ENUMERATION OF LOCALLY RESTRICTED DIGRAPHS

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Introduction. Among the unsolved problems in graphical enumeration listed in **(4)** is included the determination of the number of graphs and digraphs with a given partition. Parthasarathy **(9)** has developed a formulation for counting graphs with a given partition by making a suitable modification of the method given in **(2)** for the enumeration of graphs. We present here an analogous modification that leads to a formula for the number of digraphs with a given partition. Not surprisingly, the main combinatorial device for this purpose is provided by the classical theorem due to Pólya.

Now we give the form of Pólya's enumeration theorem **(10)** which will be used in this paper.

Let A be a permutation group of order $|A|$ acting on the set X. Let W be a function from X into any ring with zero characteristic such that for all α in A and all x in X,

$$W(x) = W(\alpha x).$$

As usual W is called a *weight function*. Let the orbits (transitivity systems) of A be denoted by $X_1, X_2, \ldots, X_n$. Because of this condition satisfied by W, we can define the weight of each orbit X_i, denoted $W(X_i)$, by $W(X_i) = W(x)$ for any x in X_i.

Then Pólya's theorem can be stated, in a slight modification of the formulation given by de Bruijn **(1)**, as follows.

THEOREM (Pólya).

$$\sum_{i=1}^{n} W(X_i) = \frac{1}{|A|} \sum_{\alpha \in A} \sum_{x=\alpha x} W(x). \tag{1}$$

1. Enumeration of locally restricted digraphs. A *directed graph* (or *digraph*) consists of a finite set V of *points* together with a prescribed collection of ordered pairs of distinct points of V. Each such ordered pair (u, v) is called a *directed line* and is usually denoted by uv. The point u is *adjacent to* v and v is *adjacent from* u. The outdegree of a point u is the number of points adjacent from u; the *indegree* is the number of points adjacent to u. The *partition* of a digraph is a sequence of ordered pairs consisting of the outdegree and indegree of each point. We arbitrarily order the ordered pairs in the partition of a digraph somewhat lexicographically, i.e. first we insist that the outdegrees

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be in descending order, then for equal outdegrees we stipulate that the indegrees be in descending order. A *locally restricted* digraph is a digraph with a given partition. For further details on digraphs see **(6)**.

Now let $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_p$ be two sets of variables, the x_i for the outdegrees and the y_j for the indegrees. The generating function that enumerates locally restricted digraphs with p points is a polynomial

$$N(x, y) = N(x_1, x_2, \ldots, x_p; y_1, y_2, \ldots, y_p)$$

in these variables such that each term

$$x_1^{s_1} x_2^{s_2} \ldots x_p^{s_p} . y_1^{t_1} y_2^{t_2} \ldots y_p^{t_p} \text{ satisfies}$$

(2) $$s_1 \geqslant s_2 \geqslant \ldots \geqslant s_p$$

and

(3) $$\text{if } s_i = s_{i+1}, \qquad \text{then } t_i \geqslant t_{i+1}.$$

The coefficient of such a term is the number of digraphs with partition $(s_1, t_1), (s_2, t_2), \ldots, (s_p, t_p)$.

For example, consider the digraph on five points in Figure 1. The term in $N(x, y)$ that corresponds to this digraph is

$$x_1^3 x_2^2 x_3^2 x_4 x_5 y_1 y_2^2 y_3^2 y_4^3 y_5.$$

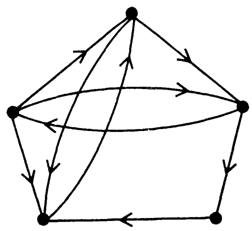

FIGURE 1

In expressing a formula for this generating function $N(x, y)$, we find it convenient to use the natural group-theoretic setting provided by the power group **(7)** as applied to this situation. The notation and definitions are now given.

