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RESEARCH ARTICLE

A value-at-risk approach to futures hedge

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Abstract
This paper examines the value-at-risk (VaR) implications of mean-variance hedging. We derive an equivalence between the VaR-based hedge and the mean-variance hedging. This method transfers the investor's subjective risk-aversion coefficient into the estimated VaR measure. As a result, we characterize the collapse probability bounds under which the VaR-based hedge could be insignificantly different from the minimum-variance hedge in the presence of estimation risk. The results indicate that the squared information ratio of futures returns is the primary factor determining the difference between the minimum-variance and VaR-based hedges.

1. Introduction

The expected utility framework is widely used in futures hedging (see [9,17]). Specifically, mean-variance hedging (MVH) with a single risk attitude parameter offers an equivalent to exponential utility hedging under the normality assumption. The MVH investor aims to maximize the hedged return minus the risk in terms of the variance multiplied by a positive risk-aversion coefficient (γ) that measures the degree of risk aversion. The larger the value of γ, the more risk-averse the investor is. Hence, the MVH decision focuses on the tradeoff between a higher return, which is desirable, and a higher variance, which is undesirable for investors with greater risk aversion. Thus, higher hedged risk should be offset by a higher return. The tradeoff depends on the degree of risk aversion.

Let us examine different index-futures combinations, assuming all the hedge ratios. We graph the VaR-based hedged boundary as the best possible portfolios on the return-variance plane. The vertex of the hedged boundary is the minimum-variance (MV) hedge with an infinite γ. Note that γ reflects the slope of the secant line (the sharpness) between a particular MVH and the MV hedge.

Classifying futures contracts as financial derivatives does not inherently make them more or less risky than other financial instruments. Indeed, futures hedging is primarily used to reduce traders' overall exposure to risk. Although the MV hedge is popular in reality, futures can be very risky since investors take speculative positions with a generous amount of leverage. For example, Aït-Sahalia and Brandt [1] p. 1316 use $\gamma = 1, 2, 5, 10, 20$ for empirical study in which $\gamma = 1, 2$ may locate the MVH far away from the MV hedge on the VaR-based hedged boundary. However, because MVH is sensitive to the risk-aversion coefficient used, it is challenging for investors to precisely state their risk attitude.

One of the popular approaches to combatting risk-aversion estimation in portfolio selection is to maximize the portfolio return subject to a value-at-risk (VaR) measure. The VaR is defined as a threshold return with a maximum collapse probability, defined as the likelihood that the hedged return falls below the threshold return. This method transforms the investor's subjective risk-aversion coefficient into an estimated VaR-based basis. For instance, Roy [22] and Telser [27] present the safety-first portfolio theory. Shefrin and Statman [26] translate the safety-first approach into a behavioral style portfolio that is the same as expected wealth optimization with a VaR constraint. Das *et al.* [10] p. 318 suggest that investors can better express their threshold return for a portfolio and maximum collapse probability below the goal than state their risk-aversion coefficient. In these authors' framework, portfolio optimization is equivalent to the safety-first optimization in Telser [27]. Bodnar et *al.* [4] also show that the mean-variance optimization with a risk-aversion coefficient coincides with the upper section of the efficient frontier in the mean-variance space. Chiu [7] presents a closed-form solution to link the mean-variance optimization and the VaR-based portfolio.

Another method employs the global minimum-variance portfolio (GMVP) instead of estimating the risk attitude. For example, Bodnar and Okhrin [3] indicate that the GMVP is statistically justified for the mean-variance portfolio with an extensive range of risk-aversion coefficients. Chiu [6] extends Bodnar and Okhrin's work to determine the γ's boundaries that significantly distinguish the MV hedge and the MVH based on the statistical errors in the presence of estimation risk.

This paper integrates two approaches into a manageable VaR-based hedge. First, we extend the VaR approach to futures hedging based on the estimated VaR constraint. This method derives the equivalence between VaR-based and mean-variance hedging that converts an investor's subjective γ into an estimated VaR measure. Second, we determine the boundaries of the collapse probabilities that significantly distinguish the MV and VaR-based hedges in the presence of estimation risk. Then, we employ the regression method to validate the significance in terms of three criteria: the hedge ratio (the regression coefficient), the hedged variance (the error's variance), and the hedged return (the regression intercept).

The paper proceeds as follows. In Section 2, we review the MV hedge and MVH. In Section 3, we present the VaR-based hedge. In Section 4, we establish the significance tests. In Section 5, we illustrate the significance tests using S&P500 futures hedging. The conclusions follow in Section 6.

2. Futures hedge

This section reviews two major methods for obtaining the futures hedge ratio. We first assume that the hedged portfolio is composed of C_{S} units long spot position and C_{F} units of the short futures position. Let S_t and F_t denote the spot and futures prices at time t, respectively. In this paper, the hedge ratio is the comparative value of futures positions ($C_{\mathrm{F}}F_t$) to the cash positions ($C_{\mathrm{S}}S_t$). Thus, the hedged return with the hedge ratio b is given as:

$$r = \frac{C_{\mathrm{S}}S_t r_{\mathrm{S},t} - C_{\mathrm{F}}F_t r_{\mathrm{F},t}}{C_{\mathrm{S}}S_t} = r_{\mathrm{S},t} - b r_{\mathrm{F},t}, \tag{1}$$

where $r_{\mathrm{S},t} = (S_{t+1} - S_t)/S_t$ and $r_{\mathrm{F},t} = (F_{t+1} - F_t)/F_t$ are one-period returns on the spot and futures positions with the expected returns vector $(\mu_{\mathrm{S}}, \mu_{\mathrm{F}})$ at date t. Using the realized returns $(r_{\mathrm{S},t}, r_{\mathrm{F},t})$ at date $t = 1, 2, \ldots, n$, we define the sample means $(\hat{\mu}_{\mathrm{S}}, \hat{\mu}_{\mathrm{F}})$, variances $(\hat{\sigma}^2_{\mathrm{S}}, \hat{\sigma}^2_{\mathrm{F}})$, and covariance $(\hat{\sigma}_{\mathrm{SF}})$ as follows:

