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P-HARMONIC DIMENSIONS ON ENDS

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Dedicated to Professor Masanori Kishi on his sixtieth birthday

Consider an *end* Ω in the sense of Heins (cf. Heins [3]): Ω is a relatively noncompact subregion of an open Riemann surface such that the relative boundary $\partial\Omega$ consists of finitely many analytic Jordan closed curves, there exist no nonconstant bounded harmonic functions with vanishing boundary values on $\partial\Omega$ and Ω has a single ideal boundary component. A *density* $P = P(z)dxdy$ $(z = x + iy)$ is a 2-form on $\Omega \cup \partial\Omega$ with nonnegative locally Hölder continuous coefficient $P(z)$. Denote by $\mathscr{P}_P(\Omega)$ the class of nonnegative solutions of the equation

$$L_P u \equiv \Delta u - Pu = 0 \quad (\text{i.e.} \quad d*du - uP = 0) \tag{1}$$

on Ω with vanishing boundary values on $\partial\Omega$. The *P-harmonic dimension* of Ω (or the *elliptic dimension* of P on Ω (cf. e.g. Nakai [8])), $\dim \mathscr{P}_P(\Omega)$ in notation, is defined to be the 'dimension' of the half module $\mathscr{P}_P(\Omega)$. The P-harmonic dimension $\dim \mathscr{P}_0(\Omega)$ for the particular $P \equiv 0$ is called simply the *harmonic dimension* of Ω (cf. Heins [3]).

We are particularly interested in the following result by Heins [3]:

THEOREM A. *Let $\{A_n\}$ be a sequence of mutually disjoint annuli in Ω satisfying that A_{n+1} separates A_n from the ideal boundary of Ω for every n. Suppose that the sum of moduli of A_n diverges. Then the harmonic dimension of Ω is one.*

A density P is said to be *finite* if $\int\int_\Omega P dxdy < \infty$. The above theorem has been generalized for finite densities P by Nakai [8] and Kawamura [4] as follows:

THEOREM B. *Let P be finite on Ω and $\{A_n\}$ be the same as in Theorem* A. *Then*

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the P-harmonic dimension of Ω is one.

The following is another generalization of Theorem A (cf. Segawa [9]):

THEOREM C. *Let $\{A_n\}$ be a sequence of mutually disjoint sets in Ω such that each A_n consists of at most N mutually disjoint annuli for a positive integer N and A_{n+1} separates A_n from the ideal boundary of Ω for every n. Suppose that the sum of moduli of A_n diverges. Then the harmonic dimension of Ω is at most N.*

The main purpose of this paper is to unify Theorems B and C to a form including both Theorems B and C as special cases. The main theorem is as follows:

MAIN THEOREM. *Let P be finite on Ω and $\{A_n\}$ be the same as in Theorem C. Then the P-harmonic dimension of Ω is at most N.*

We shall prove a bit more in Theorem 6. In Section 1, we prove a duality relation for P-harmonic dimensions (cf. Theorem 2), which plays a fundamental role for the proof of Theorem 6.

§1. Duality relation

1.1. A relatively noncompact subregion Ω of an open Riemann surface is referred to as a *general end* if the relative boundary $\partial\Omega$ of Ω consists of a finitely many disjoint analytic Jordan closed curves. In this section we assume that Ω is a general end. We denote by β the ideal boundary of Ω. Without loss of generality, we may assume that there exist an open Riemann surface R and its exhaustion $\{R_n\}_{n=0}^{\infty}$ with $\Omega = R - \bar{R}_0$. Let $e_P^{(n)}$ be the solution of the equation (1) on $\Omega \cap R_n = R_n - \bar{R}_0$ with boundary values 1 on $\partial\Omega$ and 0 on ∂R_n. Since $\{e_P^{(n)}\}$ is increasing and dominated by the constant function 1, the limit $e_P = \lim_{n\to\infty} e_P^{(n)}$ exists. Note that e_P is the solution of (1) on Ω with boundary values 1 on $\partial\Omega$ and '0 on the ideal boundary β'. The function e_P is referred to as the *P-unit* on Ω for P (cf. Nakai [7]). Obviously e_P does not depend on a choice of $\{R_n\}_{n=0}^{\infty}$. We consider the *associated operator* $\hat{L}_P$ with L_P which is introduced by Nakai (cf. [7], [8]):