Let $X = \{1, 2, \ldots, p\}$. The set of ordered pairs (i, j) of elements of X with $i \neq j$ is denoted by $X^{[2]}$. Let the symmetric group of degree p, denoted by S_p, act on X. The *reduced ordered pair group* $S_p^{[2]}$, defined in **(2)**, acts on $X^{[2]}$, and each of its permutations is induced by a permutation in S_p. That is, for each permutation α in S_p, let α' be the induced permutation in $S_p^{[2]}$. Then for all (i, j), $\alpha'(i, j) = (\alpha i, \alpha j)$.

Now let E_2 be the identity group acting on the set $Y = \{0, 1\}$. Consider the power group $E_2^{S_p^{[2]}}$ acting on $Y^{X^{[2]}}$, the functions from $X^{[2]}$ into Y. Each function f in $Y^{X^{[2]}}$ represents a digraph whose points are $1, 2, \ldots, p$ and in which i is adjacent to j whenever $f(i, j) = 1$. Two functions f_1 and f_2 in $Y^{X^{[2]}}$ represent isomorphic digraphs whenever there is a permutation γ in $E_2^{S_p^{[2]}}$ such that $\gamma f_1 = f_2$.

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Again consider the variables $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_p$ and let R be the ring of polynomials in these variables with integral coefficients. We define a function W from $Y^{X^{[2]}}$ into R as follows. For each f in $Y^{X^{[2]}}$,

$$W(f) = \prod_{(i,j)\in X^{[2]}} (x_i y_j)^{f(i,j)}. \tag{4}$$

Then in the digraph represented by f, the outdegree and indegree of the point i are given by the exponents of x_i and y_i respectively in $W(f)$.

It is convenient to define a function denoted by θ from R into itself in the following way. Suppose

$$x_1^{s_1}x_2^{s_2}\ldots x_p^{s_p}y_1^{t_1}y_2^{t_2}\ldots y_p^{t_p}$$

is any monomial in R. Choose a permutation σ in S_p so that

$$s_{\sigma(1)} \geqslant s_{\sigma(2)} \geqslant \ldots \geqslant s_{\sigma(p)} \tag{5}$$

and

$$\text{if } s_{\sigma(i)} = s_{\sigma(i+1)}, \qquad \text{then } t_{\sigma(i)} \geqslant t_{\sigma(i+1)}. \tag{6}$$

Then θ is defined on this monomial in R by

$$\theta\,(x_1^{s_1}x_2^{s_2}\ldots x_p^{s_p}y_1^{t_1}y_2^{t_2}\ldots y_p^{t_p}) = (x_1^{s_{\sigma(1)}}x_2^{s_{\sigma(2)}}\ldots x_p^{s_{\sigma(p)}}\cdot y_1^{t_{\sigma(1)}}y_2^{t_{\sigma(2)}}\ldots y_p^{t_{\sigma(p)}}).$$

Now θ is extended to all of R in the obvious way and it is clear that θ is a linear operator on R.

Let f be any function in $Y^{X^{[2]}}$. Then the monomial $\theta(W(f))$ is called the *weight of* f. Thus for convenience we have defined the weight of f as the degree sequence of the graph determined by f. From the definition of θ, it is clear that $\theta(W(f))$ is the contribution to the generating function $N(x, y)$ made by the digraph of f. It is also clear that any two functions in $Y^{X^{[2]}}$ that are equivalent under the power group $E_2^{S_p^{[2]}}$ have the same weight.

With this notation and an application of Pólya's theorem, the following result is obtained.

THEOREM 1. *The generating function which enumerates locally restricted digraphs is*

$$N(x, y) = \frac{1}{p!}\sum_{\gamma\in E_2^{S_p^{[2]}}} \theta\Big(\sum_{f=\gamma f} W(f)\Big). \tag{7}$$

Now we give a formula for $\sum_{f=\gamma f} W(f)$. Each permutation γ in $E_2^{S_p^{[2]}}$ can be written as $\gamma = (\alpha';\,(0)(1))$ where α' in $S_p^{[2]}$ is induced by α in S_p and $(0)(1)$ is the identity permutation on Y. Let Z_r and Z_s be any two distinct cycles of lengths r and s respectively in the disjoint cycle decomposition of α. In obtaining the formula for the cycle index of $S_p^{[2]}$, it was shown in **(2)** that each cycle

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Z_r contributes $r-1$ cycles, each of length r, to the cycle structure of α'. Further, it was shown that each pair of distinct cycles Z_r, Z_s contributes $2d(r,s)$ cycles each of length $rs/d(r,s)$, where $d(r,s)$ is the greatest common divisor of r and s.

