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SOME CONSEQUENCES OF TD AND sTD

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Abstract. Strong Turing Determinacy, or sTD, is the statement that for every set A of reals, if $\forall x \exists y \geq_T x (y \in A)$, then there is a pointed set $P \subseteq A$. We prove the following consequences of Turing Determinacy (TD) and sTD over ZF—the Zermelo–Fraenkel axiomatic set theory without the Axiom of Choice:

(1) ZF + TD implies $\mathrm{wDC}_{\mathbb{R}}$—a weaker version of $\mathrm{DC}_{\mathbb{R}}$.
(2) ZF + sTD implies that every set of reals is measurable and has Baire property.
(3) ZF + sTD implies that every uncountable set of reals has a perfect subset.
(4) ZF + sTD implies that for every set of reals A and every $\epsilon > 0$:
  (a) There is a closed set $F \subseteq A$ such that $\mathrm{Dim}_{\mathrm{H}}(F) \geq \mathrm{Dim}_{\mathrm{H}}(A) - \epsilon$, where $\mathrm{Dim}_{\mathrm{H}}$ is the Hausdorff dimension.
  (b) There is a closed set $F \subseteq A$ such that $\mathrm{Dim}_{\mathrm{P}}(F) \geq \mathrm{Dim}_{\mathrm{P}}(A) - \epsilon$, where $\mathrm{Dim}_{\mathrm{P}}$ is the packing dimension.

§1. Introduction.

1.1. Turing Determinacy and strong Turing Determinacy. Turing reduction $\leq_T$ is a quasi-order over reals. $x \leq_T y$, or x is Turing reducible to y, means x can be computed by a Turing machine with oracle y. The reduction naturally induces an equivalence relation $\equiv_T$. Given a real x, its corresponded Turing degree $\mathbf{x}$ is a set of reals defined as $\{y \mid y \equiv_T x\}$. We say $\mathbf{x} \leq \mathbf{y}$ if $x \leq_T y$. We use $\mathcal{D}$ to denote the set of Turing degrees. An *upper cone* of Turing degrees is a set of form $\{\mathbf{y} \mid \mathbf{y} \geq \mathbf{x}\}$ for some $\mathbf{x}$.

We say that a perfect set P is *pointed* if there is a perfect tree $T \subseteq 2^{<\omega}$ such that $[T] = P$ and for any $x \in P$, $T \leq_T x$, where $[T] = \{x \in 2^\omega \mid \forall n (x \restriction n \in T)\}$.

Definition 1.1.
- Turing Determinacy, or TD, is the statement that for every set A of Turing degrees, either A or $\mathcal{D} \setminus A$ contains an upper cone of Turing degrees.
- Strong Turing Determinacy, or sTD, is the statement that for every set A of reals, if $\forall x \exists y \geq_T x (y \in A)$, then there is a pointed set P such that $P \subseteq A$.

Clearly sTD implies TD. Martin proves the following famous theorem, which partly justifies why Turing degrees are important to set theory.

Theorem 1.2 (Martin [16]). *Over* ZF, *Axiom of Determinacy, or* AD, *implies* sTD *and so* TD.

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DEFINITION 1.3. • Countable choice axiom for sets of reals, or $\mathrm{CC}_{\mathbb{R}}$, is the statement that for every countable sequence $\{A_n\}_{n\in\omega}$ of nonempty sets of reals, there is a function $f:\omega\to\mathbb{R}$ such that for every n, $f(n)\in A_n$.
• Dependent choice axiom for sets of reals, or $\mathrm{DC}_{\mathbb{R}}$, is the statement that for every binary relation R over reals such that $\forall x\exists y R(x,y)$, there is a function $f:\omega\to\mathbb{R}$ such that for every n, $R(f(n),f(n+1))$.

Turing Determinacy is an important and very useful consequence of AD. In many situations, TD seems sufficient to be used to prove set theory theorems. The following theorem partly justifies this phenomenon. Actually Woodin shows that under $\mathrm{ZF}+\mathrm{TD}+\mathrm{DC}_{\mathbb{R}}$, every Suslin set is determined (see [28, Theorem 1.2]).

THEOREM 1.4 (Woodin [28, Theorem 1.1]). *Assume* $\mathrm{ZF}+V=L(\mathbb{R})+\mathrm{DC}_{\mathbb{R}}$. AD *is equivalent to* TD.

Moreover, as we shall see in this paper, the consequences of TD (and sTD) may be proved in a uniform way, unlike the proofs that assume AD, which often require a very genius, tricky, and case-by-case design of games.

The first result in this paper concerns the relationship between AD and Axiom of Choice, or AC. Although AD contradicts AC, Mycielski proves the following theorem.

THEOREM 1.5 (Mycielski [18]). *Over* ZF, AD *implies* $\mathrm{CC}_{\mathbb{R}}$.

The question if AD implies $\mathrm{DC}_{\mathbb{R}}$ remains open for a long time.

QUESTION 1.6 (Solovay [26]). *Over* ZF, *does* AD *imply* $\mathrm{DC}_{\mathbb{R}}$?

Kechris proves the following result.

THEOREM 1.7 (Kechris [12]). *Assume* $\mathrm{ZF}+V=L(\mathbb{R})$. AD *implies* $\mathrm{DC}_{\mathbb{R}}$.

It is unknown whether the assumption $V=L(\mathbb{R})$ can be removed. Recently, the following “unconditional” result is proved.

THEOREM 1.8 (Peng and Yu [20]). *Over* ZF, TD *implies* $\mathrm{CC}_{\mathbb{R}}$.

We will use $\mathrm{CC}_{\mathbb{R}}$ throughout the paper even without mentioning it.

The first result in this paper is a partial solution to Question 1.6. We prove in Theorem 4.3 that $\mathrm{ZF}+\mathrm{TD}$ implies $\mathrm{wDC}_{\mathbb{R}}$, a weaker version of $\mathrm{DC}_{\mathbb{R}}$ (for the definition of $\mathrm{wDC}_{\mathbb{R}}$, see Definition 4.1).

The second result in this paper is about the regularity properties of sets of reals. Although TD seems unlikely as strong as AD, a natural question is whether TD is as “useful” as AD. Sami initiated this project by proving (in [23]) that $\mathrm{ZF}+\mathrm{TD}$ implies CH, the continuum hypothesis. But it seems a rather difficult (and long-standing) question whether $\mathrm{ZF}+\mathrm{TD}(+\mathrm{DC})$ implies regularity properties for sets of reals. A progress is made in [8] by showing that the perfect set property for all Turing invariant sets of reals implies the perfect set property for all sets of reals. A partial answer (Theorem 5.1) to this question is due to Woodin and remains unpublished (email communication between Woodin and Yu in April 2021). In this paper, we give another proof of Theorem 5.1 that strong Turing Determinacy, sTD–a stronger version of TD—implies the regularity properties for sets of reals via a recursion theoretical method that is different from Woodin's.

