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Uniqueness of ground states to fractional nonlinear elliptic equations with harmonic potential

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In this paper, we prove the uniqueness of ground states to the following fractional nonlinear elliptic equation with harmonic potential,

$$(-\Delta)^s u + \left(\omega + |x|^2\right) u = |u|^{p-2}u \quad \text{in } \mathbb{R}^n,$$

where $n \geqslant 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$, $2 < p < 2n/(n-2s)^+$, $\lambda_{1,s} > 0$ is the lowest eigenvalue of $(-\Delta)^s + |x|^2$. The fractional Laplacian $(-\Delta)^s$ is characterized as $\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi)$ for $\xi \in \mathbb{R}^n$, where $\mathcal{F}$ denotes the Fourier transform. This solves an open question in [M. Stanislavova and A. G. Stefanov. J. Evol. Equ. 21 (2021), 671–697.] concerning the uniqueness of ground states.

Keywords: Uniqueness; ground states; harmonic potential; fractional elliptic equations

2020 *Mathematics Subject Classification:* 35A02; 35R11

1. Introduction

In this paper, we study the uniqueness of ground states to the following fractional nonlinear elliptic equation with harmonic potential,

$$(-\Delta)^s u + \left(\omega + |x|^2\right) u = |u|^{p-2}u \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n \geqslant 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$, $2 < p < 2_s^* := 2n/(n-2s)^+$ and $\lambda_{1,s} > 0$ is the lowest eigenvalue of $(-\Delta)^s + |x|^2$, which is defined by

$$\lambda_{1,s} := \inf_{u \in \Sigma_s} \left\{\left\langle \left((-\Delta)^s + |x|^2\right) u, u\right\rangle : \|u\|_2 = 1\right\}, \quad \Sigma_s := H^s(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2 \, \mathrm{d}x). \tag{1.2}$$

The fractional Laplacian $(-\Delta)^s$ is characterized as $\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi)$ for $\xi \in \mathbb{R}^n$, where $\mathcal{F}$ denotes the Fourier transform defined by

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} u(x) \, \mathrm{d}x.$$

For $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(1 + |\xi|^{2s}\right) |\mathcal{F}(u)|^2 \, \mathrm{d}\xi < \infty \right\}$$

equipped with the norm

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} \left(1+|\xi|^{2s}\right) |\mathcal{F}(u)|^2 \, d\xi.$$

The problem under consideration arises in the study of standing waves to the following time-dependent Schrödinger equation,

$$i\partial_t \psi + (-\Delta)^s \psi + |x|^2 \psi = |\psi|^{p-2}\psi \quad \text{in } \mathbb{R} \times \mathbb{R}^n. \tag{1.3}$$

Here a standing wave to (1.3) is a solution of the form

$$\psi(t,x) = e^{-i\omega t} u(x), \quad \omega \in \mathbb{R}.$$

It is simple to see that ψ is a solution to (1.3) if and only if u is a solution to (1.1). Equation (1.1) is of particular interest in fractional quantum mechanics and originates from the early work of Laskin [**8**, **9**].

For the case $s = 1$, the uniqueness of ground states to (1.1) was achieved in [**5**, **6**]. However, for the case $0 < s < 1$, the uniqueness of ground states to (1.1) is open so far. The aim of this paper is to make a contribution towards this direction.

In the present paper, we are only concerned with the uniqueness of ground states to (1.1), the existence of which is a simple consequence of the use of mountain pass theorem, see [**11**, Theorem 1.15], and the fact that Σ_s is compactly embedded into $L^q(\mathbb{R}^n)$ for any $2 \leqslant q < 2_s^*$, see [**1**, Lemma 3.1]. Moreover, in view of the maximum principle, we can further obtain that any ground state to (1.1) is positive. The main result of the paper reads as follows.

THEOREM 1.1. *Let $n \geqslant 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$ and $2 < p < 2_s^*$. Then ground state to (1.1) is unique up to translations.*

Due to the nonlocal feature of the fractional Laplacian operator, the well-known ODE techniques often adapted to discuss the uniqueness of ground states to nonlinear elliptic equations with $s = 1$ are not applicable to our problem. Therefore, to establish theorem 1.1, we shall make use of the scheme developed in [**3**, **4**].

REMARK 1.2. Theorem 1.1 answers an open question posed in [**10**] with respect to the uniqueness of ground states to (1.1), which also extends the uniqueness results in [**5**, **6**] for $s = 1$ to the case $0 < s < 1$.

NOTATION 1.3. *For $1 \leqslant q \leqslant \infty$, we denote by $\|\cdot\|_q$ the standard norm in the Lebesgue space $L^q(\mathbb{R}^n)$. Moreover, we use $X \lesssim Y$ to denote that $X \leqslant CY$ for some proper constant $C > 0$ and we use $X \sim Y$ to denote $X \lesssim Y$ and $Y \lesssim X$.*

2. Proof of theorem 1.1

In this section, we are going to establish theorem 1.1. To do this, we first present the nondegeneracy of ground states.

LEMMA 2.1. *Let $n \geqslant 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$ and $2 < p < 2_s^*$. Let $u \in \Sigma_s$ be a ground state to (1.1). Then the linearized operator*

$$\mathcal{L}_{+,s} := (-\Delta)^s + \left(\omega + |x|^2\right) - (p-1)|u|^{p-2}$$

has a trivial kernel.