Since $\gamma f = f$, f must be constant on each cycle in the disjoint decomposition of α'. By observing the number of times the symbols permuted by Z_r and Z_s appear in these cycles of α', we obtain the following formula:

$$(8)\quad \sum_{f=\gamma f} W(f) = \prod_{Z_r,Z_s}\left(\left(1+\prod_{i\in Z_r} x_i^{s/d(r,s)}\prod_{j\in Z_s} y_j^{r/d(r,s)}\right)^{d(r,s)} \times\left(1+\prod_{i\in Z_r} y_i^{s/d(r,s)}\prod_{j\in Z_s} x_j^{r/d(r,s)}\right)^{d(r,s)}\right)\prod_{Z_r}\left(1+\prod_{i\in Z_r} x_i y_i\right)^{r-1},$$

where the first product is over all distinct pairs of cycles Z_r, Z_s in α and the second is over all cycles of α.

Obviously Formula (7) in Theorem 1 can be modified further using the fact that $\sum_{f=\gamma f} W(f)$ depends only on the cycle structure of α. This is, of course, the procedure that is followed in actual practice.

The computation is rather cumbersome even for $p=3$, but we shall sketch some of the details briefly. We take S_3 acting on $\{1,2,3\}$. There are three possibilities for the cycle structure of a permutation in S_3. Using formula (8) for each of these cases, we obtain the following results. For $\alpha = (1)(2)(3)$,

$$\sum_{f=\gamma f} W(f) = (1+x_1y_2)(1+x_2y_1)(1+x_1y_3)(1+x_3y_1)(1+x_3y_2) \times (1+x_2y_3).$$

For $\alpha = (12)(3)$,

$$\sum_{f=\gamma f} W(f) = (1+x_1x_2y_3^2)(1+x_3^2y_1y_2)(1+x_1y_1x_2y_2)^2.$$

For $\alpha = (123)$,

$$\sum_{f=\gamma f} W(f) = (1+x_1y_1x_2y_2x_3y_3)^2.$$

Now, by applying Theorem 1, we have

$$\begin{aligned} N(x,y) &= (1/3!)\{\theta[(1+x_1y_2)(1+x_2y_1)(1+x_1y_3)(1+x_3y_1) \\ &\qquad \times (1+x_3y_2)(1+x_2y_3)] + 3\theta[(1+x_1x_2y_3^2)(1+x_3^2y_1y_2) \\ &\qquad \times (1+x_1y_1x_2y_2)^2] + 2\theta[1+x_1y_1x_2y_2x_3y_3]^2\} \\ &= 1 + x_1y_2 + [x_1x_2y_1y_2 + x_1x_2y_1y_3 + x_1^2y_2y_3 + x_1x_2y_3] \\ &\quad + [x_1^2x_2y_1y_2y_3 + x_1x_2x_3y_1y_2y_3 + x_1^2x_2y_2y_3^2 + x_1x_2y_1^2y_2y_3] \\ &\quad + [x_1^2x_2x_3y_1^2y_2y_3 + x_1^2x_2x_3y_1y_2^2y_3 + x_1^2x_2^2y_1y_2y_3^2 + x_1^2x_2x_3y_2^2y_3^2] \\ &\quad + x_1^2x_2^2x_3y_1^2y_2y_3^2 + x_1^2x_2^2x_3^2y_1^2y_2^2y_3^2. \end{aligned}$$

We conclude this section by mentioning the simple modifications of the preceding formulation necessary for the enumeration of digraphs with loops. The *ordered pair group* $S_p{}^2$, as defined in **(2)**, acts on the set X^2, the cartesian product of X with itself. Its permutations are defined in the same way as those

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of $S_p^{[2]}$. One need only replace $S_p^{[2]}$ by S_p^2 throughout and note that the contribution of Z_r to $\sum_{f=\gamma f} W(f)$ in formula (8) is $(1 + \prod_{i \in Z_r} x_i y_i)^r$.