$$\begin{aligned}
(\hat{\mu}_{\mathrm{S}}, \hat{\mu}_{\mathrm{F}}) &= \left(\frac{\sum_{t=1}^{n} r_{\mathrm{S},t}}{n}, \frac{\sum_{t=1}^{n} r_{\mathrm{F},t}}{n}\right), \\
(\hat{\sigma}^2_{\mathrm{S}}, \hat{\sigma}^2_{\mathrm{F}}) &= \left(\frac{\sum_{t=1}^{n}(r_{\mathrm{S},t} - \hat{\mu}_{\mathrm{S}})^2}{n}, \frac{\sum_{t=1}^{n}(r_{\mathrm{F},t} - \hat{\mu}_{\mathrm{F}})^2}{n}\right), \\
\hat{\sigma}_{\mathrm{SF}} &= \frac{\sum_{t=1}^{n}(r_{\mathrm{S},t} - \hat{\mu}_{\mathrm{S}})(r_{\mathrm{F},t} - \hat{\mu}_{\mathrm{F}})}{n}.
\end{aligned} \tag{2}$$

2.1. *Minimum-variance hedge*

The objective of the MV hedge is to minimize risk in terms of portfolio variance in Eq. (1):

$$\min_b \sigma^2 = \min_b(\sigma^2_{\mathrm{S}} - 2b\sigma_{\mathrm{SF}} + b^2\sigma^2_{\mathrm{F}}), \tag{3}$$

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where σ_S^2, σ_F^2, and σ_{SF} are the variances of r_S and r_F as well as the covariance between r_S and r_F. Alternatively, Ederington's [15] regression is the most practical approach employed to test the MV hedge:

$$r_{S,t} = a + br_{F,t} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2). \tag{4}$$

The ordinary least squares method estimates the MV hedge ratio, hedged variance, and hedged return:

$$\begin{aligned} \hat{b}_L &= \hat{b}_M = \frac{\hat{\sigma}_{SF}}{\hat{\sigma}_F^2}, \\ \hat{a}_L &= \hat{\mu}_M = \frac{\hat{\mu}_S \hat{\sigma}_F^2 - \hat{\mu}_F \hat{\sigma}_{SF}}{\hat{\sigma}_F^2}, \\ \hat{\sigma}_L^2 &= \frac{\sum_{t=1}^n [r_{S,t} - \hat{a}_L - \hat{b}_L r_{F,t}]^2}{n-2} = \frac{n}{n-2}\left(\frac{\hat{\sigma}_S^2 \hat{\sigma}_F^2 - \hat{\sigma}_{SF}^2}{\hat{\sigma}_F^2}\right) = \frac{n\hat{\sigma}_M^2}{n-2}. \end{aligned} \tag{5}$$

Under the returns' normality assumption, the simple regression approach also provides the sampling distributions of Eq. (5).

2.2. Mean-variance hedging

Previous studies also use intuitive expected utility maximization to determine the hedging policy based on either the quadratic utility function or the jointly normal return distribution. For example, Khoury and Martel [17] maximize the expected exponential utility to determine the hedge ratio:

$$\max_b \left[\frac{1}{\gamma}(1 - E(e^{-\gamma(r_S - br_F)}))\right]. \tag{6}$$

Assuming that the hedged return is normally distributed, this optimization is equivalent to maximizing MVH with a parameter γ (see [16])[1]

$$\max_b Q = \max_b \left[\mu_S - b\mu_F - \frac{\gamma}{2}(\sigma_S^2 - 2b\sigma_{SF} + b^2\sigma_F^2)\right]. \tag{7}$$

In the MVH framework, investors consider the risk-aversion coefficient as the essential component while associating high levels of uncertainty with a greater probability of higher returns. The following results outline several well-known MVH properties for comparison with the VaR-based hedge.

Lemma 1. *Assume that the MVH investor has a risk-aversion γ. The optimal MVH hedge ratio and the parametric system (σ_U^2, μ_U) are given by:*

$$b_U = b_M - \frac{\mu_F}{\gamma\sigma_F^2}, \quad \sigma_U^2 = \sigma_M^2 + \frac{\mu_F^2}{\gamma^2\sigma_F^2}, \quad \mu_U = \mu_M + \frac{\mu_F^2}{\gamma\sigma_F^2}. \tag{8}$$

Moreover, the slope of the secant line reflects the steepness between the MVH and MV hedges

$$\frac{\mu_M + \mu_F^2/\gamma\sigma_F^2 - \mu_M}{\sigma_M^2 + \mu_F^2/\gamma^2\sigma_F^2 - \sigma_M^2} = \gamma. \tag{9}$$

[1] Another approach treats MVH as the general continuous time semimartingale model based on the minimum mean squared error. For instance, Duffie and Richardson [13] provide closed-form solutions under the mean-variance and quadratic objectives. Schweizer [23,24] extends the MVH problem using the continuous-time martingale and proposes the variance-optimal probability measure. Pham [20] develops the semimartingale price process for continuous risk minimization and MVH. Černý and Kallsen [5] present a general semimartingale structure for the discontinuous stochastic MVH. The comprehensive reviews see Cont [8] pp. 1177–1181 and Eberlein and Kellsen [14] pp. 595–615.

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As a result, the secant line between the MVH and MV hedges has an intercept in the mean-variance plane:

$$\tilde{C} = \mu_{\mathrm{M}} - \gamma\sigma_{\mathrm{M}}^2. \tag{10}$$

Equation (8) shows that the larger the γ, the closer the MVH hedge is to the MV hedge, and vice versa. The MVH investor chooses the MV hedge as γ approaches infinity.

