

## EVENTUALLY REGULAR SEMIGROUPS

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A semigroup is said to be *eventually regular* if each of its elements has some power that is regular. Regular and group-bound semigroups are each eventually regular. *Idempotent-surjective* semigroups are semigroups such that all idempotent congruence classes contain idempotents; eventually regular semigroups are idempotent-surjective. Many results for regular semigroups also hold for eventually regular semigroups or even for idempotent-surjective semigroups and so in particular are also valid for group-bound semigroups. Lallement's lemma is generalized to eventually regular semigroups and the maximum idempotent-separating congruence on such a semigroup is found. Other congruences are considered and the results obtained are applied to yield results on biordered sets.

### 1. Introduction

In 1958 Drazin [3] introduced the concept of a pseudo-invertible element of an associative ring or semigroup. He defines a function  $x \rightarrow x'$ , "somewhat analogous to the generalized inverse function, over the elements of arbitrary finite-dimensional algebras, and even of an extensive class of associative rings" (see [3], p. 506).

Munn [11] studied further these pseudo-inverses in semigroups and found that an element  $x$  of a semigroup  $S$  is pseudo-invertible if and

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only if some power of  $x$  lies in a subgroup of  $S$ . This naturally leads to the study of group-bound semigroups (see Hall and Munn [7]) which are semigroups in which each element has some power in a subgroup or equivalently, semigroups in which all elements are pseudo-invertible.

Not all regular semigroups are group-bound. Define a semigroup to be *eventually regular* if each of its elements has some power that is regular. Thus in particular if a semigroup is regular or group-bound it is an eventually regular semigroup.

A generalization of the concept of eventually regular will also prove convenient. Define a semigroup  $S$  to be *idempotent-surjective* if whenever  $\rho$  is a congruence on  $S$  and  $a\rho$  is an idempotent congruence class in  $S/\rho$ , then  $a\rho$  contains an idempotent of  $S$ . It will be shown that all eventually regular semigroups are idempotent-surjective semigroups.

In Section 3 similarities between regular, eventually regular and idempotent-surjective semigroups are explored and it is shown that many results for regular semigroups extend to eventually regular semigroups or even to idempotent-surjective semigroups. Thus in particular these results are also valid for group-bound semigroups.

Lallement's lemma ([9], Lemma 2.2) is extended to eventually regular semigroups. In fact a stronger result will be shown namely that if  $S$  is eventually regular and  $\phi$  is a morphism from  $S$  onto a semigroup  $T$  such that, for  $c$  and  $d$  in  $S$ ,  $c\phi$  and  $d\phi$  are mutual inverses in  $T$ , then there exist  $a, b$  in  $S$  such that  $a\phi = c\phi$  and  $b\phi = d\phi$  with  $a$  and  $b$  mutual inverses in  $S$ .

We define a relation, denoted throughout this paper by  $\mu$ , on an arbitrary semigroup  $S$  and show that it is an idempotent-separating congruence. For an eventually regular semigroup  $S$ , we show that  $\mu$  is the maximum idempotent-separating congruence on  $S$  and also show (see Example 3 in Section 5) that a semigroup may have a maximum idempotent-separating congruence that is distinct from  $\mu$ .

In Section 4, some of the results obtained in Section 3 are applied to biordered sets and conditions are found on a congruence  $\rho$  on a semigroup  $S$ , such that the biordered set  $E(S/\rho)$  is isomorphic to the biordered set  $E(S)$ . It is shown that  $S/\mu$  is finite if and only if  $E(S)$  is finite and it is further shown that if  $S$  is an idempotent-surjective

semigroup then  $E(S/\mu) \cong E(S)$  .

Section 5 contains some counterexamples to possible extensions of the results to arbitrary semigroups.

## 2. Preliminaries

Whenever possible the notations and conventions of Clifford and Preston [1, 2] or Howie [8] are used. Further for any semigroup  $S$  define a relation  $V = V(S)$  (as in Hall [6]) by

$$V = \{(a, b) \in S \times S \mid aba = a \text{ and } bab = b\} ,$$

and for each  $a$  in  $S$  , denote  $\{b \in S \mid (a, b) \in V\}$  by  $V(a)$  .

A semigroup  $S$  is called *eventually regular* if for each element  $a$  in  $S$  there is a positive integer  $n$  (in general depending on  $a$  ) such that  $a^n$  is regular.

If  $T$  is a semigroup then the set of idempotents of  $T$  will be denoted by  $E(T)$  . The set  $E(T)$  will be regarded where necessary, as equipped with an appropriate structure, such as that of a biordered set, as will be clear from the context.