Proof. To prove this lemma, one can follow closely the line of the proof of [**10**, Theorem 2]. Let us now sketch the proof. First we observe that $\mathcal{L}_{+,s} \mid_{\{u\}^\perp} \geqslant 0$. On the other hand, we find that

$$\langle \mathcal{L}_{+,s} u, u\rangle = -(p-2) \int_{\mathbb{R}^n} |u|^p < 0.$$

It then follows that $\mathcal{L}_{+,s}$ has only one negative eigenvalue. From [**10**, Proposition 7], we actually know that the eigenvalue is simple. Using spherical harmonics and the representations of fractional Schrödinger operators introduced in [**10**], we can write that

$$\mathcal{L}_{+,s} = \bigoplus_{l=0}^{\infty} \mathcal{L}_{+,s,l} := \mathcal{L}_{+,s,0} \bigoplus \mathcal{L}_{+,s,\geqslant 1},$$

where the operator $\mathcal{L}_{+,s,l}$ acting on $L^2_{rad}(\mathbb{R}^n)$ is given by

$$\mathcal{L}_{+,s,l} := \left(-\partial_{rr} - \frac{n-1}{r}\partial_r + \frac{l(l+n-2)}{r^2}\right)^s \\ + \left(\omega + |x|^2\right) - (p-1)|u|^{p-2}, \quad l = 0, 1, \cdots, k, \cdots .$$

It is clear that

$$\sigma(\mathcal{L}_{+,s}) = \bigcup_{l=0}^{\infty} \sigma(\mathcal{L}_{+,s,l}), \\ \mathcal{L}_{+,s,0} < \mathcal{L}_{+,s,1} < \cdots < \mathcal{L}_{+,s,k} < \cdots .$$

At this point, to conclude the proof, we only need to verify that the second smallest eigenvalue of $\mathcal{L}_{+,s,0}$ is positive and $\mathcal{L}_{+,s,\geqslant 1} \geqslant \delta > 0$. This can be achieved by applying [**10**, Propositions 8–9]. Thus, the proof is completed. □

In order to establish theorem 1.1, we shall closely follow the strategies developed in [**3**, **4**]. For this, we now introduce some notations. Let $n \geqslant 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$ and $2 < p < 2_s^*$. Define

$$X_p := \left\{u \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) : u \text{ is radially symmetric and real-valued and } \\ \|xu\|_2 < +\infty\right\}$$

equipped with the norm

$$\|u\|_{X_p} := \|u\|_2 + \|u\|_p + \|xu\|_2.$$

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Lemma 2.2. *Let $n \geqslant 1$, $0 < s < 1$, $\omega > -\lambda_{1,s}$ and $2 < p < 2_s^*$ and $u \in X_p$ be a solution to (1.1). Then $u \in H^s(\mathbb{R}^n)$.*

Proof. First we show that $u \in H^1(\mathbb{R}^n)$. Since $v \in X_p$ be a solution to (1.1), then

$$(-\Delta)^s u + \left(\omega + |x|^2\right) u + 2\lambda u = 2\lambda u + |u|^{p-2} u, \tag{2.1}$$

where $\lambda > 0$ satisfies $\omega + \lambda > 0$. Note that

$$(-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda > (-\Delta)^s + \lambda > 0.$$

This leads to

$$0 < \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} < \left((-\Delta)^s + \lambda\right)^{-1}. \tag{2.2}$$

It then follows from Young's inequality that

$$\begin{aligned} \left\| \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} u \right\|_2 &\leqslant \left\| \left((-\Delta)^s + \lambda\right)^{-1} u \right\|_2 \\ &= \|\mathcal{K} * u\|_2 \lesssim \|u\|_2 \lesssim \|u\|_{H^{-s}}, \end{aligned} \tag{2.3}$$

where $H^{-s}(\mathbb{R}^n)$ denotes the dual space of $H^s(\mathbb{R}^n)$ and $\mathcal{K}$ is the fundamental solution to the equation

$$(-\Delta)^s u + \lambda u = 0 \tag{2.4}$$

and $\mathcal{K} \in L^1(\mathbb{R}^n)$ by [**4**, Lemma C. 1]. This indicates that the operator $((-\Delta)^s + (\omega + |x|^2) + 2\lambda)^{-1}$ maps $H^{-s}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Using dual theory, we then see that $((-\Delta)^s + (\omega + |x|^2) + 2\lambda)^{-1}$ maps $L^2(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$. Observe that

$$\left\| \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} u \right\|_{H^s} \leqslant \left\| \left((-\Delta)^s + \lambda\right)^{-1} u \right\|_{H^s} \lesssim \|u\|_{H^{-s}} \lesssim \|u\|_{p'}, \tag{2.5}$$

where the last inequality is from the dual to the Sobolev embedding $\|u\|_p \lesssim \|u\|_{H^s}$. This indicates that the operator $((-\Delta)^s + (\omega + |x|^2) + 2\lambda)^{-1}$ maps $L^{p'}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$. In fact, this can observe that

$$u = \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} \left(2\lambda u + |u|^{p-2} u\right)$$

and

$$2\lambda u + |u|^{p-2} u \in L^2(\mathbb{R}^n) + L^{p'}(\mathbb{R}^n).$$

Then the desired result follows. This completes the proof. □

Lemma 2.3. *Let $s_n \to s$ as $n \to \infty$, then $\lambda_{1,s_n} \to \lambda_{1,s}$ as $n \to \infty$.*

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Proof. To prove this, we only need to show that $A_{s_n} \to A_s$ in the norm-resolvent sense as $n \to \infty$, where

$$A_{s_n} := (-\Delta)^{s_n} + |x|^2, \quad A_s := (-\Delta)^s + |x|^2.$$

Let $z \in \mathbb{C}$ be such that Im $z \neq 0$, then

$$\begin{aligned} A_{s_n} + z = (-\Delta)^{s_n} + |x|^2 + z &= (-\Delta)^s + |x|^2 + z + (-\Delta)^{s_n} - (-\Delta)^s \\ &= \left(1 + \left((-\Delta)^{s_n} - (-\Delta)^s\right)(A_s + z)^{-1}\right)(A_s + z). \end{aligned}$$