The number of digraphs with a given partition was discussed by Katz and Powell **(8)**. They reduced this question to a formulation by Sukhatme **(11)** which gives recurrence relations for certain number-theoretic functions. The extent of Sukhatme's tables enabled Katz and Powell to compute only those locally restricted digraphs with at most 13 lines. In principle, our formulation in Theorem 1 has no such restriction, but is no less unwieldy in actual use.

2. Enumeration of locally restricted graphs. We refer to Harary **(5)** for conventional graphical terminology. The *partition* of a graph is the sequence of degrees of its points, usually written in descending order. A *locally restricted* graph is a graph with a given partition.

The generating function that enumerates locally restricted graphs with p points is a polynomial $N(x_1, x_2, \ldots, x_p)$ such that each term, $x_1^{s_1} x_2^{s_2} \ldots x_p^{s_p}$, satisfies

$$s_1 \geqslant s_2 \geqslant \ldots \geqslant s_p. \tag{9}$$

The coefficient of such a term is the number of graphs with partition $s_1, s_2, \ldots, s_p$. For example, the term in $N(x_1, \ldots, x_5)$ which corresponds to the graph in Figure 2 is $x_1^4 x_2^3 x_3^3 x_4^2 x_5^2$.

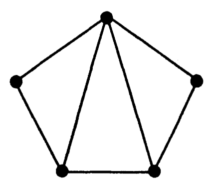

FIGURE 2

Now we review briefly the method of Parthasarathy **(9)** for the enumeration of locally restricted graphs. But again in obtaining a formula for the generating function, we use the natural group-theoretic setting provided by the power group **(7)** as applied to this situation.

Let $X = \{1, 2, \ldots, p\}$. We denote the set of unordered pairs $\{i, j\}$ of elements of X by $X^{(2)}$. The *pair group* $S_p^{(2)}$, as defined in **(2)**, acts on $X^{(2)}$, and each of its permutations is induced by a permutation in S_p. As before, E_2 is the identity group acting on $Y = \{0, 1\}$. The power group $E_2^{S_p^{(2)}}$ acts on $Y^{X^{(2)}}$. Each function f in $Y^{X^{(2)}}$ represents a graph whose points are $1, 2, \ldots, p$ and in which i and j are adjacent whenever $f(\{i, j\}) = 1$.

Corresponding to (4), the function W from $Y^{X^{(2)}}$ into the ring R is defined by

$$W(f) = \prod_{\{i,j\} \in X^{(2)}} (x_i x_j)^{f(\{i,j\})}. \tag{10}$$

Then in the graph of f, the degree of the point i is given by the exponent of x_i in $W(f)$. As before, $\theta(W(f))$ is called the weight of f.

An application of Pólya's theorem gives the following result, which was obtained by Parthasarathy **(9)** in somewhat different form.

THEOREM 2. *The generating function that enumerates locally restricted graphs is*

$$N(x_1, x_2, \ldots, x_p) = \frac{1}{p!} \sum_{\gamma \in E_2^{S_p^{(2)}}} \theta\Big(\sum_{f=\gamma f} W(f)\Big). \tag{11}$$