A congruence on a semigroup is called *idempotent-surjective* if every idempotent congruence class contains an idempotent and *idempotent-creative* otherwise. Thus a congruence  $\rho$  on a semigroup  $S$  is idempotent-surjective if and only if the natural map  $\rho^{\#}$  restricted to  $E(S)$  is a surjection of  $E(S)$  upon  $E(S/\rho)$  . A semigroup is called *idempotent-surjective* if all of its congruences are idempotent-surjective.

The empty set will be denoted by  $\emptyset$  , the identity relation on a set  $X$  will be denoted by  $1_X$  and the conjunction of  $L_a \leq L_b$  and  $R_a \leq R_b$  will be denoted by  $H_a \leq H_b$  .

## 3. Eventually regular semigroups

In this section the relationship between regular, group-bound, eventually regular and idempotent-surjective semigroups is investigated. A regular or group-bound semigroup is clearly eventually regular. By extending a version of Lallement's lemma ([9], Lemma 2.2) to eventually regular semigroups it is shown that all eventually regular semigroups are

idempotent-surjective semigroups.

In preparation for the next section, this section concludes with an investigation of idempotent-separating congruences on eventually regular semigroups including the determination of the maximum such congruence.

**THEOREM 1.** *Let  $S$  be an eventually regular semigroup and let  $\phi$  be a morphism of  $S$  onto a semigroup  $T$  with  $(c, d) \in V(T)$ . Then for any  $x, y$  in  $S$  such that  $x\phi = c$  and  $y\phi = d$  there exist elements  $a, b, r, s$  in  $S$  such that  $a = xrx$ ,  $b = ysy$ ,  $a\phi = c$ ,  $b\phi = d$  and  $(a, b) \in V(S)$ .*

*Proof.* As  $\phi$  is onto  $T$  there exist  $x, y$  in  $S$  such that  $x\phi = c$  and  $y\phi = d$ . As  $S$  is eventually regular there exists an integer  $n \geq 1$  such that  $((xy)^2)^n$  is regular. Let  $z \in V((xy)^{2n})$  and put  $a = (xy)^{2n-1}zxyx$  and  $b = y(xy)^{2n-2}zxy$  (if  $n = 1$  put  $b = yzxy$ ). It is easy to verify that  $(a, b) \in V(S)$ .

As  $z \in V((xy)^{2n})$ ,  $x\phi = c$ ,  $y\phi = d$  and  $\phi$  is a morphism it follows that  $z\phi \in V((cd)^{2n})$ . As  $(c, d) \in V(T)$ ,  $(cd)^{2n} = cd$ . Thus  $z\phi \in V(cd)$  and so will be denoted  $(cd)'$ . As  $\phi$  is a morphism,

$$\begin{aligned} a\phi &= (cd)^{2n-1}(cd)'cdc \\ &= cd(cd)'cdc, & \text{as } (c, d) \in V(T), \\ &= cdc, & \text{as } (cd)' \in V(cd), \\ &= c \end{aligned}$$

and, if  $n > 1$ ,

$$\begin{aligned} b\phi &= d(cd)^{2n-2}(cd)'cd \\ &= d(cd)'cd, & \text{as } (c, d) \in V(T), \\ &= dcd(cd)'cd, & \text{as } d = dcd, \\ &= dcd, & \text{as } (cd)' \in V(cd), \\ &= d. \end{aligned}$$

If  $n = 1$  then  $b\phi = d(cd)'cd = d$  as above. It can now be seen that  $a$  and  $b$  meet all of the requirements of the theorem.

In our terminology Lallement's lemma states that regular semigroups are idempotent-surjective. As a corollary of Theorem 1, the following generalization of Lallement's lemma holds.

**COROLLARY 2.** *Eventually regular semigroups are idempotent-surjective semigroups. Indeed if  $S$  is an eventually regular semigroup,  $\rho$  is a congruence on  $S$ , and  $w\rho$  is an idempotent of  $S/\rho$  then an idempotent  $e$  can be found in  $w\rho$  such that  $H_e \leq H_w$ .*

*Proof.* Let  $w\rho = (w\rho)^2$  in  $S/\rho$ . Then  $(w\rho, w\rho) \in V(S/\rho)$  and so by Theorem 1 (using  $x = y = w$ ) there exist  $a, b, r, s$  in  $S$  such that  $a = wrw$ ,  $b = wsw$ ,  $a\rho = w\rho = b\rho$  and  $(a, b) \in V(S)$ . Putting  $e = ab$  yields  $e = e^2$ ,  $e = wrwsw$  and  $e\rho = (ab)\rho = a\rho b\rho = w\rho w\rho = w\rho$ . Thus  $e$  is an idempotent in  $w\rho$  and  $H_e \leq H_w$  as required.