Then we see that

$$\begin{aligned} &(A_{s_n} + z)^{-1} - (A_s + z)^{-1} \\ &= (A_s + z)^{-1}\left(\left(1 + \left((-\Delta)^{s_n} - (-\Delta)^s\right)(A_s + z)^{-1}\right)^{-1} - 1\right). \end{aligned} \tag{2.6}$$

Note that

$$\left\| \left((-\Delta)^{s_n} - (-\Delta)^s\right)(A_s + z)^{-1} \right\|_{L^2 \to L^2} = o_n(1).$$

In addition, we see that $(A_s + z)^{-1}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. As a consequence, from (2.6), we can conclude that

$$\left\| (A_{s_n} + z)^{-1} - (A_s + z)^{-1} \right\|_{L^2 \to L^2} = o_n(1).$$

This completes the proof. □

Lemma 2.4. *Let $0 < s_0 < 1$ and $2 < p < 2^*_{s_0}$. Suppose that $u_0 \in X_p$ solves (2.1) with $s = s_0$ such that the linearized operator*

$$\mathcal{L}_{+,s_0} = (-\Delta)^{s_0} + \left(\omega + |x|^2\right) - (p-1)|u_0|^{p-2}$$

has a trivial kernel on $L^2_{rad}(\mathbb{R}^n)$, where $w > -\lambda_{1,s_0}$. Then there exist $\delta_0 > 0$ and a map $u \in C^1(I; X_p)$ with $I = [s_0, s_0 + \delta_0)$ such that

(i) *u_s solves (1.1) for $s \in I$, where $u_s := u(s)$ for $s \in I$.*

(ii) *There exists $\epsilon > 0$ such that u_s is the unique solution to (1.1) for $s \in I$ in the neighbourhood*

$$\left\{u \in X_p : \|u - u_0\|_{X_p} < \epsilon\right\},$$

where $u_{s_0} = u_0$.

Proof. Let $\delta_0 > 0$ be a small constant to be determined later and $\lambda_{1,s} > 0$ be the lowest eigenvalue of $(-\Delta)^s + |x|^2$ for $s \in [s_0, s_0 + \delta_0)$. Define a mapping $F : X_p \times [s_0, s_0 + \delta_0) \to X_p$ by

$$F(u, s) := u - \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1}\left(2\lambda u + |u|^{p-2}u\right),$$

where $\omega > 0$ satisfies $\omega > -\lambda_{1,s}$ and $\lambda > 0$ satisfies $\lambda_{1,s} < \lambda$ for any $s \in [s_0, s_0 + \delta_0)$. Due to $\omega > -\lambda_{1,s_0}$, by lemma 2.3, then there exists $\delta_0 > 0$ small such that $\omega > -\lambda_{1,s}$

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is valid for any $s \in [s_0,\, s_0+\delta_0)$. Moreover, observe that $\Sigma_1 \subset \Sigma_s$, then

$$\lambda_{1,s} \leqslant \inf_{u\in\Sigma_s} \left\{ \left\langle \left(-\Delta + |x|^2\right) u, u \right\rangle : \|u\|_2 = 1 \right\} \leqslant \lambda_{1,1},$$

where $\lambda_{1,1} > 0$ is defined by

$$\lambda_{1,1} := \inf_{u\in\Sigma_1} \left\{ \left\langle \left(-\Delta + |x|^2\right) u, u \right\rangle : \|u\|_2 = 1 \right\}.$$

This then justifies that there exists $\lambda > 0$ such that $\lambda_{1,s} < \lambda$ for any $s \in [s_0, s_0+\delta_0)$.

First we check that F is well-defined. As an immediate consequence of the proof of lemma 2.2, we see that $F(u,\, s) \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for any $u \in X_p$ and $s \in [s_0,\, s_0+\delta_0)$. Let us now check that $F(u,\, s) \in L^2(\mathbb{R}^n; |x|^2\, \mathrm{d}x)$ for any $u \in X_p$ and $s \in [s_0,\, s_0+\delta)$. Define

$$f := \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} \left(2\lambda u + |u|^{p-2} u\right).$$

As the proof of lemma 2.2, we find that $f \in H^s(\mathbb{R}^n)$. This further gives that

$$(-\Delta)^s f + \left(\omega + |x|^2\right) f + 2\lambda f = 2\lambda u + |u|^{p-2} u.$$

Therefore, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\,(-\Delta)^{s/2} f|^2\, \mathrm{d}x + \int_{\mathbb{R}^n} \left(\omega + |x|^2\right) |f|^2\, \mathrm{d}x \\
&\quad + 2\lambda \int_{\mathbb{R}^n} |f|^2\, \mathrm{d}x = 2\lambda \int_{\mathbb{R}^n} u f\, \mathrm{d}x + \int_{\mathbb{R}^n} |u|^{p-2} u f\, \mathrm{d}x \\
&\quad \leqslant 2\|u\|_2 \|f\|_2 + \|u\|_p^{p-1} \|f\|_p < +\infty,
\end{aligned}$$

where we used Hölder's inequality for the inequality. It then leads to the desired result.