$$\hat{L}_P u \equiv \Delta u + 2\nabla(\log e_P)\cdot\nabla u \tag{2}$$

where e_P is the P-unit on Ω. Denote by $B_P(\Omega)$ the class of *bounded* solutions of the equation

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$$\hat{L}_P u = 0 \tag{3}$$

on Ω with continuous boundary values on $\partial\Omega$. Note that $1 \in B_P(\Omega)$. To begin with we show the following

LEMMA 1. *Suppose that u belongs to $B_P(\Omega)$. Then u satisfies the following inequalities:*

$$\min_{p\in\partial\Omega} u(p) \leq \inf_{p\in\Omega} u(p) \leq \sup_{p\in\Omega} u(p) \leq \max_{p\in\partial\Omega} u(p).$$

Proof. We have only to show the last inequality since $-u$ also belongs to $B_P(\Omega)$. By adding a constant we may assume $u \geq 0$. Set $M = \max_{p\in\partial\Omega} u(p)$. Let v_n be the solution of (1) on $\Omega \cap R_n$ with boundary values u on $\partial\Omega$ and 0 on ∂R_n. Since $\{v_n\}$ is increasing and $v_n \leq Me_P$ $(n = 1,2,\ldots)$, $v = \lim_{n\to\infty} v_n$ exists and is a solution of (1). It is clear that $v/e_P \leq M$ on Ω. Thus we complete the proof if we show that $u = v/e_P$, or $ue_P = v$.

Note that $L_P(ue_P) = 0$ and $0 \leq v \leq ue_P$ on Ω. There exists a constant $C > 0$ such that $0 \leq u \leq C$, i.e. $0 \leq ue_P \leq Ce_P$. Let w_n be the solution of (1) on $\Omega \cap R_n$ with boundary values 0 on $\partial\Omega$ and e_P on ∂R_n. Then $w_n = e_P - e_P^{(n)}$. By the minimum principle, $0 \leq ue_P - v \leq Cw_n$ on $\Omega \cap R_n$. Since $\lim_{n\to\infty} e_P^{(n)} = e_P$, $\lim_{n\to\infty} w_n = 0$ and therefore $ue_P = v$. □

1.2. Let $B_P^0(\Omega)$ be the subspace of $B_P(\Omega)$ which consists of functions with the limit 0 at β:

$$B_P^0(\Omega) = \{u \in B_P(\Omega) : \lim_{p\to\beta} u(p) = 0\}.$$

Next consider the quotient space

$$\mathscr{B}_P(\Omega) = B_P(\Omega)/B_P^0(\Omega)$$

and denote by $\dim \mathscr{B}_P(\Omega)$ the dimension of the linear space $\mathscr{B}_P(\Omega)$. Our first achievement of this paper is the following duality relation for $\mathscr{P}_P(\Omega)$ and $\mathscr{B}_P(\Omega)$ (cf. Segawa [9]):

THEOREM 2. *If either $\mathscr{P}_P(\Omega)$ or $\mathscr{B}_P(\Omega)$ is of finite dimension, then the P-harmonic dimension $\dim \mathscr{P}_P(\Omega)$ coincides with $\dim \mathscr{B}_P(\Omega)$:*

$$\dim \mathscr{P}_P(\Omega) = \dim \mathscr{B}_P(\Omega).$$

The proof of the above theorem is given in no.1.4. By the definition of

$\mathscr{B}_P(\Omega)$ and the fact $1 \in \mathscr{B}_P(\Omega)$, $\dim \mathscr{B}_P(\Omega) = 1$ is equivalent that $\lim_{p\to\beta} u(p)$ exists for every u in $B_P(\Omega)$. Therefore Theorem 2 implies the following, which was originally obtained by Hayashi [3] (cf. Nakai [7]):