When applied to regular semigroups Theorem 1 reduces to Lemma 1 of Hall [6]. For group-bound semigroups, in particular for finite semigroups, Theorem 1 appears to be new.

**THEOREM 3.** *Let  $S$  be an eventually regular semigroup and  $\rho$  a congruence on  $S$  contained in  $L$ . Then for each element  $a$  in  $S$ ,  $a$  is an element of a subgroup of  $S$  if and only if  $a\rho$  is an element of a subgroup of  $S/\rho$ .*

*Proof.* Just use Theorem 1 instead of Hall's Lemma 1 in his proof of Theorem 15 in [6].

**THEOREM 4.** *Let  $S$  be an idempotent-surjective semigroup with  $E(S) \neq \square$ . Define a relation  $\sigma$  on  $S$  by*

$$\sigma = \{(ef, fe) \in S \times S \mid e^2 = e, f^2 = f\}.$$

*Let  $\sigma^*$  be the congruence generated by  $\sigma$ . Then  $E(S/\sigma^*)$  is a semi-lattice and so the set of regular elements of  $S/\sigma^*$  is an inverse semigroup. Also  $\sigma^*$  is the finest congruence that has this property and moreover, if  $\rho$  is a congruence on  $S$  with  $\sigma^* \subseteq \rho$ , then the regular elements of  $S/\rho$  form an inverse semigroup.*

*Proof.* As  $S$  is idempotent-surjective each idempotent congruence class contains an idempotent. This together with the definition of  $\sigma^*$  easily leads to a proof of the claims made in the theorem.

A corollary to this theorem is Result 3 of Hall [5], namely that if  $S$  is regular then  $\sigma^*$  is the finest inverse congruence on  $S$ .

For the definition of categorical at zero and its related concepts see

Clifford and Preston ([2], 73–80). The following result of Hall will be used.

**RESULT A** ([4], Theorem 1). *Let  $S = S^0$  be a semigroup which is categorial at zero. Define*

$$\rho_1 = \{(e, f) \in S \times S \mid e = e^2, f = f^2, ef = fe = f \neq 0\} .$$

*Let  $\rho_1^*$  be the congruence generated by  $\rho_1$ . Define by induction  $\rho_{m+1}^*$ , the congruence generated by*

$$\rho_{m+1} = \left\{ (e, f) \in S \times S \mid e\rho_m^*e^2, f\rho_m^*f^2, (ef)\rho_m^*(fe)\rho_m^*f \neq 0 \right\} .$$

*Define  $\rho = \bigcup_{n=1}^{\infty} \rho_n^*$ . Then  $\rho$  is the finest 0-restricted primitive congruence on  $S$ . (Note  $\rho_1^*$  is a 0-restricted congruence.)*

**LEMMA 5.** *Let  $S = S^0$  be a semigroup with the following properties:*

- (a)  *$S$  is categorial at zero;*
- (b)  *$S$  is idempotent-surjective;*
- (c) *each of the (local) subsemigroups  $eSe$ ,  $e = e^2$  is idempotent-surjective.*

*Let  $\rho$  and  $\rho_1$  be defined as in Result A. Then  $\rho = \rho_1^*$ .*

**Proof.** It suffices to show that  $S/\rho_1^*$  is primitive since  $\rho_1^* \subseteq \rho$  ( $S/\rho_1^*$  primitive implies that  $\rho \subseteq \rho_1^*$  by Result A). Suppose  $A = A^2$ ,  $B = B^2$  and  $0 \neq A \leq B$  in  $S/\rho_1^*$ . Then since  $S$  is idempotent-surjective there exists  $b = b^2 \in B$  and because  $BAB = A = A^2$ ,  $A \in E(B(S/\rho_1^*)B)$ . Since  $bSb$  is idempotent-surjective,  $(\rho_1^*)^{\#} \upharpoonright_{E(bSb)}$  maps  $E(bSb)$  onto  $E(B(S/\rho_1^*)B)$ . Therefore, since  $A \in E(B(S/\rho_1^*)B)$ , there exists  $a \in E(bSb)$  such that  $(a)(\rho_1^*)^{\#} = A$ . From  $a \in E(bSb)$  it follows that  $a = a^2$  and  $a \leq b$ . Note that  $a \in A$  and  $a \neq 0$  since  $A \neq 0$  in  $S/\rho_1^*$ . Thus

$0 \neq a \leq b$  , and so from the definition of  $\rho_1$  we have  $a\rho_1 b$  , whence  $A = B$  and so  $S/\rho_1^*$  is primitive.