To apply the implicit function theorem, we are going to check that F is of class C^1. First we show that $\partial F/\partial u$ exists and

$$\frac{\partial F}{\partial u} = 1 - \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} \left(2\lambda + (p-1)|u|^{p-2}\right).$$

For simplicity, we shall define

$$G(u,s) := \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} \left(2\lambda u + |u|^{p-2} u\right).$$

Indeed, it suffices to prove that $\partial G/\partial u$ exists and

$$\frac{\partial G}{\partial u} = \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} \left(2\lambda + (p-1)|u|^{p-2}\right).$$

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Observe that, for any $h \in X_p$,

$$\begin{aligned}
&\left\|G(u+h,s)-G(u,s)-\frac{\partial G}{\partial u}(u,s)h\right\|_{L^2\cap L^p}\\
&\quad=\left\|\left((-\Delta)^s+\left(\omega+|x|^2\right)+2\lambda\right)^{-1}\left(|u+h|^{p-2}(u+h)\right.\right.\\
&\quad\left.\left.-|u|^{p-2}u-(p-1)|u|^{p-2}h\right)\right\|_{L^2\cap L^p}\\
&\quad\leqslant\left\|\left((-\Delta)^s+\lambda\right)^{-1}\left(|u+h|^{p-2}(u+h)-|u|^{p-2}u-(p-1)|u|^{p-2}h\right)\right\|_{L^2\cap L^p}\\
&\quad\lesssim\left\||u+h|^{p-2}(u+h)-|u|^{p-2}u-(p-1)|u|^{p-2}h\right\|_{\frac{p}{p-1}}=o(\|h\|_p)=o(\|h\|_{L^2\cap L^p}),
\end{aligned}$$

where we used the fact that the fundamental solution $\mathcal{K}$ to (2.4) satisfies $\mathcal{K} \in L^{p/2}(\mathbb{R}^n) \cap L^{2p/p+2}(\mathbb{R}^n)$ and Young's inequality. Define

$$g:=\left((-\Delta)^s+\left(\omega+|x|^2\right)+2\lambda\right)^{-1}\left(|u+h|^{p-2}(u+h)-|u|^{p-2}u-(p-1)|u|^{p-2}h\right).$$

Since

$$|u+h|^{p-2}(u+h)-|u|^{p-2}u-(p-1)|u|^{p-2}h \in L^{p'}(\mathbb{R}^n),$$

then $g \in H^s(\mathbb{R}^n)$ by arguing as the proof of lemma 2.2. Then we write

$$(-\Delta)^s g+\left(\omega+|x|^2\right)g+2\lambda g=|u+h|^{p-2}(u+h)-|u|^{p-2}u-(p-1)|u|^{p-2}h.$$

It then follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|(-\Delta)^{s/2}g|^2\,\mathrm{d}x+\int_{\mathbb{R}^n}\left(\omega+|x|^2\right)|g|^2\,\mathrm{d}x+2\lambda\int_{\mathbb{R}^n}|g|^2\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^n}\left(|u+h|^{p-2}(u+h)-|u|^{p-2}u-(p-1)|u|^{p-2}h\right)g\,\mathrm{d}x=o(\|h\|_p)\|g\|_p.
\end{aligned}$$

Using the fact that $H^s(\mathbb{R}^n)$ is continuously embedded into $L^p(\mathbb{R}^n)$ and Young's inequality, we then obtain that

$$\|g\|_{L^2(\mathbb{R}^n;|x|^2\,\mathrm{d}x)} \lesssim o(\|h\|_2).$$

Consequently, there holds that

$$\begin{aligned}
&\left\|G(u+h,s)-G(u,s)-\frac{\partial G}{\partial u}(u,s)h\right\|_{L^2(\mathbb{R}^n;|x|^2\,\mathrm{d}x)}\\
&\quad=\left\|\left((-\Delta)^s+\left(\omega+|x|^2\right)+2\lambda\right)^{-1}\left(|u+h|^{p-2}(u+h)\right.\right.\\
&\quad\left.\left.-|u|^{p-2}u-(p-1)|u|^{p-2}h\right)\right\|_{L^2(\mathbb{R}^n;|x|^2\,\mathrm{d}x)}\\
&\quad=\|g\|_{L^2(\mathbb{R}^n;|x|^2\,\mathrm{d}x)}\lesssim o(\|h\|_2).
\end{aligned}$$

Thus, we conclude that

$$\left\|G(u+h,s)-G(u,s)-\frac{\partial G}{\partial u}(u,s)h\right\|_{X_p}\lesssim o(\|h\|_{X_p}).$$

The desired result follows.

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Next we are going to verify that $\partial F/\partial u$ is continuous. Indeed, it suffices to show that $\partial G/\partial u$ is continuous. For this aim, we shall demonstrate that, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|u - \tilde{u}\|_{X_p} + |s - \tilde{s}| < \delta$, then, for any $h \in X_p$,

$$\left\| \left(\frac{\partial G}{\partial u}(u, s) - \frac{\partial G}{\partial u}(\tilde{u}, \tilde{s}) \right) h \right\|_{X_p} < \epsilon \|h\|_{X_p}. \tag{2.7}$$

Observe that

$$\left(\frac{\partial G}{\partial u}(u, s) - \frac{\partial G}{\partial u}(\tilde{u}, \tilde{s}) \right) h = \left(A_s - A_{\tilde{s}}\right) \left(2\lambda + (p-1)|u|^{p-2}\right) h + A_{\tilde{s}} \left(2\lambda + (p-1)\left(|u|^{p-2} - |\tilde{u}|^{p-2}\right)\right) h,$$

where

$$A_s := \left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1}, \quad A_{\tilde{s}} := \left((-\Delta)^{\tilde{s}} + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1}.$$

Note that

$$\|f\|_{L^2 \cap L^p} \lesssim \left\| \left((-\Delta)^{s_p/2} + 1\right) f \right\|_2, \quad 0 < s_p := \frac{(p-2)n}{2p} < s.$$