COROLLARY 3. *The P-harmonic dimension $\dim \mathscr{P}_P(\Omega)$ is one if and only if there exists $\lim_{p\to\beta} u(p)$ for every u in $B_P(\Omega)$.*

1.3. Consider the linear space $\mathscr{E}$ generated by $\mathscr{P}_P(\Omega)$, i.e.

$$\mathscr{E} = \{h_1 - h_2 : h_1, h_2 \in \mathscr{P}_P(\Omega)\},$$

and the bilinear functional

$$(u, h) \mapsto \langle u, h\rangle = -\int_{\partial\Omega} u * dh = \int_{\partial\Omega} u \frac{\partial h}{\partial n} ds$$

defined on $B_P(\Omega) \times \mathscr{E}$ where $\partial/\partial n$ is the inner normal derivative. Let $g_n(\cdot, p)$ be the Green's function of (1) on $\Omega \cap R_n$ with pole at p for each $n \in \mathbf{N}$, the set of positive integers. Note that $g_n(\cdot, p)$ converges to the Green's function $g(\cdot, p)$ of (1) on Ω with pole at p uniformly on each compact subset in $\Omega \cup \partial\Omega$. Set

$$Q = \{h \in \mathscr{P}_P(\Omega) : \langle 1, h\rangle = 1\}.$$

We maintain

LEMMA 4. *If $u \in B_P(\Omega)$, then*

$$\limsup_{p\to\beta} u(p) = \sup \langle u, Q\rangle$$

and

$$\liminf_{p\to\beta} u(p) = \inf \langle u, Q\rangle$$

where $\langle u, Q\rangle = \{\langle u, h\rangle : h \in Q\}$.

Proof. We first show that

$$u(p)e_P(p) = -\frac{1}{2\pi}\int_{\partial\Omega} u * dg(\cdot, p) \quad (p \in \Omega) \tag{4}$$

for every $u \in B_P(\Omega)$. Suppose that $p \in \Omega \cap R_n$. Let u_n be the solution of (3) on $\Omega \cap R_n$ with boundary values u on $\partial\Omega$ and 0 on ∂R_n. By Lemma 1, u_n converges to u uniformly on each compact subset in $\Omega \cup \partial\Omega$. Observe that $u_n e_P$ is the solu-

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tion of (1) on $\Omega \cap R_n$ with boundary values u on $\partial\Omega$ and 0 on ∂R_n. Hence the Green's formula yields that $u_n(p)e_P(p) = -(1/2\pi)\int_{\partial\Omega} u * dg_n(\cdot, p)$. By letting $n \to \infty$, we have (4).

Take an arbitrary cluster value a of u at β and a sequence $\{p_n\}$ with $\lim_{n\to\infty} p_n = \beta$ and $\lim_{n\to\infty} u(p_n) = a$. Applying (4) to $1 \in B_P(\Omega)$, we see that $e_P(p_n) = -(1/2\pi)\int_{\partial\Omega} u * dg(\cdot, p_n)$, i.e.

$$-\int_{\partial\Omega} * d\frac{g(\cdot, p_n)}{2\pi e_P(p_n)} = 1.$$

From this it follows that a suitable subsequence of $\{(1/2\pi)\, g(\cdot, p_n)/e_P(p_n)\}$ converges to a function G, which belongs to Q, uniformly on each compact subset of $\Omega \cup \partial\Omega$. By (4) we also have

$$u(p_n) = -\int_{\partial\Omega} u * d\frac{g(\cdot, p_n)}{2\pi e_P(p_n)}.$$

Therefore we conclude that

$$a = -\int_{\partial\Omega} u * dG,$$

i.e. $a \in \langle u, Q\rangle$, which implies

$$\inf\langle u, Q\rangle \le \liminf_{p\to\beta} u(p) \le \limsup_{p\to\beta} u(p) \le \sup\langle u, Q\rangle.$$