Note that the application of the MVH requires knowledge of assigning the risk-aversion coefficient. However, it is difficult to state how the investor's risk attitude should be in practice. Previous studies usually attribute γ between 1 (the lowest γ) and 20 (the highest γ) to an investor. For example, Pindyck [21] p. 183 highlights that an investor's reasonable estimate of γ should range from 3 to 4. Aït-Sahalia and Brandt [1] p. 1316 use $\gamma = 2, 5, 10, 20$ for empirical study. DeMiguel *et al.* [11] p. 1944 employ $\gamma = 1, 2, 3, 4, 5, 10$ to examine asset pricing robustness.

On the other hand, different studies could present divergent γ estimates for the same portfolio. For example, an often-quoted estimate of S&P500 risk-aversion is Bliss and Panigirtzoglou's [2] p. 429 option-implied γ. Their estimates are 4.36 for the 6-week horizon and 5.22 for the 5-week horizon. Das *et al.* [10] also suggest a VaR-constraint approach instead of using vague risk-aversion coefficients to combat the risk-aversion estimation in portfolio selection. They suggest that investors are better at expressing their threshold return for a portfolio and maximum collapse probability below the goal than stating their risk-aversion coefficient.

3. Futures hedge based on the VaR criterion

This section presents the first main result and employs the VaR-constraint approach to solve the futures hedging problem.

3.1. VaR-based futures hedge

Roy's [22] safety-first rule minimizes the collapse probability with a threshold return "h". Equivalently, the VaR-constraint optimization maximizes the threshold return given a collapse probability "α" in Telser [27]. The objective value function of the futures hedge is given as:

$$P(r_{\mathrm{S}} - br_{\mathrm{F}} \le h) \le \alpha. \tag{11}$$

The feasible set of Eq. (11) must satisfy the following inequality:

$$P\left(\frac{r_{\mathrm{S}} - br_{\mathrm{F}} - (\mu_{\mathrm{S}} - b\mu_{\mathrm{F}})}{\sqrt{\sigma_{\mathrm{S}}^2 - 2b\sigma_{\mathrm{SF}} + b^2\sigma_{\mathrm{F}}^2}} \le \frac{h - (\mu_{\mathrm{S}} - b\mu_{\mathrm{F}})}{\sqrt{\sigma_{\mathrm{S}}^2 - 2b\sigma_{\mathrm{SF}} + b^2\sigma_{\mathrm{F}}^2}}\right) \le \alpha. \tag{12}$$

We apply the inverse of the cumulative normal function to Eq. (12) based on the return's normality assumption:

$$h \le (\mu_{\mathrm{S}} - b\mu_{\mathrm{F}}) + \Phi^{-1}(\alpha)\sqrt{\sigma_{\mathrm{S}}^2 - 2b\sigma_{\mathrm{SF}} + b^2\sigma_{\mathrm{F}}^2}, \tag{13}$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution function. To obtain a feasible solution for Eq. (13), we formulate the objective function of the short hedge as follows:

$$\max_b h = \max_b(\mu_{\mathrm{S}} - b\mu_{\mathrm{F}} + \Phi^{-1}(\alpha)\sqrt{\sigma_{\mathrm{S}}^2 - 2b\sigma_{\mathrm{SF}} + b^2\sigma_{\mathrm{F}}^2}). \tag{14}$$

The following result presents the main result of the VaR-based hedge.

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Theorem 1. *The necessary and sufficient conditions (NSC) for the VaR hedge are as follows:*

$$\begin{cases} \Phi^{-1}(\alpha) < -\dfrac{\mu_{\mathrm{F}}}{\sigma_{\mathrm{F}}} < 0, & \textit{if the futures' return is positive } (\hat{\mu}_{\mathrm{F}} > 0), \\ \Phi^{-1}(\alpha) < \dfrac{\mu_{\mathrm{F}}}{\sigma_{\mathrm{F}}} < 0, & \textit{if the futures' return is negative } (\hat{\mu}_{\mathrm{F}} < 0). \end{cases} \tag{15}$$

Under the NSC, the optimal VaR-based hedge ratio is given by:

$$b_{\mathrm{V}} = b_{\mathrm{M}} - \sqrt{\frac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}} \times \frac{\mu_{\mathrm{F}}}{\sigma_{\mathrm{F}}^2}, \tag{16}$$

where the hedged return and variance are given by:

$$\begin{cases} \mu_{\mathrm{V}} = \mu_{\mathrm{M}} + \sqrt{\dfrac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}} \times \dfrac{\mu_{\mathrm{F}}^2}{\sigma_{\mathrm{F}}^2}, \\ \sigma_{\mathrm{V}}^2 = \dfrac{\sigma_{\mathrm{M}}^2[\Phi^{-1}(\alpha)]^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}. \end{cases} \tag{17}$$

Proof. The first-order condition of the VaR-based optimization (14) is equivalent to solving the following identity:

$$-\mu_{\mathrm{F}}\sqrt{\sigma_{\mathrm{S}}^2 - 2b\sigma_{\mathrm{SF}} + b^2\sigma_{\mathrm{F}}^2} + \Phi^{-1}(\alpha)(-\sigma_{\mathrm{SF}} + b\sigma_{\mathrm{F}}^2) = 0.$$

Denoting the variance relative to b_{V} as $\sigma_{\mathrm{V}}^2 = \sigma_{\mathrm{S}}^2 - 2b_{\mathrm{V}}\sigma_{\mathrm{SF}} + b_{\mathrm{V}}^2\sigma_{\mathrm{F}}^2$, this implies that the optimal hedge ratio is an implicit function of the hedged variance:

$$b_{\mathrm{V}} = \frac{\Phi^{-1}(\alpha)\sigma_{\mathrm{SF}} + \mu_{\mathrm{F}}\sigma_{\mathrm{V}}}{\Phi^{-1}(\alpha)\sigma_{\mathrm{F}}^2} = b_{\mathrm{M}} + \frac{\mu_{\mathrm{F}}\sigma_{\mathrm{V}}}{\Phi^{-1}(\alpha)\sigma_{\mathrm{F}}^2}.$$