Then, by Plancherel's identity, the mean value theorem and Young's inequality, there holds that

$$\begin{aligned} &\left\| \left(A_s - A_{\tilde{s}}\right) \left(2\lambda + (p-1)|u|^{p-2}\right) h \right\|_{L^2 \cap L^p} \\ &\lesssim \left\| \left((-\Delta)^{s_p/2} + 1\right) \left(A_s - A_{\tilde{s}}\right) \left(2\lambda + (p-1)|u|^{p-2}\right) h \right\|_2 \\ &\lesssim |s - \tilde{s}| \left(\|h\|_2 + \|h\|_p + \|u\|_p^{p-2} \|h\|_p\right). \end{aligned}$$

In addition, we see that

$$\begin{aligned} &\left\| \left(A_s - A_{\tilde{s}}\right) \left(2\lambda + (p-1)|u|^{p-2}\right) h \right\|_{L^2(\mathbb{R}^n; |x|^2 \,\mathrm{d}x)} \\ &\lesssim |s - \tilde{s}| \left(\|h\|_2 + \|h\|_p + \|u\|_p^{p-2} \|h\|_p\right). \end{aligned}$$

Notice that

$$\|A_s f\|_{L^2(\mathbb{R}^n; |x|^2 \,\mathrm{d}x)} \lesssim \|f\|_2, \quad \|A_s f\|_{L^2(\mathbb{R}^n; |x|^2 \,\mathrm{d}x)} \lesssim \|f\|_{p/p-1}.$$

Further, we can conclude that

$$\left\| A_{\tilde{s}} \left(2\lambda + \left(|u|^{p-2} - |\tilde{u}|^{p-2}\right)\right) h \right\|_{X_p} \lesssim \|h\|_2 + \|h\|_p + \left\| |u|^{p-2} - |\tilde{u}|^{p-2} \right\|_{p/p-2} \|h\|_p.$$

Note that

$$\left\| |u|^{p-2} - |\tilde{u}|^{p-2} \right\|_{p/p-2} \leqslant \left\| |u - \tilde{u}|^{p-2} \right\|_{p/p-2} = \|u - \tilde{u}\|_p^{p-2}, \quad 2 < p \leqslant 3$$

and

$$\begin{aligned} \left\| |u|^{p-2} - |\tilde{u}|^{p-2} \right\|_{p/p-2} &\lesssim \left\| \left(|u|^{p-3} + |\tilde{u}|^{p-3}\right) |u - \tilde{u}| \right\|_{p/p-2} \\ &\leqslant \left(\|u\|_p^{p-3} + \|\tilde{u}\|_p^{p-3}\right) \|u - \tilde{u}\|_p, \quad p > 3. \end{aligned}$$

Consequently, from the calculations above, (2.7) holds true. This implies that $\partial F/\partial u$ is continuous. By a similar argument, we are also able to show that $\partial F/\partial s$

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exists and

$$\frac{\partial F}{\partial s} = -\left((-\Delta)^s \log(-\Delta)\right)\left((-\Delta)^s + \left(\omega + |x|^2\right) + 2\lambda\right)^{-2}\left(2\lambda u + |u|^{p-2}u\right).$$

In addition, we can prove that $\partial F/\partial s$. Thus, we have that F is of class C^1.

Now we employ the implicit function theorem to establish theorem. Note first that $F(u_0, s_0) = 0$ and

$$\frac{\partial F}{\partial u}(u_0, s_0) = 1 + K, \quad K := -\left((-\Delta)^{s_0} + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1}\left(2\lambda + (p-1)|u_0|^{p-2}\right).$$

It is simple to see that K is compact on $L^2_{rad}(\mathbb{R}^n)$. Moreover, from lemma 2.1, we have that $-1 \notin \sigma(K)$. Then $1 + K$ is invertible. Furthermore, arguing as before, we can show that $1 + K$ is bounded from X_p to X_p. This implies that $(1 + K)^{-1}$ is bounded from X_p to X_p. It then follows from the implicit function theorem that theorem holds true. This completes the proof. □

In the following, we shall consider the maximum extension of the branch u_s for $s \in [s_0, s_*)$, where $s_* > s_0$ is given by

$$s_* := \sup\Big\{s_0 < \tilde{s} < 1, u_s \in C^1([s_0, \tilde{s}); X_p), u_s \text{ satisfies the assumptions of lemma 2.4 for } s \in [s_0, \tilde{s})\Big\}.$$

Lemma 2.5. *There holds that*

$$\int_{\mathbb{R}^n} \left(w + |x|^2\right)|u_s|^2\,\mathrm{d}x \sim \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u_s|^2\,\mathrm{d}x \sim \int_{\mathbb{R}^n} |u_s|^p\,\mathrm{d}x \sim 1$$

for any $s \in [s_0, s_*)$.

Proof. Define

$$M_s := w\int_{\mathbb{R}^n} |u_s|^2\,\mathrm{d}x, \quad H_s := \int_{\mathbb{R}^n} |x|^2|u_s|^2\,\mathrm{d}x, \quad T_s := \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u_s|^2\,\mathrm{d}x, \quad V_s := \int_{\mathbb{R}^n} |u_s|^p\,\mathrm{d}x.$$

Since $u_s \in H^s(\mathbb{R}^n)$ is a solution to (1.1), then

$$T_s + M_s + H_s = V_s. \tag{2.8}$$

In addition, we have that u_s satisfies the following Pohozaev identity,

$$\frac{N-2s}{2}T_s + \frac{N}{2}M_s + \frac{N+2}{2}H_s = \frac{N}{p}V_s. \tag{2.9}$$

