also, the transport equation of the turbulent kinetic energy structure function ($\overline{(\delta q^2)} = \overline{(\delta u)^2} + \overline{(\delta v)^2} + \overline{(\delta w)^2}$) is similar to that of any scalar ϕ (e.g. Djenidi, Antonia & Tang 2022). When the Prandtl number or Schmidt number is 1, there is a 'perfect' analogy between $\overline{(\delta q^2)}$ and $\overline{(\delta\phi)^2}$. In that case, one should expect $\overline{(\delta\phi)^2}$ to behave like $\overline{(\delta q^2)}$, which according to our results should be $\overline{(\delta\phi)^2} \sim r^{2/3}$ in the inertial-convective range. The mixed velocity-scalar structure function -$\overline{\delta u(\delta\phi)^2}$ behaves like r, which can be derived easily when the molecular diffusion and the large-scale terms are dropped from the transport equation of $\overline{(\delta\phi)^2}$. As in the case of the velocity field, the intermittency of ϵ_ϕ is not incompatible with this 2/3-law for the passive scalar. The intermittency of ϵ_ϕ is fully compatible with the 4/3-law $\overline{(\delta u)(\delta\phi)^2} = -(4/3)\epsilon_\phi r$ derived from the transport equation for $\overline{(\delta\phi^2)}$ in the inertial-convective range (Yaglom 1949). Further, Danaila, Antonia & Burattini (2012) presented evidence that suggests that, at the same Re_λ, the scalar variance transfer is closer to the asymptotic value of 4/3 than its kinetic energy counterpart. On the basis of this evidence and of the relative behaviours of the u and θ spectra (Danaila & Antonia 2009), one could in fact argue that the scalar field is 'less' anomalous than the velocity field. Such an argument could, however, be fallacious, partly because the statistics of u and θ are not directly comparable, and also because the finite Reynolds number effect needs to be fully accounted for.

Funding. S.L.T. wishes to acknowledge the financial support of NSFC through grant 91952109 and the Research Grants Council of Shenzhen Government through grant RCYX20210706092046085.

Declaration of interests. The authors report no conflict of interest.

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Appendix A. Plausible derivation of a power law for $\overline{(\delta u)^n}$

We begin our derivation by assuming a dual scaling where $\overline{(\delta u)^n}$ scales with both (u_0, L) and (v_K, η) in the inertial range $\eta \ll r \ll L$ as $Re_L \to \infty$, which leads to

$$\overline{(\delta u)^n} = u_0^n f_n(r^+) = v_K^n g_n(r^*), \tag{A1}$$

where $r^+ = r/L$, $r^* = r/\eta$, and f_n and g_n are functions independent of the Reynolds number to be determined. Taking the derivative of (A1) with respect to r yields

$$\frac{\partial \overline{(\delta u)^n}}{\partial r} = v_K^n \frac{\partial g_n(r^*)}{\partial r} = u_0^n \frac{\partial f_n(r^+)}{\partial r}. \tag{A2}$$

Using the variable change $r = r^*\eta = r^+L$, we obtain

$$\frac{v_K^n}{\eta} \frac{\partial g_n(r^*)}{\partial r^*} = \frac{u_0^n}{L} \frac{\partial f_n(r^+)}{\partial r^+}. \tag{A3}$$

We now multiply both sides of (A3) by r, and rearrange terms to obtain

$$r^* \frac{\partial g_n(r^*)}{\partial r^*} = r^+ \frac{u_0^n}{v_K^n} \frac{\partial f_n(r^+)}{\partial r^+}. \tag{A4}$$

Using (A1) in (A4), we arrive at

$$\frac{r^*}{g_n(r^*)} \frac{\partial g_n(r^*)}{\partial r^*} = \frac{r^+}{f_n(r^+)} \frac{\partial f_n(r^+)}{\partial r^+}. \tag{A5}$$

Since in the inertial range $\eta/L \to 0$, we can treat r^* and r^+ as independent variables, which implies that each side of (A5) is a constant; we denote that constant by ζ_n. Integrating each side of (A5) yields the solutions

$$g_n(r^*) = B_n r^{*\zeta_n} \tag{A6}$$

and

$$f_n(r^+) = C_n r^{+\zeta_n}, \tag{A7}$$

where B_n and C_n are constants of integration.

Interestingly, if $u_0^n f_n(r^+)$ and $v_K^n g_n(r^*)$ are interpreted as the outer and inner expressions for $\overline{(\delta u)^n}$ when $r^+ \to 0$ and $r^* \to \infty$, respectively, then (A1) indicates that there exists a range of scales, the inertial range, where both solutions overlap. In that respect, the above derivation of (1.1) is, as already pointed out by Tennekes & Lumley (1972) in the context of the spectrum of u, akin to the matching method used to derive the log law (or law of the wall) in a turbulent boundary layer or turbulent channel and pipe flows (Millikan 1939; Tennekes & Lumley 1972; Barenblatt & Goldenfeld 1995; McKeon & Morrison 2007).

It is also relevant to point out that Gamard & George (2000) applied a matching method in the spectral domain using the same two scaling sets as considered here, and recovered, in the limit of infinitely large Reynolds number, the $-5/3$-law, i.e. $E(k) \sim k^{-5/3}$, where $E(k)$ is the three-dimensional energy spectrum, and k is the three-dimensional wavenumber. Lundgren (2002) applied such a method to the Kármán–Howarth equation (Kármán & Howarth 1938) to derive the $2/3$-law (1.2).

Some caution is warranted. The above derivation of (A6) and (A7) is based solely on the validity of the dual scaling (u_0, L) and (v_K, η) for $\overline{(\delta u)^n}$. We can only note that this dual scaling is consistent with $\zeta_n = n/3$.

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