**THEOREM 6.** *If  $S = S^0$  is an eventually regular semigroup which is categorical at zero and  $\rho$  and  $\rho_1$  are defined as in Result A then  $\rho = \rho_1^*$  .*

*Proof.* If  $S$  is an eventually regular semigroup and  $e = e^2 \in S$  then the subsemigroup  $eSe$  is also eventually regular because if  $a \in eSe$  , with  $a^n$  regular in  $S$  and  $x \in V(a^n)$  , then  $exe \in V(a^n) \cap eSe$  . Therefore both  $S$  and  $eSe$  are idempotent-surjective by Corollary 2 and so the result follows from Lemma 5.

This generalizes the first assertion of Theorem 2 of [4] from regular to eventually regular semigroups.

**LEMMA 7.** *Let  $\sigma$  be an idempotent-separating congruence on an eventually regular semigroup  $S$  and let  $(e, b) \in \sigma$  with  $e = e^2$  . Then  $H_e \leq H_b$  .*

*Proof.* Since  $b\sigma$  is an idempotent congruence class and  $S$  is eventually regular, by Corollary 2 there exists an idempotent  $f$  in  $b\sigma$  such that  $H_f \leq H_b$  . However since  $\sigma$  is idempotent-separating and  $(e, f) \in \sigma$  it follows that  $e = f$  , whence  $H_e \leq H_b$  .

**DEFINITION.** For any semigroup  $S$  define a relation  $\mu$  by 
$$\mu = \{(a, b) \in S \times S \mid \text{if } x \in S \text{ is regular then each of } xRax, xRxb \text{ implies } xaHxb, \text{ and each of } xLax, xLbx \text{ implies } aHbx\} .$$

**THEOREM 8.** *For any semigroup  $S$  the relation  $\mu$  is an idempotent-separating congruence.*

*Proof.* It is clear that the relation  $\mu$  is both reflexive and symmetric. Suppose  $(a, b) \in \mu$  and  $(b, c) \in \mu$  . Let  $x$  be regular and let  $xRax$  . From the definition of  $\mu$  , we then have  $xaHxb$  . Since  $H \subseteq R$  it follows that  $xRxb$  , whence again from the definition of  $\mu$  ,  $xbHxc$  . Hence  $xaHxc$  . Similarly,  $xLax$  with  $x$  regular implies  $aHbx$  . Thus by the symmetry of the conditions on  $a$  and  $c$  ,  $(a, c) \in \mu$

and so  $\mu$  is transitive.

Take  $(a, b) \in \mu$  and  $y \in S$ . Let  $xRx(ay)$  with  $x$  regular. In any semigroup  $xRxa y$  implies  $xRxa$  and so it follows from the definition of  $\mu$  that  $xaHxb$ . As  $xa yRxa$  there exists  $u$  in  $S^1$  such that  $xa y u = xa$ . As  $xaHxb$  there exists  $v \in S^1$  such that  $xb = vxa$ . Thus  $(xby)u = (vxa)yu = v(xabu) = vxa = xb$  and so  $xbRxb y$ , whence  $xbyRxbRxaRcRxa y$ . Since  $xaHxb$ , it follows that  $xaLxb$ ; hence  $xa yLxby$ , since  $L$  is a right congruence. Thus  $x(ay)Hx(by)$ .

Now let  $xL(ay)x$  with  $x$  regular. Then  $xlyx$  and so  $yx$  is regular. Thus  $yxLayx$  with  $yx$  regular from which it follows that  $a(yx)Hb(yx)$ , since  $(a, b) \in \mu$ . Hence  $(ay)Hx(by)x$ . By symmetry of the conditions on  $a$  and  $b$  it follows that  $(ay, by) \in \mu$ . Dually,  $(ya, yb) \in \mu$  and so  $\mu$  is compatible.

To see that the congruence  $\mu$  is idempotent-separating consider  $(e, f) \in \mu$  with  $e^2 = e$ ,  $f^2 = f$ . Then by putting  $x = e$  in the definition for  $\mu$ ,  $eRee$  implies  $eHef$ , and similarly  $fLff$  implies  $eHf$ . Thus  $eHf$  and so  $e = f$ .

That  $\mu$  is a congruence may also be seen by checking that  $\mu = \ker \phi$  where  $\phi$  is the following representation of  $S$  (verbal communication from T.E. Hall). Let  $X$  be the set of regular  $L$ -classes of  $S$  and  $Y$  be the set of regular  $R$ -classes of  $S$ . Then define  $\phi : S \rightarrow PT_X \times PT_Y^*$  by  $s\phi = (\rho_s, \lambda_s)$ , where  $\rho_s : L_x \rightarrow L_{xs}$  if  $x$  is regular and  $xRcs$ , and is undefined otherwise, and  $\lambda_s : R_x \rightarrow R_{sx}$  if  $x$  is regular and  $xLsx$ , and is undefined otherwise. (Note  $PT_Y^*$  denotes the dual of the semigroup  $PT_Y$ .)