A basis for a class $\mathcal{C}$ of linearly ordered sets is a collection $\mathcal{B} \subseteq \mathcal{C}$ such that for each $L_1 \in \mathcal{C}$, there is an $L_2 \in \mathcal{B}$ such that L_2 is isomorphic to a subset of L_1. Investigating basis for linear ordering is a very active area in set theory today. For example, Moore [17] proves that under proper forcing axioms, PFA, a five-element basis for uncountable linear orders exists. But it seems that basis theorems for linear orderings under AD remain untouched. In this paper, we prove a basis theorem for linear orderings over $\mathbb{R}$ under the assumption ZF + TD + $\mathrm{DC}_{\mathbb{R}}$ + "every uncountable set of reals has a perfect subset" by showing that for every linear order $\leq_L$ over $\mathbb{R}$, there is an order-preserving embedding from $(2^\omega, \leq)$ to $(\mathbb{R}, \leq_L)$. In other words, $\{(2^\omega, \leq)\}$ is a basis for $\{(\mathbb{R}, \leq_L) \mid \leq_L$ is a linear ordering over $\mathbb{R}\}$.

The last result in this paper is an application of recursion theory to fractal geometry theory. Besicovitch and Davis [1, 5] prove that for every analytic set, its Hausdorff dimension can be approximated arbitrarily close by the Hausdorff dimension of its closed subsets. Joyce and Preiss [11] prove a similar result for packing dimension. Recently Slaman [25] proves that both Besicovitch–Davis and Joyce–Preiss theorems fail for some Π^1_1-set under the assumption $V = L$. However, we prove in Theorem 6.5 that both theorems hold for arbitrary sets of reals over ZF + sTD. So the phenomenon can be viewed as a new regularity property for the sets of reals. This weakens the assumption of the following result in [3].

THEOREM 1.9 (Crone, Fishman, and Jackson [3]). *Assume* ZF + AD + $\mathrm{DC}_{\mathbb{R}}$. *For every set A and $\epsilon > 0$, there is a closed set $F \subseteq A$ such that* $\mathrm{Dim}_{\mathrm{H}}(F) \geq \mathrm{Dim}_{\mathrm{H}}(A) - \epsilon$.

The proof of this theorem is direct and uses some rather deep results from set theory. However, we believe that our proof is simpler and, most importantly, it does not depend on $\mathrm{DC}_{\mathbb{R}}$.

1.2. Point to set principle. *Relativization* opens a door between recursion theory and other mathematical branches. In recursion theory, for a real x, a relativization to x, roughly speaking, is a way to add prefix x- to every appearance of any notion in the statement. Then if a notion is defined in recursion theory, its relativization is defined *naturally*. And if a theorem in recursion theory is proved, then its relativization also follows *naturally*. For example, every continuous function is a recursive function relative to a real, and a Borel set is a hyperarithmeitc set relative to a real. From this point of view, one may apply recursion theory results to analysis.

The "point to set" principle is a more concrete way, by using relativization, to apply recursion theory to other areas of mathematics. Generally speaking, the principle says that a set A having certain property is equivalent to that it contains some special points. Such argument can be dated back to Sacks, who (in [21]) gave a recursion theoretical proof of the classical result that every analytic set is measurable. For one more example, given a relativizable algorithmic randomness notion Γ (such as Martin-Löf and Schnorr), we have the following fact.

FACT 1.10. *Assume* ZF + $\mathrm{CC}_{\mathbb{R}}$. *A set $A \subseteq \mathbb{R}$ is null if and only if there is some real x such that no $\Gamma(x)$-random real is in A.*

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So a set A is not null is equivalent to say that for every real x, there is a $\Gamma(x)$-random real in A. One may also replace randomness with genericity and obtain the corresponding results. In this paper, we apply some recent results in recursion theory and algorithmic randomness theory to descriptive set theory and fractal geometry theory. Especially some deep results concerning the lowness properties for various recursion theory notations turned to be crucial to our proof. The so-called "lowness properties" is a kind of property preserving some algorithmic property that was considered as very unique in algorithmic randomness theory. For example, a real x is low for Turing jump (or just low) if $x' \equiv_T \emptyset'$, and a real x is called low for Schnorr random (for the definition of Schnorr randomness, see the paragraphs below Theorem 4.3) if every Schnorr random real is Schnorr random relative to x, etc. Ironically, different from the "slowdown" properties of themselves, these notions will be used to prove some "speedup" results. We expect to see more such applications in the near future.

In [15], a specific theorem (i.e., Theorem 6.6) is called the "point to set" principle. It can be viewed as a dual principle to the randomness in geometric measure theory.

We organize the paper as follows. In Section 2, we give some terminologies and notions. In Section 3, we sketch a recursion theoretical reformulation of Sami's proof that ZF + TD implies CH. The result will be used in Section 5. In Section 4, we prove $\mathrm{wDC}_{\mathbb{R}}$ within ZF + TD. In Section 5, we prove that ZF + sTD implies regularity properties for sets of reals. In the same section, we also prove a basis theorem for linear orderings over sets of reals within ZF + TD + $\mathrm{DC}_{\mathbb{R}}$ + "every uncountable set of reals has a perfect subset." In Section 6, we prove that the Besicovitch–Davis theorem holds for every set of reals within ZF + sTD.

§2. Terminologies and notions. We assume that readers have some knowledge of descriptive set theory and recursion theory. The major references are [2, 6, 10, 14, 19, 22].

2.1. Set theory. We assume that readers have some knowledge of axiomatic set theory. ZF is the Zermelo–Fraenkel axiom system. AD is the axiom of Determinacy.

When we say that $T \subseteq 2^{<\omega}$ is a tree, we mean that T is a tree without dead nodes. $[T]$ is the collection of infinite paths through T. Given any $x \in \omega^\omega$ and natural number n, we use $x \upharpoonright n$ to denote an initial segment of x with length n. In other words, $x \upharpoonright n$ is a finite string $\sigma \in \omega^{<\omega}$ of length n such that for any $i < n$, $\sigma(i) = x(i)$.

2.2. Recursion theory. We use $\leq_T$ to denote Turing reduction and $\leq_h$ to denote hyperarithmetic reduction. We use Φ^x to denote a Turing machine with oracle x. Sometimes we also say that Φ^x is a recursive functional. We fix an effective enumeration $\{\Phi^x_e\}_{e\in\omega}$ of recursive functionals.

We use Kleene's $\mathcal{O}$ and we write ω_1^{CK} for the least non-recursive ordinal, ω_1^x for the least ordinal not recursive in x.

We say a set A ranges Turing degrees cofinally if for every real x, there is some $y \geq_T x$ in A. We use x' to denote the Turing jump relative to x. More generally, if $\alpha < \omega_1^x$, then $x^{(\alpha)}$ is that α-th Turing jump of x. Then, $x \leq_h y$ if $x \leq_T y^{(\alpha)}$ for some $\alpha < \omega_1^y$.

The following fact is folklore and a sketched proof can be found in [20].

LEMMA 2.1. *Assume* ZF. *For any Turing degree* $\mathbf{x}$, *there is a family of Turing degrees* $\{\mathbf{y}_r \mid r \in \mathbb{R}\}$ *satisfying the following properties*:

(1) *For any* $r \in \mathbb{R}$, $\mathbf{x} < \mathbf{y}_r$.
(2) *For any* $r_0 \neq r_1 \in \mathbb{R}$ *and* $\mathbf{z} < \mathbf{y}_{r_0}, \mathbf{y}_{r_1}$, *we have that* $\mathbf{z} \leq \mathbf{x}$.
(3) *For any* $\mathbf{z} \geq \mathbf{x}''$, *the Turing double jump of* $\mathbf{x}$, *there is an infinite set* $C_{\mathbf{z}} \subset \mathbb{R}$ *such that* $\mathbf{y}_r'' = \mathbf{z}$ *for any* $r \in C_{\mathbf{z}}$.