Next we show that

(5) $$\liminf_{p\to\beta} u(p) \le \inf\langle u, Q\rangle \le \sup\langle u, Q\rangle \le \limsup_{p\to\beta} u(p).$$

Suppose that $h \in Q$ and $u \in B_P(\Omega)$. Let h_{nm} be the solution of (1) on $R_m - \bar{R}_n$ $(m > n)$ with boundary values h on ∂R_n and 0 on ∂R_m. The Green's formula yields that

$$\langle u, h\rangle = -\int_{\partial\Omega} u * dh = -\int_{\partial\Omega} u_m e_P * dh = \int_{\partial R_n} u_m e_P * dh - h * d(u_m e_P)$$
$$= \int_{\partial R_n} u_m e_P * dh - h_{nm} * d(u_m e_P)$$

and

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$$\int_{\partial R_n} u_m e_P * dh_{nm} - h_{nm} * d(u_m e_P) = \int_{\partial R_m} u_m e_P * dh_{nm} - h_{nm} * d(u_m e_P) = 0$$

where u_m is defined at the beginning of the proof. Therefore, by letting $m \to \infty$, we have

$$\langle u, h \rangle = \int_{\partial R_n} u e_P * d(h - h_n) \tag{6}$$

where $h_n = \lim_{m \to \infty} h_{nm}$. Applying (6) to $u = 1$, we also have

$$\int_{\partial R_n} e_P * d(h - h_n) = \langle 1, h \rangle = -\int_{\partial \Omega} * dh = 1. \tag{7}$$

Hence (6) and (7) imply that

$$\inf_{p \in \partial R_n} u(p) \leq \inf \langle u, Q \rangle \leq \sup \langle u, Q \rangle \leq \sup_{p \in \partial R_n} u(p).$$

Thus (5) follows from the above.

The proof is herewith complete. □

1.4. *Proof of Theorem* 2. By definition, the dimension $\dim \mathscr{E}$ of the linear space $\mathscr{E}$ coincides with $\dim \mathscr{P}_P(\Omega)$.

Consider the $\mathscr{E}$-kernel ($B_P(\Omega)$-kernel resp.)

$$K_1 = \bigcap_{h \in \mathscr{E}} \{u \in B_P(\Omega) : \langle u, h \rangle = 0\}$$

$$(K_2 = \bigcap_{u \in B_P(\Omega)} \{h \in \mathscr{E} : \langle u, h \rangle = 0\} \text{ resp.})$$

of the bilinear functional $(u, h) \mapsto \langle u, h \rangle$. By virtue of Lemma 4, it is easily seen that $K_1 = B_P^0(\Omega)$, and hence $\mathscr{B}_P(\Omega) = B_P(\Omega)/K_1$. Since $\{u|_{\partial\Omega} : u \in B_P(\Omega)\} = C(\partial\Omega)$, it follows from $h \in K_2$ that $\partial h / \partial n \equiv 0$ on $\partial\Omega$. Combining this with the fact $h \equiv 0$ on $\partial\Omega$, we have $K_2 = \{0\}$ (cf. e.g. Miranda [6]). Therefore we can consider $\mathscr{B}_P(\Omega) = B_P(\Omega)/K_1$ ($\mathscr{E} = \mathscr{E}/K_2$ resp.) to be a subspace of $\mathscr{E}^*$ ($\mathscr{B}_P(\Omega)^*$ resp.) where we denote by X^* the conjugate space of a linear space X. In particular we have

$$\dim \mathscr{B}_P(\Omega) \leq \dim \mathscr{E}^*$$

and

$$\dim \mathscr{E} \leq \dim \mathscr{B}_P(\Omega)^*.$$

Hence we have

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$$\dim \mathscr{B}_P(\Omega) = \dim \mathscr{E} = \dim \mathscr{P}_P(\Omega),$$

since linear spaces of finite dimension are isomorphic to their conjugate spaces. □