To obtain the optimal hedge ratio, we first need to solve the hedged variance relative to b_{V}:

$$\begin{aligned} \sigma_{\mathrm{V}}^2 &= \sigma_{\mathrm{S}}^2 - 2\sigma_{\mathrm{SF}}\left(\frac{\Phi^{-1}(\alpha)\sigma_{\mathrm{SF}} + \mu_{\mathrm{F}}\sigma_{\mathrm{V}}}{\Phi^{-1}(\alpha)\sigma_{\mathrm{F}}^2}\right) + \left(\frac{\Phi^{-1}(\alpha)\sigma_{\mathrm{SF}} + \mu_{\mathrm{F}}\sigma_{\mathrm{V}}}{\Phi^{-1}(\alpha)\sigma_{\mathrm{F}}^2}\right)^2 \sigma_{\mathrm{F}}^2 \\ &= \frac{\sigma_{\mathrm{S}}^2\sigma_{\mathrm{F}}^2 - \sigma_{\mathrm{SF}}^2}{\sigma_{\mathrm{F}}^2} + \frac{\mu_{\mathrm{F}}^2\sigma_{\mathrm{V}}^2}{[\Phi^{-1}(\alpha)]^2\sigma_{\mathrm{F}}^2} \\ &= \sigma_{\mathrm{M}}^2 + \frac{\mu_{\mathrm{F}}^2\sigma_{\mathrm{V}}^2}{[\Phi^{-1}(\alpha)]^2\sigma_{\mathrm{F}}^2}. \end{aligned}$$

Solving the above equation, we obtain:

$$\sigma_{\mathrm{V}}^2 = \frac{\sigma_{\mathrm{M}}^2[\Phi^{-1}(\alpha)]^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}.$$

Under the NSC, there are two possibilities of the optimal VaR-based hedge. For case (A) with $\Phi^{-1}(\alpha) < -\mu_{\mathrm{F}}/\sigma_{\mathrm{F}} < 0$ or $\Phi^{-1}(\alpha) < \mu_{\mathrm{F}}/\sigma_{\mathrm{F}} < 0$, the VaR-based hedge ratio and the hedged return are reduced to:

$$b_{\mathrm{V}} = b_{\mathrm{M}} - \sqrt{\frac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}} \times \frac{\mu_{\mathrm{F}}}{\sigma_{\mathrm{F}}^2},$$

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and

$$\mu_{\mathrm{V}} = \mu_{\mathrm{S}} - b_{\mathrm{V}}\mu_{\mathrm{F}} = \mu_{\mathrm{M}} + \sqrt{\frac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}} \times \frac{\mu_{\mathrm{F}}^2}{\sigma_{\mathrm{F}}^2}.$$

Additionally, the second-order condition confirms the optimality since:

$$\frac{\partial^2 h}{\partial b^2} = \frac{\Phi^{-1}(\alpha)}{\sigma_{\mathrm{V}}^3}\left[\sigma_{\mathrm{V}}^2\sigma_{\mathrm{F}}^2 - \left(\frac{\mu_{\mathrm{F}}\sigma_{\mathrm{V}}}{\Phi^{-1}(\alpha)}\right)^2\right] = \frac{\sigma_{\mathrm{F}}^2[\Phi^{-1}(\alpha)]^2 - \mu_{\mathrm{F}}^2}{\Phi^{-1}(\alpha)\sigma_{\mathrm{V}}} < 0.$$

For case (B) with $\Phi^{-1}(\alpha) > \mu_{\mathrm{F}}/\sigma_{\mathrm{F}} > 0$ or $\Phi^{-1}(\alpha) > -\mu_{\mathrm{F}}/\sigma_{\mathrm{F}} > 0$, then the hedge ratio is:

$$b_{\mathrm{V}} = b_{\mathrm{M}} + \sqrt{\frac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}} \times \frac{\mu_{\mathrm{F}}}{\sigma_{\mathrm{F}}^2}.$$

However, the second-order condition is not met since:

$$\frac{\partial^2 h}{\partial b^2} = \frac{\Phi^{-1}(\alpha)}{\sigma_{\mathrm{V}}^3}\left[\sigma_{\mathrm{V}}^2\sigma_{\mathrm{F}}^2 - \left(\frac{\mu_{\mathrm{F}}\sigma_{\mathrm{V}}}{\Phi^{-1}(\alpha)}\right)^2\right] = \frac{\sigma_{\mathrm{F}}^2[\Phi^{-1}(\alpha)]^2 - \mu_{\mathrm{F}}^2}{\Phi^{-1}(\alpha)\sigma_{\mathrm{V}}} > 0.$$

This completes the proof. □

Comparing the MVH variance in Eqs. (8) to (17), we observe the equivalence between the VaR-based and MVH hedges:

$$\frac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)} = \frac{1}{\gamma^2}.$$

Solving this identity, we obtain the inverse relation between α and γ:

$$\gamma = \sqrt{\frac{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}{\sigma_{\mathrm{M}}^2}}. \tag{18}$$

More importantly, the threshold return h relative to b_{V} is linked to the intercept of the line passing through the MV and VaR-based hedges.