§3. On Sami's theorem.

THEOREM 3.1 (Sami [23]). ZF + TD + DC *proves* CH.

In this section, we sketch a recursion theoretical proof of Theorem 3.1 to show that DC can be removed from the assumption, which was also observed by Sami. So we can give another proof of the following result.

PROPOSITION 3.2 (Sami, email communication between Sami and Yu in June 2021). ZF + TD *proves* CH.

PROOF. Given an uncountable set $A \subseteq \mathbb{R}$, by Lemma 2.1, for any real x, there is a real $y >_T x$ such that there is some real $r \in A$ Turing below y'' but not below y. So, by TD, there is some real z_0 such that for any $y \geq_T z_0$, there is some real $r \in A$ Turing below y'' but not below y.

Now it is simple to construct a $\Sigma^1_1(z_0)$ set B so that:

(i) For any $y \leq_h z_0$ and $x \in B$, we have that $y \leq_T x$.
(ii) For any $x_0 \neq x_1 \in B$, if $y \leq_h x_0, x_1$, then $y \leq_h z_0$.

To see the existence of such B, first note that the set $\{y \mid \forall r \leq_h z_0 (r \leq_T y)\}$ is an uncountable $\Sigma^1_1(z_0)$-set. Then one may construct a perfect set $P \subseteq B$ so that any two different members from P form a minimal pair over z_0 in the hyperarithmetic degree sense.

Now for every real $x \in B$, let $y_x = \Phi_e^{x''}$ where e is the least n such that $\Phi_n^{x''}$ is in A and not Turing below x. For any $x_0 \neq x_1 \in B$, if $y_{x_0} = y_{x_1}$, then by (ii), $y_{x_0} = y_{x_1} \leq_h z_0$. By (i), we have that $y_{x_0} = y_{x_1} \leq_T x_0$, which is a contradiction.

So $x \mapsto y_x$ is a 1–1 map from B to A. It is known that every uncountable analytic set has a perfect subset and so A has the same power as $\mathbb{R}$. ⊣

From the proof of Proposition 3.2, we can see the following fact that will be used later. Actually by the remark after the proof of [23, Theorem 1.3], Sami proves that f_n below can be chosen to be continuous. But we only need this weaker version here.

LEMMA 3.3 (Sami [23]). *Assume* ZF + TD. *For every uncountable set* A *of reals, there is a perfect set* B *of reals and a sequence of arithmetical functions* $\{f_n\}_{n\in\omega}$ *from* B *to* $\mathbb{R}$ *such that* $B \subseteq \bigcup_{n\in\omega} f_n^{-1}(A)$. *Moreover, restricted to* B, f_n *is* 1–1 *for every* n.

PROOF. Fix an effective enumeration of Turing functional $\{\Phi_n\}_{n\in\omega}$. Going to a subset, assume that in the proof of Proposition 3.2, B is perfect. Define $f_n : B \to \mathbb{R}$

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by

$$f_n(x) = \begin{cases} \uparrow, & (\exists m \Phi_n^{x''}(m) \text{ is not defined}) \vee (\Phi_n^{x''} \leq_T x), \\ \Phi_n^{x''}, & \text{otherwise.} \end{cases} \tag{1}$$

Clearly f_n is arithmetical for every n. We have that $B \subseteq \bigcup_{n\in\omega} f_n^{-1}(A)$. Moreover, if $x \in B$ and $f_n(x)$ is defined, then $f_n(x) \leq_T x'' \wedge f_n(x) \not\leq_T x$. Then by the same reason as in the proof of Proposition 3.2, f_n must be 1–1 on B. So $\{f_n\}_{n\in\omega}$ is as required. ⊣

§4. Weakly dependent choice. Throughout the section, we work within ZF + TD.

Definition 4.1. Weakly dependent choice for sets of reals, or $\mathrm{wDC}_{\mathbb{R}}$, is the statement that for every binary relation R over $\mathbb{R}$ with the property that the set $\{y \mid R(x, y)\}$ has positive inner measure for any real x, there is a sequence $\{x_n\}_{n\in\omega}$ of reals such that $\forall n R(x_n, x_{n+1})$.

Remark. $\mathrm{wDC}_{\mathbb{R}}$ is not a consequence of ZF. To see this, let V be a model of ZFC+GCH and $\kappa = \aleph_\omega^V$. Let G be $Col(\omega, <\kappa)$-generic over V. Let $A_n = \kappa^\omega \cap V[G \cap Col(\omega, <\aleph_n^V)]$. Let

$$M = HOD^{V[G]}_{\bigcup A_n \cup \{\langle A_n : n<\omega\rangle\}}.$$

Then in M:

- $2^\omega \cap M = 2^\omega \cap (\bigcup A_n)$. [See, e.g., the proof of [24, Theorem 6.69].]
- A_n is countable.
- Every countable subset of reals intersects at most finitely many $A_{n+1} \setminus A_n$'s.

For every $x \in 2^\omega \cap M$, let $n(x)$ be the least n such that $x \in A_n$. Then in M,

$$R = \{(x, y) \in 2^\omega \times 2^\omega : n(x) < n(y)\}$$

witnesses the failure of $\mathrm{wDC}_{\mathbb{R}}$.

Proposition 4.2. $\mathrm{wDC}_{\mathbb{R}}$ *does not imply* $\mathrm{CC}_{\mathbb{R}}$.

Proof. To see this, let $V = L[A]$ for some set A of ordinals, $\mathbb{P}$ be the random forcing over $2^{\omega\times\omega}$, G be $\mathbb{P}$-generic over V, and $r_G : \omega \times \omega \to 2$ be the induced random real. Let $x_n = r_G(n, \cdot) : \omega \to 2$ for each $n < \omega$ and $X = \{x_n : n < \omega\}$. Let

$$M = HOD^{V[G]}_{\{A\}\cup X\cup\{X\}}.$$

First note that any bijection $\pi : \omega \to \omega$ will induce a homeomorphism $\tilde{\pi}$ of $2^{\omega\times\omega}$ to itself by: $\tilde{\pi}(x)(n, m) = x(\pi(n), m)$. Moreover, $\tilde{\pi}$ preserves measure and X. Then the following hold in M.