§2. Proof of Main Theorem

2.1. In this section, we give a proof of Main Theorem in terms of extremal length.

Hereafter we assume that Ω is a *parabolic* end: i.e. there exist no non-constant bounded harmonic functions on Ω with vanishing boundary values on $\partial\Omega$. The following was proved by Nakai [8] essentially:

PROPOSITION 5. *If Ω is parabolic and P is finite on Ω, then every bounded solution of* (3) *on Ω has finite Dirichlet integral on $\Omega - R_1$.*

For the proof we refer to Nakai [8] and Kawamura [4].

2.2. We denote by $\lambda(\Gamma)$ the extremal length of a curve family Γ in Ω. For the definition and details of extremal length we refer to e.g. Ahlfors and Sario [1]. For every positive integer n, let $\Gamma_n(\Omega)$ be the totality of 1-cycles γ in Ω such that γ consists of at most n closed curves and separates $\partial\Omega$ from the ideal boundary β. The following is the main achievement of this paper (cf. Shiga [10]):

THEOREM 6. *Suppose that P is finite on Ω. If the extremal length $\lambda(\Gamma_N(\Omega))$ is zero for an $N \in \mathbf{N}$, then the P-harmonic dimension* $\dim \mathscr{P}_P(\Omega)$ *is at most N.*

Proof. Set $\Gamma_1 = \Gamma_1(\Omega)$ and $\Gamma_n = \Gamma_n(\Omega) - \Gamma_{n-1}(\Omega)\,(n = 2,3,\cdots)$. Since $\Gamma_N(\Omega) = \cup_{n=1}^{N} \Gamma_n$, there exists a $\nu \in \mathbf{N}$ such that $\nu \leq N$ and $\lambda(\Gamma_\nu) = 0$. We shall show that $\dim \mathscr{P}_P(\Omega)$ is at most ν. Take arbitrary $\nu + 1$ functions $u_1,\cdots,u_{\nu+1}$ in $B_P(\Omega)$. By virtue of Theorem 2, we have only to show that a nonzero linear combination $c_1u_1 + \cdots + c_{\nu+1}u_{\nu+1}$ of $u_1,\cdots,u_{\nu+1}$ belongs to $B_P^0(\Omega)$.

Consider the *'density'* ρ on Ω such that

$$\rho\,|dz| = \begin{cases} \sum_{i=1}^{\nu+1} |\nabla u_i|\,|dz| & \text{on } \Omega - R_1 \\ 0 & \text{on } \Omega \cap R_1. \end{cases}$$

By Schwarz' inequality and Proposition 5, we have

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(8)
$$\int\int_{\Omega}\rho^2 dxdy \leq (\nu+1)\sum_{i=1}^{\nu+1} D_{\Omega-R_1}(u_i) < \infty,$$

where $D_{\Omega-R_1}(u_i) = \int\int_{\Omega-R_1} |\nabla u_i|^2\, dxdy$. Set $\Gamma_\nu^m = \{\gamma \in \Gamma_\nu : \gamma \cap R_m = \phi\}$. By means of $\lambda(\Gamma_\nu) = 0$, we have $\lambda(\Gamma_\nu^m) = 0$ for every $m \in \mathbf{N}$ (cf. Kusunoki [5]). Therefore, because of (8), we can find a sequence $\{\gamma_n\}$ in Γ_ν such that γ_n converges to the ideal boundary β and $\lim_{n\to\infty}\int_{\gamma_n}\rho\,|dz| = 0$. In particular, we obtain

(9)
$$\lim_{n\to\infty}\int_{\gamma_n} |\nabla u_i|\,|dz| = 0 \quad (i = 1,\cdots,\nu+1).$$