**THEOREM 9.** *The following are equivalent for a congruence  $\sigma$  on a semigroup  $S$  :*

- (i)  $\sigma \subseteq \mu$  ;
- (ii) for all  $e \in E(S)$ , for all  $b$  in  $S$ ,  $(e, b) \in \sigma$  implies  $H_e \subseteq H_b$  ;
- (iii) for all regular elements  $a$  in  $S$ , for all  $b$  in  $S$ ,



$$(a, b) \in \sigma \text{ implies } H_a \leq H_b ;$$

and if  $S$  is eventually regular then each of the above is equivalent to

(iv)  $\sigma$  is idempotent-separating.

Proof. That (i) implies (ii) follows easily from the definition of  $\mu$ .

We now show that (ii) implies (iii). Assume (ii) and let  $(a, b) \in \sigma$  with  $a$  regular. Take  $a' \in V(a)$ . Then since  $(aa', ba') \in \sigma$  and  $aa' \in E(S)$  by (ii),  $H_{aa'} \leq H_{ba'}$ . Therefore  $R_a = R_{aa'} \leq R_{ba'} \leq R_b$  and a dual argument to the above yields  $L_a \leq L_b$ , whence  $H_a \leq H_b$ .

We now show that (iii) implies (i). Assume (iii) and take  $(a, b) \in \sigma$ . Let  $xRxa$  with  $x$  regular. Then as  $(xa, xb) \in \sigma$  with  $xa$  regular by (iii),  $H_{xa} \leq H_{xb}$ . Therefore  $R_x = R_{xa} \leq R_{xb} \leq R_x$ , whence  $xRxb$  and  $xb$  is regular. Therefore, by (iii) again,  $H_{xb} \leq H_{xa}$  and so  $H_{xa} = H_{xb}$ . A dual argument shows  $xLax$  implies  $axHbx$ . These implications together with symmetry show that  $(a, b) \in \mu$ , whence  $\sigma \subseteq \mu$ .

The equivalence of (i), (ii) and (iii) is now clear and that (i) implies (iv) is trivial. If  $S$  is eventually regular then (iv) implies (ii) by Lemma 7. Thus if  $S$  is eventually regular then all four conditions are equivalent. (The equivalence of (i) and (ii) has also been noted by D. Easdown in an unpublished note.)

**COROLLARY 10.** *Let  $R$  denote the set of regular elements of an arbitrary semigroup  $S$  and denote the maximum congruence contained in  $H$  on  $S$  by  $H^b$ . Then*

- (a)  $H^b \subseteq \mu$ ,
- (b) if  $a, b \in R$  and  $(a, b) \in \mu$  then  $aHb$ ,
- (c)  $H^b \cap (R \times R) \subseteq \mu \cap (R \times R) \subseteq H \cap (R \times R)$ ,
- (d) if  $S$  is regular then  $\mu = H^b$ .

Proof. (a) is clear; (b) follows from the equivalence of (i) and (iii) in Theorem 9; (c) follows from (a) and (b); and (d) follows easily

from (c).

**THEOREM 11.** *If  $S$  is an eventually regular semigroup then  $\mu$  is the maximum idempotent-separating congruence on  $S$ .*

**Proof.** The proof follows immediately from Theorem 9.

Note that, as Example 3 in Section 5 shows, there exist semigroups which have a maximum idempotent-separating congruence not equal to  $\mu$ .

#### 4. Biordered sets

Let  $\rho$  be a congruence on a semigroup  $S$ . The results of the previous section can in some cases be used to determine when the idempotents of  $S$  and of  $S/\rho$  form isomorphic biordered sets. We also show that, if  $S$  is an eventually regular semigroup, then its biordered set of idempotents is an  $M$ -biordered set (for definition, see below).

First, some background material on biordered sets will be useful. For a definition of a biordered set, its related axioms and concepts see Nambooripad [12] or Meakin [10]. The notation of [12] and [10] will be used and is as follows.