(1) M satisfies $\mathrm{wDC}_{\mathbb{R}}$. [Let R be a binary relation in M satisfying the assumption of $\mathrm{wDC}_{\mathbb{R}}$. Find $a \in V$, $x_0, \dots, x_{m-1} \in X$ and formula φ such that xRy iff $V[G] \vDash \varphi(x, y, a, x_0, \dots, x_{m-1}, X)$. We may assume that there is no occurrence of x_i in φ since, e.g., we may view $L[A, x_0, \dots, x_{m-1}]$ as the ground model and $r_G \restriction_{[m,\omega)\times\omega}$ as the random real over $L[A, x_0, \dots, x_{m-1}]$. So

$$xRy \text{ iff } V[G] \vDash \varphi(x, y, a, X).$$

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Recall that $\mathbb{R} \cap V$ has full outer measure in $V[G]$. Consequently, for any $x \in \mathbb{R} \cap V$, there is some $y \in V$ such that xRy. We will be done if $R \cap (\mathbb{R} \cap V)^2 \in V$. Now it suffices to prove that for any $x, y \in \mathbb{R} \cap V$, for any $p, q \in \mathbb{P}$,

$$p \Vdash \varphi(x, y, a, \dot{X}) \text{ iff } q \Vdash \varphi(x, y, a, \dot{X}).$$

Suppose towards a contradiction that $p \Vdash \varphi(x, y, a, \dot{X})$ and $q \Vdash \neg\varphi(x, y, a, \dot{X})$. Find finite unions of basis open sets O_p and O_q such that for some n:

- $\mu(O_p) = \mu(O_q) = \epsilon$ for some $\epsilon > 0$;
- $\mu(O_p \setminus p) + \mu(O_q \setminus q) < \epsilon^2$;
- $O_p = \bigcup_{i<k} O_{\sigma_i}$ and each $\sigma_i : n \times \omega \to 2$ is a finite partial map;
- $O_q = \bigcup_{i<k^*} O_{\tau_i}$ and each $\tau_i : n \times \omega \to 2$ is a finite partial map.

Fix a bijection $\pi : \omega \to \omega$ such that $\pi[[0, n)] \subset [n, \omega)$. Then

$$\mu(p \cap \tilde{\pi}(q)) \geq \mu(O_p \cap \tilde{\pi}[O_q]) - \mu(O_p \setminus p) - \mu(\tilde{\pi}[O_q \setminus q]) > 0.$$

But recall that $\tilde{\pi}[\dot{X}] = \dot{X}$. So

$$p \cap \tilde{\pi}(q) \Vdash \varphi(x, y, a, \dot{X}) \wedge \neg\varphi(x, y, a, \dot{X}).$$

A contradiction.]

(2) There is no injection from ω to X. [Suppose otherwise, f is an injection. Then for some $a \in V$, $x_0, \dots, x_{m-1} \in X$ and formula φ, for any $(n, x) \in \omega \times X$, $f(n) = x$ iff $V[G] \vDash \varphi(n, x, a, x_0, \dots, x_{m-1}, X)$. Choose n such that $f(n) = x_k \notin \{x_0, \dots, x_{m-1}\}$ and $p \in G$ such that $p \Vdash \varphi(n, \dot{x}_k, a, \dot{x}_0, \dots, \dot{x}_{m-1}, \dot{X})$. It is straightforward to find a bijection $\pi : \omega \to \omega$ such that π is identity on $\{0, \dots, m-1\}$, $\pi(k) \neq k$ and $\mu(p \cap \tilde{\pi}[p]) > 0$. But then $p \cap \tilde{\pi}[p] \Vdash \dot{x}_k = \dot{f}(n) = \dot{x}_{\pi(k)}$. A contradiction.]

So M does not satisfy $\mathrm{CC}_{\mathbb{R}}$. To see this, find a sequence of disjoint rational intervals $\langle I_n : n < \omega \rangle$ such that $I_n \cap X \neq \emptyset$ for each n. Then by (2), $\langle I_n \cap X : n < \omega \rangle$ admits no choice function. ⊣

If we use Cohen forcing instead of random forcing in the above argument, then we conclude that the category version of $\mathrm{wDC}_{\mathbb{R}}$ does not imply $\mathrm{CC}_{\mathbb{R}}$. But we do not know if $\mathrm{CC}_{\mathbb{R}}$ implies $\mathrm{wDC}_{\mathbb{R}}$.

THEOREM 4.3. ZF + TD *implies* $\mathrm{wDC}_{\mathbb{R}}$.

We remark that if "having positive inner measure" is replaced by having Baire property and non-meager in the definition of $\mathrm{wDC}_{\mathbb{R}}$, then the theorem still holds.

A Schnorr test relative to x is an x-recursive sequence of x-recursive open sets $\{V_n\}_{n\in\omega}$ such that $\forall n \mu(V_n) = 2^{-n}$. A real r is called *x-Schnorr random* if $r \notin \bigcap_{n\in\omega} V_n$ for any Schnorr test $\{V_n\}_{n\in\omega}$ relative to x. If x is recursive, then we simply use Schnorr randomness instead of x-Schnorr randomness. It is not difficult to see that there is a Schnorr random $r \leq_T \emptyset'$. And it is clear from the definition that if r is x-Schnorr random and $z \leq_T x$, then r is also z-Schnorr random. Also note that if $x \geq_T \emptyset'$ and r is x-Schnorr random, then x is Turing incomparable with r.

A real x is called *low for Schnorr random* if every Schnorr random real is Schnorr random relative to x. The following theorem, which was proved by Sacks forcing, is due to Terwijn and Zambella.

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$$\begin{array}{ccccccc} r_0 \leq_T \Phi_{e_0}^{y_0''} & \xrightarrow{\quad R \quad} & r_1 \leq_T \Phi_{e_0}^{y_1''} & \xrightarrow{\quad R \quad} & r_2 \leq_T \Phi_{e_0}^{y_2''} & \xrightarrow{\quad R \quad} & \dots \\ | & & | & & | & & \\ y_0\text{-Schnorr Random} & & y_0''\text{-Schnorr Random} & & y_1''\text{-Schnorr Random} & & \cdots \end{array}$$

FIGURE 1. $R(r_n, r_{n+1})$.

THEOREM 4.4 (Terwijn and Zambella [27]). *For every real $y \geq_T \emptyset''$, there is a real x low for Schnorr random such that $x'' \equiv_T y$.*

PROOF OF THEOREM 4.3. Fix a binary relation R as stated in $\mathrm{wDC}_{\mathbb{R}}$. To prove $\mathrm{wDC}_{\mathbb{R}}$, we may assume that for any real x, the set $R_x = \{y \mid R(x,y)\}$ is upward closed under Turing reduction (and so R_x is co-null for any x). I.e., for any y and z, if $y \leq_T z$ and $y \in R_x$, then $z \in R_x$. To see this, we may define a new relation $\tilde{R}$ so that $\tilde{R}(x,y)$ if and only if $\forall z_0 \leq_T x \exists z_1 \leq_T y\ R(z_0, z_1)$. Then for every real x, the set $\tilde{R}_x = \{y \mid \tilde{R}(x,y)\}$ is upward closed under Turing reduction and has positive measure, and so co-null. Moreover, if there is a sequence $\{y_n\}_{n\in\omega}$ such that $\forall n \tilde{R}(y_n, y_{n+1})$. Then we build a sequence $\{x_n\}_{n\in\omega}$ so that $\forall n R(x_n, x_{n+1})$ as follows.

First let $x_0 = y_0$. By the definition of $\tilde{R}$, we may choose the least m_1 such that $\Phi_{m_1}^{y_1}$ is defined and $R(x_0, \Phi_{m_1}^{y_1})$. Let $x_1 = \Phi_{m_1}^{y_1}$. Generally, if x_n is defined, then $x_n \leq_T y_n$. So by the definition of $\tilde{R}$, we may choose the least index m_{n+1} such that $\Phi_{m_{n+1}}^{y_{n+1}}$ is defined and $R(x_n, \Phi_{m_{n+1}}^{y_{n+1}})$. Set $x_{n+1} = \Phi_{m_{n+1}}^{y_{n+1}}$. Then we have that $\forall n R(x_n, x_{n+1})$.