Combining (2.8) and (2.9), we see that

$$sT_s - H_s = \frac{N(p-2)}{2p}V_s. \tag{2.10}$$

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It follows from (2.8) and (2.10) that

$$s_0 M_s + (1+s_0)H_s \leqslant sM_s + (1+s)H_s = \frac{2ps - N(p-2)}{2p}V_s < \frac{2ps_* - N(p-2)}{2p}V_s$$

and

$$s_* M_s + (1+s_*)H_s > sM_s + (1+s)H_s = \frac{2ps - N(p-2)}{2p}V_s \geqslant \frac{2ps_0 - N(p-2)}{2p}V_s.$$

Consequently, we have that $M_s + H_s \sim V_s$ for any $s \in [s_0, s_*)$. It follows from (2.8) and (2.10) that

$$(1+s_0)T_s \leqslant (1+s)T_s + M_s = \frac{N(p-2)+2p}{2p}V_s$$

and

$$(1+s_*)T_s > (1+s)T_s + M_s = \frac{N(p-2)+2p}{2p}V_s.$$

This leads to $T_s \sim V_s$ for any $s \in [s_0, s_*)$. Therefore, we obtain that

$$M_s + H_s \sim T_s \sim V_s \tag{2.11}$$

for any $s \in [s_0, s_*)$. Since $2 < p < p_{s_0}$, there exists $0 < \theta < 1$ such that $p = 2\theta + (1-\theta)p_{s_0}$. From Gagliardo–Nirenberg's inequality and Hölder's inequality, we then get that

$$V_s \leqslant M_s^{\theta} \left(\int_{\mathbb{R}^n} |u_s|^{p_{s_0}} \, \mathrm{d}x \right)^{(1-\theta)} \lesssim (M_s + H_s)^{\theta} \left(\int_{\mathbb{R}^n} |(-\Delta)^{s_0/2} u_s|^2 \, \mathrm{d}x \right)^{p_{s_0}(1-\theta)/2}. \tag{2.12}$$

In addition, there holds that

$$\int_{\mathbb{R}^n} |(-\Delta)^{s_0/2} u_s|^2 \, \mathrm{d}x \leqslant \left(\int_{\mathbb{R}^n} |u_s|^2 \, \mathrm{d}x \right)^{s - s_0/s} \left(\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u_s|^2 \, \mathrm{d}x \right)^{s_0/s}. \tag{2.13}$$

Utilizing (2.11), (2.12) and (2.13) then implies that

$$M_s + H_s \sim T_s \sim V_s \gtrsim 1$$

for any $s \in [s_0, s_*)$. Arguing as the proof of [**4**, Lemma 8.2], we can obtain that $V_s \lesssim 1$ for any $s \in [s_0, s_*)$. This in turn implies that

$$M_s + H_s \sim T_s \sim V_s \lesssim 1$$

for any $s \in [s_0, s_*)$. This completes the proof. □

LEMMA 2.6. *Let $n \geqslant 1$, $s_0 \leqslant s \leqslant 1$, $\omega > -\lambda_{1,s_0}$ and $2 < p < 2^*_{s_0}$. Suppose that $u_s \in X_p$ is a ground state to (1.1). Then there exists $\mu_s > 0$ such that*

$$\liminf_{\sigma \to s^-} \mathcal{L}_{+,\sigma} \mid_{\{u_\sigma\}^\perp} \geqslant \mu_s. \tag{2.14}$$

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Proof. Define

$$\alpha_s := \inf\{\langle \mathcal{L}_{+,s}f, f\rangle : f \perp u_s, \|f\|_2 = 1\}. \tag{2.15}$$

Obviously, we have that $\alpha_s \geqslant 0$. First we shall verify that $\alpha_s > 0$ is attained. Let $\{f_k\}$ be a minimizing sequence to (2.15) such that $f_k \perp u_s$, $\|f_k\|_2 = 1$ and $\langle \mathcal{L}_{+,s}f_k, f_k\rangle = \alpha_s + o_k(1)$. Observe that $\{f_k\}$ is bounded in Σ_s. Therefore, there exists a function $f \in \Sigma_s$ such that $f_k \rightharpoonup f$ in Σ_s and $f_k \to f$ in $L^q(\mathbb{R}^n)$ for any $q \in [2, 2_s^*)$ as $n \to \infty$. This leads to $f \perp u_s$, $\|f\|_2 = 1$ and $\langle \mathcal{L}_{+,s}f, f\rangle = \alpha_s$. Contrarily, we assume that $\alpha_s = 0$. When $s < 1$, using the fact that $Ker[\mathcal{L}_{+,s}] = \{0\}$ by lemma 2.1 and arguing as the proof of [**10**, Proposition 6], we are able to reach a contradiction. This in turn shows that $\alpha_s > 0$ and

$$\langle \mathcal{L}_{+,s}u, u\rangle \geqslant \alpha_s\|u\|_2^2, \quad \forall\ u \perp u_s.$$

While $s = 1$, using the fact that $Ker[\mathcal{L}_{+,1}] = \{0\}$ and following the spirit of the proof of [**10**, Proposition 6], we can also derive that $\alpha_1 > 0$ and

$$\langle \mathcal{L}_{+,1}u, u\rangle \geqslant \alpha_1\|u\|_2^2, \quad \forall\ u \perp u_1.$$

Thus, the proof is completed. □

LEMMA 2.7. *Let $u_{s_0} > 0$ be a solution to (1.1) with $s = s_0$. Then, for any $s \in [s_0, s_*)$, there holds that $u_s(x) > 0$ for $x \in \mathbb{R}^n$ and $u_s(x) \lesssim |x|^{-n}$ for $|x| \gtrsim 1$.*

Proof. In the spirit of the proof of [**4**, Lemma 8.3], we need to verify that the operator $\mathcal{L}_{-,s}$ enjoys the Perron–Frobenius type property, where

$$\mathcal{L}_{-,s} := (-\Delta)^s + \left(\omega + |x|^2\right) - |u|^{p-2}.$$

In addition, we need to check that $\mathcal{L}_{-,\tilde{s}} \to \mathcal{L}_{-,s}$ as $\tilde{s} \to s$ in norm-resolvent sense.