$$\begin{aligned}
h &= (\mu_{\mathrm{S}} - b_{\mathrm{V}}\mu_{\mathrm{F}} + \Phi^{-1}(\alpha)\sqrt{\sigma_{\mathrm{S}}^2 - 2b_{\mathrm{V}}\sigma_{\mathrm{SF}} + b_{\mathrm{V}}^2\sigma_{\mathrm{F}}^2})\\
&= \mu_{\mathrm{M}} + \sqrt{\frac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}} \times \frac{\mu_{\mathrm{F}}^2}{\sigma_{\mathrm{F}}^2} + \Phi^{-1}(\alpha)\sqrt{\frac{\sigma_{\mathrm{M}}^2[\Phi^{-1}(\alpha)]^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}}\\
&= \mu_{\mathrm{M}} + \sqrt{\frac{\sigma_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}} \times (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2 - [\Phi^{-1}(\alpha)]^2)\\
&= \mu_{\mathrm{M}} - \sqrt{\frac{[\Phi^{-1}(\alpha)]^2 - (\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}{\sigma_{\mathrm{M}}^2}} \times \sigma_{\mathrm{M}}^2\\
&= \mu_{\mathrm{M}} - \gamma\sigma_{\mathrm{M}}^2\\
&= \tilde{C}.
\end{aligned} \tag{19}$$

To understand the VaR-based hedge more fully, we now graph the VaR-based hedge geometry. Substituting different values of α into a parametric system (17) is equivalent to incorporating γ into a parametric system (8), and Figure 1 shows the VaR-based hedge boundary. Note that the curve's vertex is

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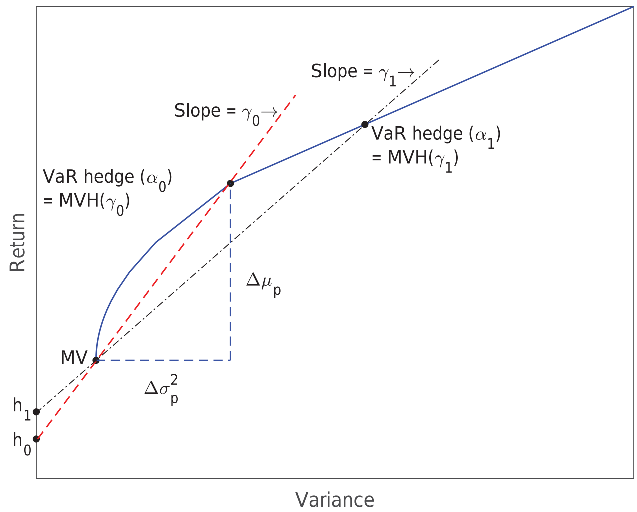

Figure 1. *The VaR-based hedge boundary. This figure depicts the equivalence between the VaR-based hedge and the MVH. The MV hedge with infinite risk-aversion localizes at the curve vertex. The MVH hedge (γ_0) and the MVH hedge (γ_1) denote the hedges in which the investor establishes the futures hedges using γ_0 and γ_1, respectively. The slope of the line connecting the MV hedge and the MVH hedge (γ_0) equals the risk-aversion coefficient γ_0 with a threshold return h_0. Similarly, the slope of the line connecting the MV hedge and the MVH hedge (γ_1) is equal to the risk-aversion coefficient γ_1 with a threshold return h_1. It is evident that the smoother the line connecting the MV and MVH hedges is, the lower the risk-aversion coefficient associated with the investor, the higher the risk tolerance, and the farther away the MVH hedge from the MV hedge, and vice versa.*

the MV hedge with infinite risk-aversion. There are several implications about the hedge boundary. First, there are two possibilities for the futures positions. For the case of positive (negative) futures returns, it is intuitive that the VaR-based investor will short fewer (more) futures hedges than the MV hedge. On the other hand, Eq. (17) indicates that a higher reward (in terms of μ_F^2/σ_F^2) leads the VaR-based hedge away from the MV hedge but also increases the portfolio's risk ($\mu_F^2/\gamma^2\sigma_F^2$).

Second, as mentioned earlier, VaR-based (MVH) investors may take a significant non-MV hedge with speculative positions. However, it is well known that transaction costs always impact hedging effectiveness. Because a particular VaR-based (MVH) hedge corresponds to one unique hedge ratio, for simplicity, we consider its location on the hedged boundary as the representative hedged cost. Following the usual analysis of the hedged boundary, we employ the comparative risk-return approach to analyze the hedged impact. In this vein, Eqs. (18) and (19) show the inverse relation between α and γ. Figure 1 shows that the more aggressive VaR-based hedge (α_1) is equivalent to an MVH hedge (γ_1), in which the slope of the secant line between the MVH and MV hedges equals the risk-aversion coefficient (γ_1). The value of γ also reflects the relative risk-return hedging performance: the steeper the secant line, the more efficient the VaR-based (MVH) hedge is. Likewise, compared to the aggressive VaR-based hedge (α_1), the MVH hedge (γ_0) has a higher risk-aversion coefficient ($\gamma_0 > \gamma_1$). In this situation, the conservative investor would build the VaR-based hedge (α_0) with the lower risk tolerance parameter ($\alpha_0 < \alpha_1$) and the lower threshold return ($h_0 < h_1$) since:

$$\begin{aligned}\alpha_0 &= P(r_S - br_F \leq \mu_M - \gamma_0\sigma_M^2)\\ &= P(r_S - br_F \leq h_0)\\ &\leq P(r_S - br_F \leq h_1)\end{aligned}$$

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$$= P(r_S - br_F \leq \mu_M - \gamma_1 \sigma_M^2)$$
$$= \alpha_1. \tag{20}$$

Although the MV hedge is popular in reality from the perspective of risk reduction, futures also allow investors to take a riskier position for speculative purposes. Equation (20) indicates that the greater the risk-aversion coefficient, the more conservative the VaR-based (MVH) investor is, and the lower the collapse probability given a lower threshold return. In other words, both the return of the VaR-based hedge (γ_1) and its overall exposure to risk increase when the investor takes a speculative non-MV position with a generous amount of leverage. Nevertheless, its corresponding return would not offset the higher hedged risk such that the hedged boundary presents the concave-down property. Using the speculative hedge leads to a relatively inefficient risk-return performance, and the tradeoff depends on the investor's degree of risk aversion. This paper transfers the subjective γ's determination into the estimated VaR-based judgment.