Now fix a real z. Note that by the assumption on R, $\bigcap_{y \leq_T z'} R_y$ is co-null. Then by Fact 1.10, there is a real $z_0 \geq_T z'$ such that for every $y \leq_T z'$ and every z_0-Schnorr random r, $R(y,r)$. Also by relativizing Theorem 4.4 to z, there is a real $x >_T z$ low for z-Schnorr random such that $x'' \geq_T z_0$. So for every $y \leq_T z'$ and every x''-Schnorr random r, $R(y,r)$. Also note that there is a z-Schnorr random, and so x-Schnorr random, real $r \leq_T z'$. Since $x'' \geq_T z'$, there is some index of Turing functional e such that $\Phi_e^{x''} = z'$. For every number $e \in \omega$, define the set

$$\begin{aligned} A_e = \{x \mid \exists r (r \text{ is } x\text{-Schnorr random } \wedge r \leq_T \Phi_e^{x''}) \\ \wedge \forall r_0 \leq_T \Phi_e^{x''} \forall r_1 (r_1 \text{ is } x''\text{-Schnorr random} \rightarrow R(r_0, r_1))\}. \end{aligned}$$

Then by the discussion above, $\bigcup_{e\in\omega} A_e$ ranges Turing degrees cofinally. So there must be some e_0 such that A_{e_0} ranges Turing degrees cofinally. By TD, there is some x_0 such that every $y \geq_T x_0$ is $\equiv_T$ equivalent to some y_0 in A_{e_0}. We may assume that $x_0 \in A_{e_0}$. Recursively in $x_0^{(\omega)}$, we first find a sequence of reals

$$\{y_n \in A_{e_0} \mid n < \omega \wedge y_n \equiv_T x_0^{(2n)}\}.$$

Then find a sequence of reals $\{r_n\}_{n\in\omega}$ so that for every n, $r_n \leq_T \Phi_{e_0}^{y_n''}$ is $y_n \equiv_T x_0^{(2n)}$-Schnorr random. Note that for every n, $r_n \leq_T \Phi_{e_0}^{y_n''}$ and r_{n+1} is $x_0^{(2n+2)} \equiv_T y_n''$-Schnorr random (see Figure 1). So by the definition of A_{e_0}, $R(r_n, r_{n+1})$. ⊣

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The reason we choose Schnorr randomness, instead of Martin-Löf randomness that is the standard randomness notion, is that every low for Martin-Löf random real is Turing below $\emptyset'$. So for any real x low for Martin-Löf random, there is no way to make the Turing jumps of x be very high.

§5. Regularity properties of sets of reals. In this section, we prove some regularity properties for sets of reals under $\mathrm{ZF} + \mathrm{sTD}(+\mathrm{DC}_{\mathbb{R}})$. Woodin already considered sTD long time ago. All the results in Sections 5.1 and 5.2 have been known to him (email communication between Woodin and Yu in April 2021).

THEOREM 5.1 (Woodin, unpublished). (1) $\mathrm{ZF} + \mathrm{sTD}$ *implies that every set is measurable and has Baire property.*
(2) $\mathrm{ZF} + \mathrm{sTD}$ *implies that every uncountable set of reals has a perfect subset.*

5.1. The proof of part (1). We only prove that every set is measurable and leave the second part to readers.

It suffices to prove that for any set A, if every measurable subset of A is null, then A must be null. Now suppose, for a contradiction, that every measurable subset of A is null but A is not null. Then, by Fact 1.10 with Schnorr randomness, for every real z, there is a z-Schnorr random real z_0 in A. By Theorem 4.4 relative to z, there is a real $x \geq_T z$ low for z-Schnorr random and $x'' \geq_T z_0$.

Now for every $e \in \omega$, let

$$B_e = \{x \mid \Phi_e^{x''} \in A \text{ is an } x\text{-Schnorr random real}\}.$$

The argument in the previous paragraph shows that $\bigcup_{e\in\omega} B_e$ ranges Turing degrees cofinally. Then there is some e_0 such that B_{e_0} ranges Turing degrees cofinally. By sTD, there is a pointed subset $P \subseteq B_{e_0}$.

Let

$$C = \{r \mid \exists x \in P(\Phi_{e_0}^{x''} = r)\}.$$

C is an analytic set and so measurable. Since P is a pointed set, by the definition of B_{e_0} and Fact 1.10 with Schnorr randomness, C is not null. This is absurd.

REMARK. Clearly the proof can be localized if we assume $\mathrm{CC}_{\mathbb{R}}$, as pointed out by one of the referees. For example, assuming $\mathrm{ZF} + \mathrm{CC}_{\mathbb{R}}$, if sTD holds for $\boldsymbol{\Sigma}^1_n$-sets, then every $\boldsymbol{\Sigma}^1_n$-set is measurable.

5.2. The proof of part (2). We first prove the following lemma.

LEMMA 5.2. *Assume* $\mathrm{ZF} + \mathrm{sTD}$. *For every perfect set P of reals and every partition $P = \bigcup_{n<\omega} B_n$, there exists n such that B_n has a perfect subset.*

PROOF. Clearly we may assume that $P = 2^\omega$ via a homeomorphism. Then for some n, B_n ranges Turing degrees cofinally. By sTD, B_n contains a perfect subset. ⊣

PROOF OF PART (2) OF THEOREM 5.1. Suppose that A is uncountable. By Lemma 3.3, we may fix a perfect set B and a sequence of functions $\{f_n\}_{n\in\omega}$ as in the lemma. Then by Lemma 5.2, we can choose a perfect $Q \subset f_n^{-1}[A]$ for some n. Now $f_n[Q]$

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is an uncountable analytic subset of A. So $f_n[Q]$ and hence A contains a perfect subset. $\dashv$

REMARK. We would like to thank the referee who pointed out to us that the proof derived the result that ZF + TD + "every set of reals has Baire property" implies every uncountable set of reals has a perfect subset.

Here we mention another approach, within $\mathrm{ZF} + \mathrm{sTD} + \mathrm{DC}_{\mathbb{R}}$, to get a perfect subset due to Sami. A set A of reals is called *Bernstein* if neither A nor $\mathbb{R} \setminus A$ has a perfect subset. Notice that the nonexistence of a Bernstein set implies that for every perfect set P and its subset $A \subseteq P$, either A or $P \setminus A$ has a perfect subset. Sami observed the following relationship between the existence of a Bernstein set and perfect subset property.

LEMMA 5.3 (Sami [23]). *Assume* $\mathrm{ZF} + \mathrm{TD} + \mathrm{DC}_{\mathbb{R}}$. *If there is no Bernstein set, then every uncountable set of reals has a perfect subset.*

PROOF. Suppose that A is uncountable. By Lemma 3.3, we may fix a perfect set B and a sequence of functions $\{f_n\}_{n\in\omega}$ as in the lemma.

Let $T^0 \subseteq 2^{<\omega}$ be a perfect tree such that $[T^0] = B$. We will inductively choose a decreasing sequence of perfect trees $T^0 \supset T^1 \supset \cdots \supset T^n \supset \cdots$ until the procedure terminates. Suppose that T^n has been chosen and we choose T^{n+1}.

Case (1). There is some perfect tree $T^* \subseteq T^n$ such that f_n is defined on every member in $[T^*]$ and $f_n([T^*]) \subseteq A$. Fix such T^*. Then $f_n([T^*]) \subseteq A$ is an uncountable analytic set. Thus A must have a perfect subset. The procedure terminates and we are done.