Define $H := (-\Delta)^s + |x|^2$, which generates a semigroup e^{-tH} with positive integral kernel. Then we have that e^{-tH} acting on $L^2(\mathbb{R}^n)$ is positivity improving. Next we show that $w + |u|^{p-2}$ belongs to Kato class, i.e.

$$\lim_{\lambda\to\infty} \left\|(H+\lambda)^{-1}\left(\omega + |u|^{p-2}\right)\right\|_{L^\infty\to L^\infty} = 0. \tag{2.16}$$

Note that $H + \lambda > (-\Delta)^s + \lambda$, then

$$(H+\lambda)^{-1} < \left((-\Delta)^s + \lambda\right)^{-1}.$$

Let $\mathcal{K}$ be the fundamental solution to the equation

$$(-\Delta)^s u + \lambda u = 0.$$

Then we have that

$$\mathcal{K}(x) = \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathcal{H}(x,t)\,\mathrm{d}t,$$

where

$$\mathcal{H}(x,t) := \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi - t|\xi|^{2s}}\,\mathrm{d}\xi.$$

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From $(A4)$ in [2, Appendix A], we find that

$$0 < \mathcal{H}(x,t) \lesssim \min\left\{t^{-n/2s}, t|x|^{-n-2s}\right\}.$$

This gives that, for any $q \geqslant 1$,

$$\begin{aligned}\|\mathcal{H}\|_q &\lesssim \left(\int_{|x|\leqslant t^{1/2s}} t^{-nq/2s}\,\mathrm{d}x\right)^{1/q} + \left(\int_{|x|\geqslant t^{1/2s}} t^q|x|^{-(n+2s)q}\,\mathrm{d}x\right)^{1/q}\\ &\lesssim t^{-n/2s(1-(1/q))}.\end{aligned}$$

It then follows that

$$\|\mathcal{K}\|_q \leqslant \int_0^{+\infty} \mathrm{e}^{-\lambda t}\,\|\mathcal{K}(\cdot,t)\|_q\,\mathrm{d}t \lesssim \int_0^{+\infty} \mathrm{e}^{-\lambda t} t^{-n/2s(1-(1/q))}\,\mathrm{d}t \lesssim \lambda^{n/2s(1-(1/q))-1},$$

where $q \geqslant 1$ satisfies

$$\frac{n}{2s}\left(1-\frac{1}{q}\right) < 1.$$

Using Young's inequality, we then get that, for any $f \in L^\infty(\mathbb{R}^n)$,

$$\|\left((-\Delta)^s+\lambda\right)^{-1}\left(\omega+|u|^{p-2}\right)f\|_\infty \lesssim \lambda^{-1}\omega\|f\|_\infty + \lambda^{n/2s(1-(2/q))-1}\|f\|_\infty,$$

which readily yields that

$$\|\left((-\Delta)^s+\lambda\right)^{-1}\left(\omega+|u|^{p-2}\right)\|_{L^\infty\to L^\infty} = o_\lambda(1).$$

Thus, (2.16) holds true and the desired result follows. Arguing as the proof of [3, Lemma C.2], we conclude that the operator $\mathcal{L}_{-,s}$ enjoys Perron–Frobenius type property.

Next we prove the convergence of the operator in norm-resolvent sense. Observe first that

$$\begin{aligned}\mathcal{L}_{-,\tilde{s}} + z &= (-\Delta)^s + \left(\omega+|x|^2\right) - |u|^{p-2} + z + (-\Delta)^{\tilde{s}} - (-\Delta)^s\\ &= \left(1 + \left((-\Delta)^{\tilde{s}} - (-\Delta)^s\right)\left(\mathcal{L}_{-,s}+z\right)^{-1}\right)\left(\mathcal{L}_{-,s}+z\right).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\left(\mathcal{L}_{-,s}+z\right)^{-1} - \left(\mathcal{L}_{-,\tilde{s}}+z\right)^{-1}\\ &\quad= \left(\mathcal{L}_{-,s}+z\right)^{-1}\left(1-\left(1+\left((-\Delta)^{\tilde{s}}-(-\Delta)^s\right)\left(\mathcal{L}_{-,s}+z\right)^{-1}\right)^{-1}\right).\end{aligned}$$

As the proof of lemma 2.3, we can show that

$$\left\|\left(\mathcal{L}_{-,s}+z\right)^{-1} - \left(\mathcal{L}_{-,\tilde{s}}+z\right)^{-1}\right\|_{L^2\to L^2} \to 0, \quad \text{as } \tilde{s}\to s.$$

This indicates that $\mathcal{L}_{-,\tilde{s}} \to \mathcal{L}_{-,s}$ in the norm-resolvent sense as $\tilde{s} \to s$. Thus, the proof is completed. □

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Lemma 2.8. *Let* $\{s_n\} \subset [s_0,\, s_*)$ *be a sequence such that* $s_n \to s_*$ *as* $n \to \infty$ *and* $u_{s_n} > 0$ *for any* $n \in \mathbb{N}$*. Then there exists* $u_* \in X_p$ *such that* $u_{s_n} \to u_*$ *in* X_p *as* $n \to \infty$*. Moreover, there holds that* $u_* > 0$ *and it solves the equation*

$$(-\Delta)^{s_*} u_* + \left(\omega + |x|^2\right) u_* = u_*^{p-1}. \tag{2.17}$$