4. Significance tests for α

The collapse probability plays an important role in determining the VaR-based hedge. This section presents the second main result. According to Eq. (16), the VaR-based hedge is approximately equal to the MV hedge when the collapse probability is very low. Thus, we are interested in testing the following null hypothesis:

$$H_0 : \text{The MV hedge equals to the VaR-based hedge with a lower } \alpha. \tag{21}$$

To determine a bound of α in which the VaR-based hedge is statistically equivalent to the MV hedge, we perform the null hypothesis (21) in terms of: (A) the hedge ratio (the regression coefficient), (B) the hedged variance (the error's variance), and (C) the hedged return (the regression intercept).

4.1. The hedge ratio test

The hedge ratio represents the hedged cost that also measures hedged effectiveness.[2] Thus, we are first concerned about the equality test of hedge ratios over different collapse probabilities. Note that the VaR-based hedge ratio in Eq. (16) could be more or less than the MV hedge ratio. If futures returns are positive, we could possibly build a VaR-based hedge that uses a lower hedge ratio but is statistically equivalent to the MV hedge. In this case, the hedges other than the MV hedge always have smaller estimates of the hedge ratio. Thus, hypothesis (21) is given by:

$$H_{A0} : b = b_0 \quad \text{versus} \quad H_{A1} : b > b_0, \quad \text{if } \hat{\mu}_F > 0. \tag{22}$$

In another circumstance, the return is negative, and we build a VaR-based hedge that costs more but is statistically equivalent to the MV hedge. The hypothesis (21) is rewritten as:

$$H_{A0} : b = b_0 \quad \text{versus} \quad H_{A1} : b < b_0, \quad \text{if } \hat{\mu}_F < 0. \tag{23}$$

The following result summarizes the significance test based on the hedge ratio.

[2] The Edertington effectiveness and its corresponding test statistic are defined as follows:

$$\hat{E} = 1 - \frac{\hat{\sigma}^2}{\hat{\sigma}_S^2} \quad \text{and} \quad \frac{\hat{E}\sqrt{n-2}}{\sqrt{1-\hat{E}^2}} \sim t_{n-2}.$$

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Theorem 2. *To test the VaR-based hedge ratio in (22)–(23), the insignificant boundary of* α *is given as:*

$$\alpha \leq \begin{cases} \alpha^* \equiv \Phi\left(-\sqrt{\dfrac{n-2+t^2_{n-2;\delta/2}}{t^2_{n-2;\delta/2}}} \times \dfrac{\hat{\mu}_{\mathrm{F}}}{\hat{\sigma}_{\mathrm{F}}}\right), \ \textit{if}\ \hat{\mu}_{\mathrm{F}} > 0, \\ \alpha^* \equiv \Phi\left(\sqrt{\dfrac{n-2+t^2_{n-2;\delta/2}}{t^2_{n-2;\delta/2}}} \times \dfrac{\hat{\mu}_{\mathrm{F}}}{\hat{\sigma}_{\mathrm{F}}}\right), \quad \textit{if}\ \hat{\mu}_{\mathrm{F}} < 0. \end{cases} \tag{24}$$

Proof. The standard regression theory (see [12]) provides the general sampling distribution for characterizing the regression coefficient's t-statistic with $n-2$ degrees of freedom:

$$t = \frac{\hat{b}_L - b_0}{\sqrt{\hat{\sigma}^2_L / \sum_{t=1}^n (r_{\mathrm{F},t} - \hat{\mu}_{\mathrm{F}})^2}} \sim t(n-2).$$

Thus, the endpoints of the $(1-\delta)\times 100\%$ confidence interval for the hedge ratio are:

$$\hat{b}_L \pm t_{n-2;\delta/2}\sqrt{\frac{\hat{\sigma}^2_L}{\sum_{t=1}^n (r_{\mathrm{F},t} - \hat{\mu}_{\mathrm{F}})^2}} = \hat{b}_L \pm t_{n-2;\delta/2}\sqrt{\frac{\hat{\sigma}^2_{\mathrm{M}}}{(n-2)\hat{\sigma}^2_{\mathrm{F}}}}.$$

Setting the VaR-based hedge ratio to be less than the confidence endpoint:

$$\sqrt{\frac{\hat{\sigma}^2_{\mathrm{M}}(\mu^2_{\mathrm{F}}/\sigma^4_{\mathrm{F}})}{[\Phi^{-1}(\alpha)]^2 - (\mu^2_{\mathrm{F}}/\sigma^2_{\mathrm{F}})}} \leq t_{n-2;\delta/2}\sqrt{\frac{\hat{\sigma}^2_{\mathrm{M}}}{(n-2)\hat{\sigma}^2_{\mathrm{F}}}}.$$

Straightforward rearrangement of this inequality yields a range of α in Eq. (24) over which the VaR-based hedge ratio is significantly different from the MV hedge ratio. □

4.2. The hedged variance test

We next focus on the equality test from the risk-reduction view. Because the hedges other than the MV hedge always have a larger hedged variance, we assess whether the VaR-based hedge significantly increases the amount of variance compared with the MV hedge. In this case, the null hypothesis (14) is reduced to:

$$H_{B0} : \sigma^2 = \sigma^2_0 \quad \text{versus} \quad H_{B1} : \sigma^2 < \sigma^2_0. \tag{25}$$

The following result summarizes the significance test based on the hedged variance.