Case (2). Otherwise. Then by the assumption of no Bernstein set, choose T^{n+1} to be a perfect subtree of T^n such that for every $x \in [T^{n+1}]$, either $f_n(x)$ is not defined or $f_n(x) \notin A$.

Either we stop at Case (1) of some n, then we find a perfect subset of A. Or else, the construction goes through all of n's. Then $\bigcap_{n<\omega}[T^n]$ is non-empty. Moreover, for every $x \in \bigcap_{n<\omega}[T^n]$ and every n, either $f_n(x)$ is not defined or $f_n(x) \notin A$. This contradicts the fact that $B \subseteq \bigcup_{n\in\omega} f_n^{-1}(A)$.

Thus we must stop at Case (1) of some n and so A must have a perfect subset. $\dashv$

sTD implies that every set is measurable and so there is no Bernstein set. Thus $\mathrm{ZF} + \mathrm{sTD} + \mathrm{DC}_{\mathbb{R}}$ implies every uncountable set of reals has a perfect subset.

5.3. An application of regularity properties to linear orderings over $\mathbb{R}$.

LEMMA 5.4. *Assume* $\mathrm{ZF} + \mathrm{CC}_{\mathbb{R}}$ + *"every set of reals is measurable." Given any linear order $\leq_L$ of $\mathbb{R}$ and any non-null set $A \subseteq \mathbb{R}$, the collection of all $x \in A$ such that either $\{y \in A \mid y \leq_L x\}$ is null or $\{y \in A \mid x \leq_L y\}$ is null is a null set.*

PROOF. Given a linear order $\leq_L$ over $\mathbb{R}$, let $A \subseteq \mathbb{R}$ be any non-null set. Fix a non-null set $B \subseteq A$. By Fubini's theorem, the set $\{(x, y) \mid x \leq_L y \wedge x \in B \wedge y \in B\}$ is measurable and has positive measure. Let

$$L^B = \{x \in B \mid \{y \in A \mid y \leq_L x\} \text{ is null}\}$$

be a subset of B. Then by Fubini's theorem again, the set $B \setminus L^B$ is not null. So the set

$$L^A = \{x \in A \mid \{y \in A \mid y \leq_L x\} \text{ is null}\}$$

is a null subset of A.

By the same method, the set

$$R^A = \{x \in A \mid \{y \in A \mid x \leq_L y\} \text{ is null}\}$$

is also a null subset of A. ⊣

Finally, we have the following basis theorem for linear orderings over $\mathbb{R}$ under $\mathrm{ZF} + \mathrm{sTD}$. In what follows, we use $\leq$ to denote the usual order on reals. Since the lexicographic order on 2^ω is order isomorphic to the real order on the Cantor set through the standard bijection, we also use $\leq$ to denote this order on 2^ω.

THEOREM 5.5. *Assume* $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$ + *"every set of reals is measurable." For every linear order* $\leq_L$ *over* $\mathbb{R}$, *there is an order-preserving embedding from* $(2^\omega, \leq)$ *to* $(\mathbb{R}, \leq_L)$.

PROOF. First we set $P_\emptyset = [0, 1]$.

By Lemma 5.4, there is a real $x \in P_\emptyset$ such that both the sets $\{y \in A \mid y \leq_L x\}$ and $\{y \in A \mid x \leq_L y\}$ have positive measure. So both of them have disjoint perfect subsets P_0 and P_1 with positive measure, respectively. Moreover, we may require that for any $i \in \{0, 1\}$ and $y, z \in P_i$, $|y - z| \leq 2^{-1}$.

Now, by an induction, it is not difficult to construct a sequence $\{P_\sigma\}_{\sigma \in 2^{<\omega}}$ of perfect sets so that:

- If σ extends τ, written as $\sigma \succ \tau$, then $P_\sigma \subset P_\tau$ has positive measure.
- If σ and τ are incompatible, then $P_\sigma \cap P_\tau = \emptyset$.
- If for some n, $\sigma \upharpoonright_n = \tau \upharpoonright_n$ and $\sigma(n) < \tau(n)$, then $\forall x \in P_\sigma \forall y \in P_\tau (x \leq_L y)$.
- For any σ and $x, y \in P_\sigma$, $|x - y| \leq 2^{-|\sigma|}$.

Define $f : 2^\omega \to \mathbb{R}$ so that $f(x)$ is the unique real in $\bigcap_n P_{x \upharpoonright n}$. Then f is an order-preserving embedding from $(2^\omega, \leq)$ to $(\mathbb{R}, \leq_L)$. ⊣

One may wonder what happens to Lemma 5.4 under $\mathrm{ZF} + \mathrm{TD}$. Since it is unknown whether $\mathrm{ZF} + \mathrm{TD}$ implies that every set of reals is measurable, we have to use a more involved argument.

DEFINITION 5.6. A linear order $(L, \leq_L)$ is *locally countable* if for any $l \in L$, the set $\{x \leq_L l \mid x \in L\}$ is countable.

A typical locally countable order is $(\omega_1, \leq)$.

For a set A of reals that is closed under Turing equivalence relation, a real x is a *minimal upper bound* of A if:

- every member of A is recursive in x; and
- there is no real $y <_T x$ such that every member of A is recursive in y.

By a classical theorem in recursion theory (see Theorem 4.11 in [14]), every countable set of reals has a minimal upper bound.

LEMMA 5.7. *Assume* $\mathrm{ZF} + \mathrm{TD}$. *There is no uncountable set* $A \subseteq \mathbb{R}$ *with a locally countable linear order over* A.

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PROOF. By Proposition 3.2, it suffices to prove that there is no locally countable linear order on $\mathbb{R}$.

Suppose not. Let $(\mathbb{R}, \leq_L)$ be a locally countable order. For every real x, let I_x be the Turing downward closure of the set $\{z \mid z \leq_L x\}$. I.e.,

$$I_x = \{s \mid \exists z \leq_L x(s \leq_T z)\}.$$

Obviously $x \leq_L y$ implies $I_x \subseteq I_y$.

Note that for any real z, there is a real x such that $z \in I_x$. So there is a real $z_0 \geq_T z$ such that z_0 is a minimal upper bound of I_x. By TD, there is a real z_1 such that every real $z_2 \geq_T z_1$ is a minimal upper bound over I_x for some x.

For every real z, let

$$M_z = \{x \mid z \text{ is a minimal upper bound of } I_x)\}$$

and

$$N_z = \bigcup_{x \in M_z} I_x.$$

Note that M_{z_2} is nonempty for every $z_2 \geq_T z_1$. We have the following fact:

- For any $z_2, z_3 \geq_T z_1$, either $N_{z_3} \subseteq N_{z_2}$ or $N_{z_2} \subseteq N_{z_3}$. [To see this, suppose that $N_{z_3} \not\subseteq N_{z_2}$. Then there must be some $x_3 \in M_{z_3}$ such that for any $x_2 \in M_{z_2}$, $x_3 \not\leq_L x_2$. In other words, $x_2 \leq_L x_3$ for any $x_2 \in M_{z_2}$. So $N_{z_2} \subseteq N_{z_3}$.]