Proof. From lemma 2.5, we know that u_{s_n} is bounded in Σ_{s_0}. Thus, there exists $u_* \in \Sigma_{s_0}$ such that $u_{s_n} \rightharpoonup u_*$ in Σ_{s_0} and $u_{s_n} \to u_*$ in $L^q(\mathbb{R}^n)$ for any $q \in [2,\, 2^*_{s_0})$. Since $u_{s_n} > 0$, then $u_* \geqslant 0$. It follows from lemma 2.5 that $u_* \neq 0$. Note that

$$u_{s_n} = \left((-\Delta)^{s_n} + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} \left(2\lambda u_{s_n} + u_{s_n}^{p-1}\right).$$

Since $u_{s_n} \to u_*$ in $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ as $n \to \infty$, then

$$u_* = \left((-\Delta)^{s_*} + \left(\omega + |x|^2\right) + 2\lambda\right)^{-1} \left(2\lambda u_* + u_*^{p-1}\right).$$

This implies that u_* solves (2.17) and $u_{s_n} \to u_*$ in X_p as $n \to \infty$. Thus, the proof is completed. □

Lemma 2.9. *Let* $u_0 \in X_p$ *be a ground state to* (1.1) *with* $s = s_0$*. Then its maximum branch* u_s *with* $s \in [s_0,\, s_*)$ *extends to* $s_* = 1$*.*

Proof. Define

$$\mathcal{L}_{+,s} := (-\Delta)^s + \left(\omega + |x|^2\right) - (p-1)|u_s|^{p-2}.$$

Reasoning as the proof of the norm-resolvent convergence of $\mathcal{L}_{-,s}$ in lemma 2.7, we can also show that $\mathcal{L}_{+,\tilde{s}} \to \mathcal{L}_{+,s}$ in the norm-resolvent sense as $\tilde{s} \to s$. This gives that

$$\mathcal{N}_{-,rad}(\mathcal{L}_{+,s}) = \mathcal{N}_{-,rad}(\mathcal{L}_{+,s_0}) = 1, \quad s \in [s_0, s_*).$$

Let $\{s_n\} \subset [s_0,\, s_*)$ be such that $s_n \to s_*$. Since $u_0 \in X_p$ is a ground state to (1.1) with $s = s_0$, then $u_0 > 0$. In view of lemma 2.7, then $u_{s_n} > 0$. From lemma 2.8, we know that there exists $u_* > 0$ solving (2.17). Note that $\mathcal{L}_{+,s_n} \to \mathcal{L}_{+,s_*}$ in the norm-resolvent sense as $n \to \infty$. By the lower semicontinuity of the Morse index, we have that

$$1 = \liminf_{n\to\infty} \mathcal{N}_{-,rad}(\mathcal{L}_{-,s_n}) \geqslant \mathcal{N}_{-,rad}(\mathcal{L}_{+,s_*}).$$

This implies that $\mathcal{N}_{-,rad}(\mathcal{L}_{+,s_*}) \leqslant 1$. On the other hand, since u_* solves (2.17), then we see that

$$\langle u_*, \mathcal{L}_{+,s_*} u_* \rangle = -(p-2) \int_{\mathbb{R}^n} |u_*|^p \,\mathrm{d}x < 0.$$

Thus, we conclude that $\mathcal{N}_{-,rad}(\mathcal{L}_{+,s_*}) = 1$, which yields that u_* is a ground state to (2.17). As a result, we have that $s_* = 1$. On the other hand, by the nondegeneracy of $\mathcal{L}_{+,s_*}$, then u_s can be extended beyond s_*. This is impossible and the proof is completed. □

Now we are ready to prove theorem 1.1.

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Proof of theorem 1.1. Let $n \geqslant 1$, $0 < s_0 < 1$ and $2 < p < 2^*_{s_0}$. Let $u_{s_0} > 0$ and $\tilde{u}_{s_0} > 0$ be two different ground states to (1.1) with $s = s_0$, which are indeed radially symmetric. From lemma 2.1, we obtain that the associated linearized operators around u_{s_0} and $\tilde{u}_{s_0}$ are nondegenerate. Then, by lemmas 2.4 and 2.9, we have that $u_s \in C^1([s_0, 1); X_p)$ and $\tilde{u}_s \in C^1([s_0, 1); X_p)$. Moreover, by the local uniqueness of solutions derived in lemma 2.4, we get that $u_s \neq \tilde{u}_s$ for any $s \in [s_0, 1)$. It follows from lemma 2.8 that there exist $u_* \in X_p$ and $\tilde{u}_* \in X_p$ such that $u_s \to u_*$ and $\tilde{u}_s \to \tilde{u}_*$ in X_p as $s \to 1^-$. In addition, $u_* > 0$ and $\tilde{u}_* > 0$ solve (2.17) with $s_* = 1$. Thanks to [**5**, Theorem 1.3] and [**6**, Theorem1.2], then we have that $u_* = \tilde{u}_*$. This implies that $\|u_s - \tilde{u}_s\|_{X_p} \to 0$ as $s \to 1^-$. Note that the linearized operator $\mathcal{L}_{+,1}$ around u_* is nondegenerate, see [**7**, Theorem 0.2]. Remark that, from the proof of [**7**, Theorem 0.2], it is simple to see that the result also holds true for $n = 1$. Then, by the implicit function theorem, there exists a unique branch $\hat{u}_s \in C^1((1-\delta, 1]; X_p)$ solving (1.1) with $\hat{u}_1 = u^*$ for some $\delta > 0$. This contradicts with $u_s \neq \tilde{u}_s$ for any $s \in [s_0, 1)$. Thus, the proof is completed. □

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Competing interest

None.

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