Let  $S$  be a semigroup and  $E \neq \square$  be its set of idempotents. Define  $\omega^L = \{(e, f) \in E \times E \mid ef = e\}$ ,  $\omega^R = \{(e, f) \in E \times E \mid fe = e\}$ ,  $\omega = \omega^L \cap \omega^R$ ,  $\kappa = \omega^L \cup \omega^R$ ,  $L = \omega^L \cap (\omega^L)^{-1}$ ,  $R = \omega^R \cap (\omega^R)^{-1}$  and  $D_E = \kappa \cup \kappa^{-1}$ . Also for  $e \in E$  and  $\gamma = \omega^L$  or  $\omega^R$ , put  $\gamma(e) = \{f \in E \mid f\gamma e\}$ . Then the partial binary algebra  $E$  with domain  $D_E$  (that is, the product  $ef$  is defined in  $E$  if and only if  $(e, f) \in D_E$ ) is a biordered set ([12], Theorem 1.1). A mapping  $\theta : E \rightarrow E'$  is a morphism (called bimorphism by Nambooripad and Meakin) of the biordered set  $E$  into the biordered set  $E'$  if  $(e, f) \in D_E$  implies  $(e\theta, f\theta) \in D_{E'}$ , and  $(ef)\theta = (e\theta)(f\theta)$ . The biordered sets  $E$  and  $E'$  are isomorphic (denoted by  $E \cong E'$ ) if there exists a bijection  $\theta : E \rightarrow E'$  such that both  $\theta$  and  $\theta^{-1}$  are morphisms.

**DEFINITIONS.** Let  $E$  be a biordered set. For  $e, f \in E$ , define  $M(e, f) = \omega^L(e) \cap \omega^R(f)$ . Define  $E$  to be an  $M$ -biordered set if

$M(e, f) \neq \square$  for all  $e, f \in E$  and a semigroup  $S$  to be an  $M$ -semigroup if  $E(S)$  is an  $M$ -biordered set. Define along the lines of Meakin [10], an arbitrary biordered set  $E$  to be rectangular if for all  $e, f \in E$  there exists an element  $g$  in  $E$  such that  $eRg$  and  $gLf$ . Then Meakin's Proposition 1.3 of [10] can be modified to give

**RESULT B.** *For an arbitrary biordered set  $E$  the following are equivalent:*

- (a)  $E$  is a rectangular biordered set;
- (b)  $E$  is an  $M$ -biordered set and  $\omega = 1_E$ ;
- (c)  $E$  is the biordered set of some rectangular band.

**LEMMA 12.** *Let  $\sigma$  be an idempotent-surjective congruence contained in  $\mu$  on a semigroup  $S$ . Suppose  $A, B \in S/\sigma$  with  $A = A^2$  and  $B = B^2$ . Then*

- (i) *if  $AB = B$  in  $S/\sigma$ , there exist  $a = a^2 \in A$  and  $b = b^2 \in B$  and, for any such idempotents  $a$  and  $b$ , the equation  $ab = b$  holds in  $S$ ; and*
- (ii) *if  $AB = A$  in  $S/\sigma$ , there exist  $a = a^2 \in A$  and  $b = b^2 \in B$  and, for any such idempotents  $a$  and  $b$ , the equation  $ab = a$  holds in  $S$ .*

**Proof.** Since  $\sigma$  is an idempotent-surjective congruence there exist  $a = a^2 \in A$  and  $b = b^2 \in B$ . Let  $AB = B$ . Then  $(ab)\sigma = b\sigma$ . Thus  $(ab, b) \in \sigma$ , whence since  $\sigma \subseteq \mu$ ,  $(ab, b) \in \mu$ . By taking  $x = b$  in the definition of  $\mu$ , from  $bLbb$ , we have  $bbHabb$ , that is,  $bHab$ , from which it follows easily that  $ab = b$ . Similarly, if  $AB = A$ , then for  $a$  and  $b$  as before,  $ab = a$ .

**LEMMA 13.** *Let  $\sigma$  be an idempotent-surjective congruence contained in  $\mu$  on a semigroup  $S$ . Then  $E(S) \cong E(S/\sigma)$ ; furthermore  $\sigma^{\sharp}|_{E(S)}$  is such an isomorphism.*

**Proof.** If  $E(S) = \square$  there is nothing to prove. Otherwise, since  $\sigma$  is an idempotent-surjective congruence which is idempotent-separating

(since  $\sigma \subseteq \mu$ ),  $\sigma \upharpoonright_{E(S)} = \alpha$ , say, is a bijection from  $E(S)$  upon  $E(S/\sigma)$ . By Lemma 12,  $\alpha^{-1}$  preserves the biordered set relations of  $E(S/\sigma)$ ; that, conversely  $\alpha$  preserves the biordered set relations of  $E(S)$  is clear. Therefore both  $\alpha$  and  $\alpha^{-1}$  are morphisms and so  $E(S) \cong E(S/\sigma)$ .