Now fix a pair of minimal covers $z_2 \not\equiv_T z_3$ of z_1 (i.e., for $i \in \{2, 3\}$, $z_i >_T z_1$ but there is no real y strictly between z_1 and z_i in the Turing reduction order sense. For the existence of such a pair, see Lemma 2.1). By the fact above, WLOG, we may assume $N_{z_2} \subseteq N_{z_3}$ and fix some $x \in M_{z_2}$. Then every real in $I_x \subseteq N_{z_2} \subseteq N_{z_3}$ is recursive in both z_2 and z_3. So every real in I_x is recursive in z_1. This contradicts the fact that z_2 is a minimal upper bound of I_x and $z_1 <_T z_2$. ⊣

COROLLARY 5.8. *Assume* ZF + TD. *For every uncountable set $A \subseteq \mathbb{R}$ and linear order $\leq_L$ over A, there are uncountably many reals $x \in A$ such that both $\{y \in A \mid y \leq_L x\}$ and $\{y \in A \mid x \leq_L y\}$ are uncountable.*

PROOF. Given a linear order $\leq_L$ over $\mathbb{R}$, let

$$L = \{x \in A \mid \{y \in A \mid y \leq_L x\} \text{ is countable}\}$$

and

$$R = \{x \in A \mid \{y \in A \mid x \leq_L y\} \text{ is countable}\}.$$

By Lemma 5.7, both L and R are countable. So there are uncountably many reals $x \in A$ such that both $\{y \mid y \leq_L x\}$ and $\{y \mid x \leq_L y\}$ are uncountable. ⊣

Now we may obtain the following result.

THEOREM 5.9. *Assume* ZF + TD + $\mathrm{DC}_{\mathbb{R}}$. *The following are equivalent.*

1. *Every uncountable set of reals has a perfect subset.*
2. *For every linear order $\leq_L$ over $\mathbb{R}$, there is an order-preserving embedding from $(2^\omega, \leq)$ to $(\mathbb{R}, \leq_L)$.*

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PROOF. (1)⇒(2). The argument of Theorem 5.5 works here. Just replace "set with positive measure" by "uncountable set."

(2)⇒(1). Fix an uncountable set of reals A. By Proposition 3.2, $|A| = |\mathbb{R}|$. So $(A, \leq)$ is order isomorphic to $(\mathbb{R}, \leq_L)$ for some $\leq_L$. By (2), there is an order-preserving map from $(2^\omega, \leq)$ to $(\mathbb{R}, \leq_L)$ and hence $(A, \leq)$.

Fix $\pi : 2^\omega \to A$ that preserves order and so is monotonic. Then π is continuous on all but countably many points. In particular, π is continuous (and injective) on a perfect subset P. So $\pi[P]$ is a perfect subset of A. ⊣

§6. Regularity properties for dimension theory. For the notions and terminologies in fractal geometry, we follow the book [7].

Given a non-empty $U \subseteq \mathbb{R}$, the *diameter* of U is

$$diam(U) = |U| = \sup\{|x - y| : x, y \in U\}.$$

Given any set $E \subseteq \mathbb{R}$ and $d \geq 0$, let

$$\mathcal{H}^d(E) = \lim_{\delta\to 0} \inf\{\sum_{i<\omega} |U_i|^d : \{U_i\} \text{ is an open cover of } E \wedge \forall i\ |U_i| < \delta\},$$

$$\mathcal{P}_0^d(E) = \lim_{\delta\to 0} \sup\{\sum_{i<\omega} |B_i|^d : \{B_i\} \text{ is a collection of disjoint balls of radii at most } \delta \text{ with centres in } E\},$$

and

$$\mathcal{P}^d(E) = \inf\{\sum_{i<\omega} \mathcal{P}_0^d(E_i) \mid E \subseteq \bigcup_{i<\omega} E_i\}.$$

DEFINITION 6.1. Given any set E:

(1) The *Hausdorff dimension* of E, or $\mathrm{Dim_H}(E)$, is

$$\inf\{d \mid \mathcal{H}^d(E) = 0\}.$$

(2) The *Packing dimension* of E, or $\mathrm{Dim_P}(E)$, is

$$\inf\{d \mid \mathcal{P}^d(E) = 0\}.$$

By the same reason as in the Lebesgue measure, it can be proved with $\mathrm{ZF} + \mathrm{CC}_\mathbb{R}$ that for every Borel set B and $\epsilon > 0$, there is a closed set $F \subseteq B$ such that $\mathrm{Dim_H}(F) > \mathrm{Dim_H}(B) - \epsilon$.

THEOREM 6.2 (Besicovitch [1] and Davis [5]). *For every analytic set A and every $\epsilon > 0$, there is a closed set $F \subseteq A$ such that* $\mathrm{Dim_H}(F) \geq \mathrm{Dim_H}(A) - \epsilon$.

THEOREM 6.3 (Joyce and Preiss [11]). *For every analytic set A and every $\epsilon > 0$, there is a closed set $F \subseteq A$ such that* $\mathrm{Dim_P}(F) \geq \mathrm{Dim_P}(A) - \epsilon$.

However Slaman proves that both Theorems 6.2 and 6.3 may fail even for some Π^1_1 set under certain assumptions.

THEOREM 6.4 (Slaman [25]). *Suppose that the set of constructible reals is not null, then there is a Π^1_1 set C with* $\mathrm{Dim_H}(C) = 1$ *but for any Borel $F \subset C$*, $\mathrm{Dim_P}(F) = 0$.

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We prove that Theorems 6.2 and 6.3 both remain true for all sets of reals under ZF + sTD. As pointed out by one of the referees, the following result can be localized similarly as the localization of Theorem 5.1.

THEOREM 6.5. ZF + sTD *implies that for every set of reals A and every $\epsilon > 0$:*

(1) *There is a closed set $F \subseteq A$ such that* $\mathrm{Dim}_{\mathrm{H}}(F) \geq \mathrm{Dim}_{\mathrm{H}}(A) - \epsilon$.
(2) *There is a closed set $F \subseteq A$ such that* $\mathrm{Dim}_{\mathrm{P}}(F) \geq \mathrm{Dim}_{\mathrm{P}}(A) - \epsilon$.

To show the theorem, we use the "point-to-set" method.

Some more facts from algorithmic randomness theory are needed. Let K denote the prefix-free Kolmogorov complexity. We use K^x to denote the prefix-free Kolmogorov complexity with oracle, which is a real, x. The following "point to set" style theorem is due to Lutz and Lutz. A similar form was also discovered by Cutler. See [4, Theorem 1.4].

THEOREM 6.6 (Lutz and Lutz [15]). *For every set $A \subseteq \mathbb{R}$,*

$$\mathrm{Dim}_{\mathrm{H}}(A) = \inf_{x\in\mathbb{R}} \sup_{y\in A} \underline{\lim}_{n\to\infty} \frac{K^x(y\restriction n)}{n}$$

and

$$\mathrm{Dim}_{\mathrm{P}}(A) = \inf_{x\in\mathbb{R}} \sup_{y\in A} \overline{\lim}_{n\to\infty} \frac{K^x(y\restriction n)}{n}.$$

The following lowness property is crucial to our proof.

THEOREM 6.7 (Herbert [9]; Lempp, Miller, Ng, Turetsky, Weber [13]).