Theorem 3. *To test the VaR-based hedged variance (16), the insignificant boundary of* α *relative to the* $(1-\delta)\times 100\%$ *confidence level is given as:*

$$\alpha \leq \begin{cases} \alpha^{**} \equiv \Phi\left(-\dfrac{\hat{\mu}_{\mathrm{F}}}{\hat{\sigma}_{\mathrm{F}}}\sqrt{\dfrac{n}{n-\chi^2_{n-2;(1-\delta/2)}}}\right), \ \textit{if}\ \hat{\mu}_{\mathrm{F}} > 0, \\ \alpha^{**} \equiv \Phi\left(\dfrac{\hat{\mu}_{\mathrm{F}}}{\hat{\sigma}_{\mathrm{F}}}\sqrt{\dfrac{n}{n-\chi^2_{n-2;(1-\delta/2)}}}\right), \quad \textit{if}\ \hat{\mu}_{\mathrm{F}} < 0. \end{cases} \tag{26}$$

Proof. Seber [25] p. 97 indicates that the sampling distribution of $\hat{\sigma}^2_L$ is of the form:

$$\frac{(n-2)\hat{\sigma}^2_L}{\sigma^2} = \frac{n\hat{\sigma}^2_{\mathrm{M}}}{\sigma^2} \sim \chi^2_{n-2}.$$

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Thus, the $(1-\delta)\times 100\%$ confidence interval for the hedged variance is:

$$\left[\frac{n\hat{\sigma}_{\mathrm{M}}^2}{\chi^2_{n-2;\delta/2}}, \frac{n\hat{\sigma}_{\mathrm{M}}^2}{\chi^2_{n-2;(1-\delta/2)}}\right].$$

The VaR-based hedged variance varies according to the value of α. We evaluate the statistical insignificance using the χ^2 statistic, which tests the equality of the MV and VaR-based hedge variances. Assume that the VaR-based hedge's variance $\hat{\sigma}_{\mathrm{V}}^2$ must be less than the upper bound of the confidence interval:

$$\frac{\hat{\sigma}_{\mathrm{M}}^2[\Phi^{-1}(\alpha)]^2}{[\Phi^{-1}(\alpha)]^2-(\hat{\mu}_{\mathrm{F}}^2/\hat{\sigma}_{\mathrm{F}}^2)} \le \frac{n\hat{\sigma}_{\mathrm{M}}^2}{\chi^2_{n-2;(1-\delta/2)}}.$$

Solving this inequality, we obtain a range of α in Eq. (26) over which the VaR-based hedged variance is significantly different from the MV hedged variance. □

4.3. The hedged return test

Additionally, we intuitively test the equivalence based on the hedged returns. Because the MVH is located at the upper hedge boundary, we assess whether the MVH will significantly increase the return compared to the MV hedge. In this case, hypothesis (14) is reduced to:

$$H_{B0}: \mu = \mu_0 \quad \text{versus} \quad H_{B1}: \mu < \mu_0. \tag{27}$$

The following result summarizes the significance test based on the hedged return.

Theorem 4. *To test the VaR-based hedged variance (27), the insignificant boundary of α is given as:*

$$\alpha \le \begin{cases} \alpha^{***} \equiv \Phi\left(-\sqrt{\dfrac{t^2_{n-2;\delta/2}(1+\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)+(n-2)(\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}{t^2_{n-2;\delta/2}(1+\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}}\times\dfrac{\hat{\mu}_{\mathrm{F}}}{\hat{\sigma}_{\mathrm{F}}}\right), & \text{if } \hat{\mu}_{\mathrm{F}}>0, \\ \alpha^{***} \equiv \Phi\left(\sqrt{\dfrac{t^2_{n-2;\delta/2}(1+\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)+(n-2)(\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}{t^2_{n-2;\delta/2}(1+\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}}\times\dfrac{\hat{\mu}_{\mathrm{F}}}{\hat{\sigma}_{\mathrm{F}}}\right), & \text{if } \hat{\mu}_{\mathrm{F}}>0. \end{cases} \tag{28}$$

Proof. The simple regression theory (see [12] p. 27) characterizes that the regression coefficient is t-distributed with $n-2$ degrees of freedom:

$$\frac{\hat{a}_L - a}{\sqrt{\hat{\sigma}_L^2[(1/n)+(\hat{\mu}_{\mathrm{F}}^2/\sum_{t=1}^n(r_{\mathrm{F},t}-\hat{\mu}_{\mathrm{F}})^2)]}} \sim t(n-2).$$

Thus, the endpoints of the $(1-\delta)\times 100\%$ confidence interval for the hedged return are given by:

$$\hat{a}_L \pm t_{n-2;\delta/2}\sqrt{\hat{\sigma}_L^2\left[\frac{1}{n}+\frac{\hat{\mu}_{\mathrm{F}}^2}{\sum_{t=1}^n(r_{\mathrm{F},t}-\hat{\mu}_{\mathrm{F}})^2}\right]} = \hat{\mu}_{\mathrm{M}} \pm t_{n-2;\delta/2}\sqrt{\frac{\hat{\sigma}_{\mathrm{M}}^2}{n-2}\left(1+\frac{\hat{\mu}_{\mathrm{F}}^2}{\hat{\sigma}_{\mathrm{F}}^2}\right)}.$$

Setting the VaR-based hedge's return to be less than the confidence endpoint:

$$\sqrt{\frac{\hat{\sigma}_{\mathrm{M}}^2}{[\Phi^{-1}(\alpha)]^2-(\mu_{\mathrm{F}}^2/\sigma_{\mathrm{F}}^2)}}\times\frac{\mu_{\mathrm{F}}^2}{\sigma_{\mathrm{F}}^2} \le t_{n-2;\delta/2}\sqrt{\frac{\hat{\sigma}_{\mathrm{M}}^2}{n-2}\left(1+\frac{\hat{\mu}_{\mathrm{F}}^2}{\hat{\sigma}_{\mathrm{F}}^2}\right)}$$

and solving, we obtain a range of α in Eq. (28). □

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Table 1. *Summary statistics.*

	Number	Stock index Return	Stock index Variance	Futures Return	Futures Variance	Other parameters Covariance	Other parameters IR
2000–2009	522	−0.0143%	7.5948%	−0.0154%	7.9215%	7.6260%	−0.0055
2010–2019	522	0.2229%	3.7290%	0.2240%	3.7959%	3.7109%	0.1150
2000–2019	1044	0.1043%	5.6706%	0.1043%	5.8674%	5.6772%	0.0431

This table reports the key statistics for the S&P500 stock index and the corresponding index futures. It decomposes the estimation period into two subsamples. Specifically, both stock index and index futures increase from negative weekly returns during the 2000–2009 subperiod to positive weekly returns during the 2010–2019 subperiod. All the sample data are downloaded from the website, https://investing.com/indices/indices-futures.