**COROLLARY 14.** *Let  $\sigma$  be an idempotent-separating congruence on an eventually regular semigroup  $S$ . Then  $E(S/\sigma) \cong E(S)$  and so in particular  $E(S/\mu) \cong E(S)$ .*

*Proof.* By Theorem 11,  $\sigma \subseteq \mu$ , and by Corollary 2,  $\sigma$  is idempotent-surjective. Thus, by Lemma 13,  $E(S/\sigma) \cong E(S)$ .

**THEOREM 15.** *Let  $S$  be any semigroup. Then  $S/\mu$  is finite if and only if  $E(S)$  is finite. Furthermore, if  $\mu$  is an idempotent-surjective congruence, then  $E(S/\mu) \cong E(S)$ .*

*Proof.* If  $E(S)$  is finite then there are only finitely many distinct regular  $L$  or  $R$ -classes. Thus the sets  $X$  and  $Y$  mentioned in Theorem 8 are finite and the morphism  $\phi$  of Theorem 8 has a finite image. Since  $\mu = \ker \phi$ , it follows that  $S/\mu$  is finite. The converse follows from Theorem 8 and the latter assertion is immediate from Lemma 13.

**COROLLARY 16.** *Let  $S$  be an eventually regular semigroup. Then  $E(S) \cong E(T)$ , for some finite semigroup  $T$ , if and only if  $E(S)$  is finite.*

*Proof.* If  $E(S)$  is finite the result follows from Corollary 14 and Theorem 15 by taking  $T = S/\mu$ . The converse is immediate.

Another observation obtainable from Theorem 15 is that the class of biordered sets arising from finite semigroups coincides with the class of biordered sets arising from semigroups with only finitely many idempotents and for which  $\mu$  is an idempotent-surjective congruence. Example 4 of Section 5 is of a finite biordered set that comes from no finite semigroup and Example 3 is of a semigroup for which  $\mu$  is not an idempotent-surjective congruence and  $E(S/\mu) \not\cong E(S)$ .

The sufficient conditions mentioned in the previous few results are in some cases necessary as shown by the next theorem.

**THEOREM 17.** *Let  $S$  be an eventually regular semigroup with  $E(S)$  finite. Then for any congruence  $\rho$  on  $S$ ,  $E(S/\rho) \cong E(S)$  if and only if  $\rho \subseteq \mu$ .*

*Proof.* If  $\rho$  is not contained in  $\mu$  then by Theorem 11,  $\rho$  is not idempotent-separating. This in conjunction with Corollary 2 yields  $|E(S/\rho)| < |E(S)|$ . Therefore  $E(S/\rho) \not\cong E(S)$ . The converse follows from Corollary 14.

The requirement that  $E(S)$  be finite is clearly needed since, for example, any infinite chain  $C$  has  $\mu = 1_C$  and has non-trivial congruences  $\rho$  such that  $C \cong C/\rho$ . Theorem 17 also has the following immediate corollary.

**COROLLARY 18.** *If  $\rho$  is a congruence on a finite semigroup  $S$  then  $E(S/\rho) \cong E(S)$  if and only if  $\rho \subseteq \mu$ .*

**PROPOSITION 19.** *Any eventually regular semigroup is an  $M$ -semigroup.*

*Proof.* Let  $S$  be an eventually regular semigroup (so  $E(S) \neq \square$ ). Let  $e, f$  be elements of  $E(S)$ ; then there exists  $n \geq 1$  such that  $((fe)^2)^n$  is regular. Take  $x \in V((fe)^{2n})$  and put  $k = (fe)^n x (fe)^n$ . Then  $k^2 = k$ ,  $ke = k$  and  $fk = k$  so  $k \in M(e, f)$ .

Example 4, below, shows that there is an  $M$ -semigroup whose biordered set is not the biordered set of any eventually regular semigroup. Thus the class of  $M$ -biordered sets is more extensive than the class of biordered sets arising from eventually regular semigroups. From Proposition 19, a necessary condition for a finite biordered set  $E$  to come from a finite semigroup is that  $E$  be an  $M$ -biordered set; but again Example 4 shows that this is not sufficient.

### 5. Examples

Not all eventually regular semigroups are regular or group-bound as shown by Example 1.

**EXAMPLE 1.** Let  $T$  and  $R$  be eventually regular semigroups and  $S = T \times R$  be the direct product of  $T$  and  $R$ . It is easy to verify that if  $T$  is regular or group-bound then  $S$  is eventually regular. Thus if  $T$  is a group-bound semigroup that is not regular (for example let  $T$  be

the two element null semigroup) and if  $R$  is a regular semigroup that is not group-bound (for example, let  $R$  be the bicyclic semigroup) then  $S$  is an eventually regular semigroup that is neither regular nor group-bound.