- *There is a perfect tree $T \subseteq 2^{<\omega}$ recursive in $\emptyset'$ such that for any real $x \in [T]$,*

$$\forall y \in \mathbb{R}\left(\underline{\lim}_{n\to\infty} \frac{K(y\restriction n)}{n} = \underline{\lim}_{n\to\infty} \frac{K^x(y\restriction n)}{n}\right).$$

- *There is a perfect tree $T \subseteq 2^{<\omega}$ recursive in $\emptyset'$ such that for any real $x \in [T]$,*

$$\forall y \in \mathbb{R}\left(\overline{\lim}_{n\to\infty} \frac{K(y\restriction n)}{n} = \overline{\lim}_{n\to\infty} \frac{K^x(y\restriction n)}{n}\right).$$

Now we are ready to prove our major theorem of this section.

PROOF OF THEOREM 6.5.

(1). Suppose that $A \subseteq \mathbb{R}$ with $\mathrm{Dim}_{\mathrm{H}}(A) > 0$. Fix any $\epsilon > 0$. By Theorem 6.6, for every real z, there is some real $x \in A$ such that

$$\underline{\lim}_{n\to\infty} \frac{K^z(x\restriction n)}{n} > \mathrm{Dim}_{\mathrm{H}}(A) - \frac{\epsilon}{2}.$$

By Theorem 6.7 relative to z, there is a real $y >_T z$ such that

$$\underline{\lim}_{n\to\infty} \frac{K^y(x\restriction n)}{n} = \underline{\lim}_{n\to\infty} \frac{K^z(x\restriction n)}{n} > \mathrm{Dim}_{\mathrm{H}}(A) - \frac{\epsilon}{2} \wedge y' >_T x.$$

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So there must be some e_0 such that the set

$$B_{e_0} = \left\{ y \mid \Phi_{e_0}^{y'} \in A \wedge \underline{\lim}_{n\to\infty} \frac{K^y(\Phi_{e_0}^{y'}\upharpoonright n)}{n} > \mathrm{Dim}_{\mathrm{H}}(A) - \frac{\epsilon}{2} \right\}$$

ranges Turing degrees cofinally. By sTD, there is a pointed set $P \subseteq B_{e_0}$.

Then the set

$$C = \{x \mid \exists y \in P(\Phi_{e_0}^{y'} = x)\}$$

is an analytic subset of A. By Theorem 6.6,

$$\mathrm{Dim}_{\mathrm{H}}(C) > \mathrm{Dim}_{\mathrm{H}}(A) - \frac{\epsilon}{2}.$$

By Theorem 6.2, C has a closed subset F such that

$$\mathrm{Dim}_{\mathrm{H}}(F) > \mathrm{Dim}_{\mathrm{H}}(C) - \frac{\epsilon}{2}.$$

Thus

$$\mathrm{Dim}_{\mathrm{H}}(F) > \mathrm{Dim}_{\mathrm{H}}(A) - \epsilon.$$

(2). The same proof as (1) except replacing Hausdorff dimension with packing dimension. We leave the details to readers. ⊣

To continue our study, we need the following folklore technique lemma of which we sketch a proof for the completeness.

LEMMA 6.8 (Folklore). *Assume* ZF + sTD. *If* $f : \mathbb{R} \to Ord$ *is a degree invariant* (*i.e.*, $x \equiv_T y \implies f(x) = f(y)$) *map such that* $f(x) < \omega_1^x$, *then there is an ordinal* α *such that* $f(x) = \alpha$ *over an upper cone of Turing degrees.*

PROOF. Fix such a map f. Since there are countably many recursive functionals, by sTD, there is some recursive functional Φ such that Φ^x codes a linear order for every real x, and a pointed set P such that $f(x) \cong \Phi^x$ for any $x \in P$. Let T be a tree representing P such that $\forall x \in P(T \leq_T x)$. Then the set

$$\{\Phi^x \mid x \in P\}$$

is a $\Sigma_1^1(T)$ set and so Φ^x represents an ordinal smaller than ω_1^T for any $x \in P$ by Σ_1^1-boundedness relative to T (see [2]). By sTD again, there must be some $\alpha < \omega_1^T$ and a pointed set $Q \subseteq P$ such that $f(x) = \alpha$ for any $x \in Q$. This finishes the proof. ⊣

Crone, Fishman, and Jackson also proved the following result.

THEOREM 6.9 (Crone, Fishman, and Jackson [3]). *Assume* ZF + AD + DC. *If* $A = \bigcup_{\alpha<\kappa} A_\alpha$ *for some ordinal* κ, *then* $\mathrm{Dim}_{\mathrm{H}}(A) = \sup\{\mathrm{Dim}_{\mathrm{H}}(A_\alpha) \mid \alpha < \kappa\}$.

We may provide an "elementary" proof of the following weaker result under ZF + sTD.

THEOREM 6.10. *Assume* ZF + sTD. *If* $A = \bigcup_{\alpha<\omega_1} A_\alpha$, *then*

$$\mathrm{Dim}_{\mathrm{H}}(A) = \sup\{\mathrm{Dim}_{\mathrm{H}}(A_\alpha) \mid \alpha < \omega_1\} \text{ and } \mathrm{Dim}_{\mathrm{P}}(A) = \sup\{\mathrm{Dim}_{\mathrm{P}}(A_\alpha) \mid \alpha < \omega_1\}.$$

Proof. Let $r = \mathrm{Dim}_{\mathrm{H}}(A)$ and for every real x,

$$\gamma_x = \min\left\{\gamma \mid \sup_{y\in\bigcup_{\alpha<\gamma} A_\alpha} \underline{\lim}_{n\to\infty}\frac{K^x(y\restriction n)}{n} \geq r\right\}.$$

By Theorem 6.6, γ_x is defined for every real x.

For any real z, by Theorem 6.7 and the assumption, there is a real $x >_T z$ such that $\gamma_x = \gamma_z$ but $\omega_1^{x'} > \gamma_z$. So

$$\gamma_x = \gamma_z < \omega_1^{x'} = \omega_1^x.$$

In other words, $x \mapsto \gamma_x$ is a degree invariant function such that $\gamma_x < \omega_1^x$ over an upper cone of Turing degrees. Then by Lemma 6.8, $x \mapsto \gamma_x$ is a constant, say η, over an upper cone. Then, by the countability of η, for every $m \in \omega$, there must be some $\alpha_m < \eta$ such that the set $\{x \mid \sup_{y\in A_{\alpha_m}} \underline{\lim}_{n\to\infty}\frac{K^x(y\restriction n)}{n} \geq r - \frac{1}{m}\}$ ranges Turing degrees cofinally. Then by Theorem 6.6, $\mathrm{Dim}_{\mathrm{H}}(A_{\alpha_m}) \geq r - \frac{1}{m}$. So

$$\mathrm{Dim}_{\mathrm{H}}(A) = \sup\{\mathrm{Dim}_{\mathrm{H}}(A_\alpha) \mid \alpha < \eta\} = \sup\{\mathrm{Dim}_{\mathrm{H}}(A_\alpha) \mid \alpha < \omega_1\}.$$

We leave the proof of the second part to readers. ⊣

Some remarks:

- The conclusion of Theorem 6.10 can also be proved within $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$. Furthermore, ω_1 can be replaced by any cardinal $\kappa < 2^{\aleph_0}$.
- We don't know the consistency strength of the conclusions of Theorem 6.5.

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