Now we give a formula for $\sum_{f=\gamma f} W(f)$, again following Parthasarathy. Each permutation γ in $E_2^{S_p^{(2)}}$ can be written as $\gamma = (\alpha'; (0)(1))$ where α' in $S_p^{(2)}$ is induced by α in S_p. The formula for the cycle index of $S_p^{(2)}$, which appears in **(2)**, shows the contribution to the cycle structure of α' made by each cycle Z_r of α and by each pair Z_r, Z_s of distinct cycles of α. Since $\gamma f = f$, the function f must be constant on each cycle in the disjoint cycle decomposition of α'. By observing the number of times the symbols permuted by the Z_r and Z_s appear in these cycles, the following formula is obtained:

$$\sum_{f=\gamma f} W(f) = \prod_{Z_r, Z_s} \Big(1 + \prod_{i \in Z_r} x_i^{s/d(r,s)} \prod_{j \in Z_s} x_j^{r/d(r,s)}\Big)^{d(r,s)} \times \prod_{\substack{Z_r \\ r \text{ even}}} \Big(1 + \prod_{i \in Z_r} x_i\Big)\Big(1 + \prod_{i \in Z_r} x_i^2\Big)^{(r-2)/2} \prod_{\substack{Z_r \\ r \text{ odd}}} \Big(1 + \prod_{i \in Z_r} x_i^2\Big)^{(r-1)/2}, \tag{12}$$

where the first product is over all distinct pairs of cycles Z_r, Z_s in α and the others are over all cycles of α.

Obviously Formula (11) in Theorem 2 can be modified further, using the fact that $\sum_{f=\gamma f} W(f)$ depends only on the cycle structure of α.

The computation is easily done for $p = 3$ and we give details here. We take S_3 acting on $\{1, 2, 3\}$. There are three possibilities for the cycle structure of a permutation in S_3. Using (12), we obtain the following results.
For $\alpha = (1)(2)(3)$,

$$\sum_{f=\gamma f} W(f) = (1 + x_1 x_2)(1 + x_2 x_3)(1 + x_1 x_3).$$

For $\alpha = (12)(3)$,

$$\sum_{f=\gamma f} W(f) = (1 + x_1 x_2 x_3^2)(1 + x_1 x_2).$$

For $\alpha = (123)$,

$$\sum_{f=\gamma f} W(f) = (1 + x_1^2 x_2^2 x_3^2).$$

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Now by applying Theorem 2 we have:

$$\begin{aligned} N(x_1, x_2, x_3) &= (1/3!)\{\theta(1 + x_1 x_2)(1 + x_2 x_3)(1 + x_1 x_3) \\ &\qquad + 3\theta(1 + x_1 x_2 x_3^2)(1 + x_1 x_2) + 2\theta(1 + x_1^2 x_2^2 x_3^2)\} \\ &= (1/3!)\{(1 + 3x_1 x_2 + 3x_1^2 x_2 x_3 + x_1^2 x_2^2 x_3^2) \\ &\qquad + 3(1 + x_1 x_2 + x_1^2 x_2 x_3 + x_1^2 x_2^2 x_3^2) + 2(1 + x_1^2 x_2^2 x_3^2)\} \\ &= 1 + x_1 x_2 + x_1^2 x_2 x_3 + x_1^2 x_2^2 x_3^2. \end{aligned}$$

Thus no two different graphs with $p = 3$ points have the same partition and the same is true for $p = 4$. The first occurrence of a partition belonging to two different graphs involves $p = 5$. There are in all three pairs of such graphs; the two pairs shown in Figures 3 and 4, and the complements of the graphs of Figure 4.

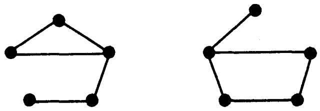

FIGURE 3

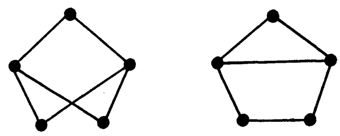

FIGURE 4

In principle only, one can determine from formula (11) whether a given partition of an even number is graphical (belongs to at least one graph), or graphical and simple (corresponds to a unique graph), and in general the multiplicity of a given graphical partition (the number of different graphs to which it belongs).

Parthasarathy **(9)** has also enumerated bicoloured graphs with a given bipartition by an entirely analogous modification of the method of Harary **(3)**. Similarly one can now take any of several solved enumeration problems and modify it to obtain a formula for the number of such graphs with a given partition. These include connected graphs, rooted graphs, etc. However, one must realize that this method only gives formal solutions to these counting problems and does not conveniently yield exact numbers or orders of magnitude.

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