By definition, every γ_n consists of exactly ν closed curves $\gamma_{n1},\cdots,\gamma_{n\nu}$. Accordingly (9) implies that there exist a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ and vectors $\mathbf{v}_i = (a_{i1},\cdots,a_{i\nu}) \in \mathbf{R}^\nu$ $(i = 1,\cdots,\nu+1)$ such that

(10)
$$\lim_{k\to\infty}\max_{p\in\gamma_{n_kj}} |u_i(p) - a_{ij}| = 0 \quad (j = 1,\cdots,\nu).$$

Evidently we can find $(c_1,\cdots,c_{\nu+1}) \in \mathbf{R}^{\nu+1} - \{(0,\cdots,0)\}$ such that $\sum_{i=1}^{\nu+1} c_i\mathbf{v}_i = (0,\cdots,0)$. Therefore, (10) yields that

(11)
$$\lim_{k\to\infty}\max_{p\in\gamma_{n_k}} \left|\sum_{i=1}^{\nu+1} c_i u_i(p)\right| = 0.$$

Since each γ_n separates $\partial\Omega$ from the ideal boundary β, it follows from Lemma 1 and (11) that $\lim_{p\to\beta}\sum_{i=1}^{\nu+1} c_i u_i = 0$. This completes the proof. □

2.3. *Proof of Main Theorem.* Main Theorem is easily verified from Theorem 6 as follows. Assume that $\{A_n\}$ is the same as in Theorem C. Set $A_n = \cup_{j=1}^{\nu(n)} A_{nj}$ where A_{nj}'s are mutually disjoint annuli and $\nu(n) \leq N$. Let Λ_n be the totality of 1-cycles γ in A_n such that $\gamma = \cup_{j=1}^{\nu(n)}\gamma_{nj}$ where each γ_{nj} is a closed curve in A_{nj} and separates two boundary components of A_{nj}. Set $\Gamma = \cup_{n=1}^\infty \Lambda_n$. Note that $\Gamma \subset \Gamma_N(\Omega)$. By virtue of Theorem 6, we have only to show that $\lambda(\Gamma) = 0$.

It is well-known that $\lambda(\Lambda_n) = 2\pi/\operatorname{mod} A_n$, where $\operatorname{mod} A_n$ is the modulus of A_n (cf. Ahlfors and Sario [1]). Since A_n's are mutually disjoint, we see $\lambda(\Gamma)^{-1} \geq \sum_{n=1}^\infty \lambda(\Lambda_n)^{-1}$. Hence, from the assumption $\sum_{n=1}^\infty \operatorname{mod} A_n = \infty$ it follows that $\lambda(\Gamma) = 0$. □

REFERENCES

[1] L. Ahlfors and L. Sario, Riemann Surfaces, Princeton, 1960.

[2] K. Hayashi, Les solutions positives de l'equation $\Delta u = Pu$ sur une surface de Riemann, Kōdai Math. Sem. Rep., **13** (1961), 20–24.

[3] M. Heins, Riemann surfaces of infinite genus, Ann. of Math., **55**(1952), 296–317.

[4] M. Kawamura, Picard principle for finite densities on Riemann surfaces, Nagoya Math. J., **67** (1977), 35–40.

[5] Y. Kusunoki, On Riemann's period relations on open Riemann surfaces, Mem. Coll. Sci. Kyoto Univ. Ser. A Math., **30** (1956), 1–22.

[6] C. Miranda, Partial Differential Equations of Elliptic Type, Springer, 1970.

[7] M. Nakai, Picard principle and Riemann theorem, Tôhoku Math. J., **28** (1976), 277–292.

[8] M. Nakai, Picard principle for finite densities, Nagoya Math. J., **70** (1978), 7–24.

[9] S. Segawa, A duality relation for harmonic dimensions and its applications, Kodai Math., J., **4** (1981), 508–514.

[10] H. Shiga, On harmonic dimensions and bilinear relations on open Riemann surfaces, J. Math. Kyoto Univ., **21** (1981), 861–879.

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