Many of the results obtained in the two previous sections are not true for arbitrary semigroups as shown by the following three examples.

**EXAMPLE 2.** As a nontrivial example of the ideas developed in the previous sections let  $S$  be the semigroup given by Table 1.  $S$  is not regular but  $S$  is eventually regular since  $S$  is finite ( $e, f, 0, a^2$  are regular). The  $\mu$ -classes are  $\{e\}, \{f\}$ , and  $\{0, a\}$ . Note that each idempotent  $\mu$ -class contains an idempotent, that is,  $\mu$  is an idempotent-surjective congruence (cf. Corollary 2).

$$E(S) = \{e, f, 0\}, \quad E(S/\mu) = \{e\mu, f\mu, 0\mu\} \quad \text{and} \quad E(S) \cong E(S/\mu)$$

(cf. Theorem 15).

Note that  $S$  eventually regular,  $(x, y) \in \mu$  with  $x$  regular does not imply that  $y$  is also regular as can be seen from  $(0, a) \in \mu$  above.

TABLE 1

	$e$	$f$	$a$	$0$
$e$	$e$	$a$	$a$	$0$
$f$	$0$	$f$	$0$	$0$
$a$	$0$	$a$	$0$	$0$
$0$	$0$	$0$	$0$	$0$

**EXAMPLE 3.** Let  $F$  be a free semigroup on a set  $X$ . Let  $S$  be the semigroup constructed from  $F$  by adjoining two idempotent elements  $e, 1$  such that  $ew = we = lw = wl = w$  for all  $w$  in  $F$  and  $e1 = 1e = e$ . Then  $S$  is not eventually regular. For this semigroup  $S$ ,  $\mu$  has as its congruence classes  $F, \{e\}$  and  $\{1\}$ . Note that  $F$  is an idempotent element of  $S/\mu$  but contains no idempotent of  $S$  (cf. Corollary 2).

$S$  has another idempotent-separating congruence  $\rho$  with congruence classes  $F \cup e$  and  $\{1\}$ . Thus  $\mu$  is strictly contained in  $\rho$  which is in fact the maximum idempotent-separating congruence on  $S$  (cf. Theorem 11).

Moreover  $E(S) = \{e, 1\}$  with  $e1$ .  $E(S/\mu) = \{F, \{e\}, \{1\}\}$  with

$\mathcal{R}\omega\{e\}\omega\{1\}$  so certainly  $E(S) \not\cong E(S/\mu)$ . Thus the second assertion of Corollary 14 is not true for arbitrary semigroups. The congruence  $\rho$  is idempotent-surjective and shows that both the latter assertion in Corollary 2 and Lemma 7 are false for idempotent-surjective congruence classes of arbitrary semigroups.

Finally the congruence  $\sigma$  on  $S$  whose congruence classes are  $\{1, e\}\sigma$  and  $F$  is not idempotent-separating but  $E(S/\sigma) = \{F, \{e, 1\}\}$  with  $\mathcal{R}\omega\{1\}$  and so  $E(S/\sigma) \cong E(S)$ . In this case there was both a collapse and a creation of idempotents which in effect mutually cancelled each other.

**EXAMPLE 4** (Hall). Let  $R$  be the free semigroup on the idempotent generators  $e, f$ . Extend  $R$  to a semigroup  $P$  by adjoining idempotents  $g, h$  satisfying  $gr = rg = g$ ,  $hr = rh = h$  for all  $r \in R$  and with  $gRh$ . Let  $E$  be the biordered set of  $P$ . Then  $E = \{e, f, g, h\}$ , for which the relations  $gwe, gwf, hwe, hwf$  and  $gRh$  hold, is an  $M$ -biordered set and so  $P$  is an  $M$ -semigroup. However  $E$  is not the biordered set of an eventually regular semigroup.

**Proof.** If  $E = E(S)$  with  $S$  eventually regular then, as in the proof of Proposition 19, there exists  $k = (fe)^n x(fe)^n \in M(e, f)$  with  $x \in V((fe)^{2n})$ .  $M(e, f) = \{g, h\}$  so without loss of generality  $k = g$  (by symmetry). As  $g = (fe)^n x(fe)^n$  and  $gwe$  and  $gwf$  it follows that  $g = (fe)^n (fe)^n x(fe)^n (fe)^n = (fe)^{2n}$ . From  $g = (fe)^{2n}$  and  $hwe$  and  $hwf$  it follows that  $h = h(fe)^{2n} = hg$ , whereas, from  $hRg$  we have  $hg = g$ , a contradiction.

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