5. Illustration

Stock index futures are usually considered leading indicators to determine market sentiment. In this section, we employ the S&P500 stock index and the corresponding index futures to illustrate the safety-first hedge.

5.1. Estimation

We begin by providing an overall view of the estimated parameters. The full estimation period 2000–2019 is decomposed into two subperiods. The subperiods 2000–2009 and 2010–2019 have a total of 522 weekly returns for each index and index futures. All of the historical returns are downloaded from the website, https://investing.com/indices/indices-futures. Table 1 reports the summary statistics. The results show that both stock index and index futures increase from negative weekly returns during the 2000–2009 subperiod to positive weekly returns during the 2010–2019 subperiod. In particular, the 2000–2009 subperiod has a relatively small squared IR $(-0.0055)^2$ compared to the 2010–2019 subperiod's squared IR (0.1150^2).

To investigate the impact of futures' IR, Table 2 illustrates the safety-first hedge ratio, return, variance, and risk-aversion coefficient at 5%, 10%, 20%, and 30% collapse probability. Panel A shows that the safety-first hedge ratio is an increasing function of α when the 2000–2009 subperiod has negative futures returns. In this case, the safety-first hedge has more spot positions since the futures return brings an inverse effect to the hedged portfolio. On the other hand, Panel B indicates that the safety-first hedge ratio is a decreasing function of α when the 2010–2019 subperiod has positive futures returns. Panel C has a similar result as Panel B. In addition, Panel B shows that the γ at the 5%, 10% collapse probability is close to Bliss and Panigirtzoglou's estimates of 4.36 and 5.22.

5.2. Significant test between the MV and safety-first hedges

To compute the bound of α in which the safety-first hedges are statistically equivalent to the MV hedge, this section performs significance tests. Table 3 reports the estimated collapse probability bounds at the 5% significance level. The corresponding threshold returns (h) and risk-aversion coefficients (γ) are reported in parentheses and brackets, respectively. For the case of the 2000–2009 sample, column 2 reports the results using the hedge ratio $P(r_p \leq -0.0314) = 47.464\%$ and $\gamma = 0.1259$; column 3 reports the results using the hedged variance $P(r_p \leq -0.0069) = 49.375\%$ and $\gamma = 0.0292$; column 4 reports the results based on the hedged return $P(r_p \leq -0.0003) = 49.782\%$ and $\gamma = 0.0007$. In this subsample, there are wide regions in which the α bounds (and γ bounds) of the safety-first hedge could be insignificantly different from the MV hedge.

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Table 2. *Estimated safety-first hedges at different collapse probabilities.*

Collapse probability	Hedge ratio	Hedged return	Hedged variance	Threshold	Risk-aversion
		Panel A: 2000–2009			
MV hedge	0.962697	0.000461	0.253303		
5% Safety-first hedge	0.963291	0.000470	0.253306	−0.827376	3.268172
10% Safety-first hedge	0.963459	0.000473	0.253307	−0.644527	2.546315
15% Safety-first hedge	0.963639	0.000476	0.253310	−0.521160	2.059280
20% Safety-first hedge	0.963857	0.000479	0.253313	−0.423111	1.672197
25% Safety-first hedge	0.964144	0.000484	0.253319	−0.338993	1.340112
30% Safety-first hedge	0.964559	0.000490	0.253330	−0.263451	1.041884
		Panel B: 2010–2019			
MV hedge	0.977618	0.003902	0.101162		
5% Safety-first hedge	0.966180	0.006464	0.101659	−0.517982	5.158864
10% Safety-first hedge	0.962914	0.007196	0.101983	−0.402065	4.013024
15% Safety-first hedge	0.959398	0.007983	0.102423	−0.323712	3.238496
20% Safety-first hedge	0.955108	0.008944	0.103086	−0.261275	2.621301
25% Safety-first hedge	0.949380	0.010227	0.104189	−0.207487	2.089602
30% Safety-first hedge	0.940937	0.012118	0.106270	−0.158832	1.608636
		Panel C: 2000–2019			
MV hedge	0.967588	0.003343	0.177356		
5% Safety-first hedge	0.963035	0.003818	0.177478	−0.689128	3.904410
10% Safety-first hedge	0.961743	0.003952	0.177557	−0.536061	3.041360
15% Safety-first hedge	0.960358	0.004097	0.177663	−0.432761	2.458913
20% Safety-first hedge	0.958681	0.004272	0.177822	−0.350631	1.995834
25% Safety-first hedge	0.956465	0.004503	0.178082	−0.280130	1.598326
30% Safety-first hedge	0.953263	0.004837	0.178560	−0.216756	1.240997

This table illustrates the S&P500 safety-first hedge for distinct collapse probabilities. Panel A shows that the safety-first hedge ratio is an increasing (decreasing) function of α (γ) when the futures return is negative. Panel B indicates that the safety-first hedge ratio is a decreasing (increasing) function of α (γ) when the futures return is positive. Panel C reports the full sample results that are similar to Panel B.

For the case of the 2010–2019 subsample, column 2 reports the results using the hedge ratio $P(r_p \leq -0.4205) = 9.022\%$ and $\gamma = 4.1945$; column 3 reports the results using the hedged variance $P(r_p \leq -0.0945) = 37.064\%$ and $\gamma = 0.9731$; column 4 reports the results based on the hedged return $P(r_p \leq -0.0456) = 42.331\%$ and $\gamma = 0.4891$. In this subsample, there are large differences in the three insignificant α bounds (and γ bounds). Note that the results also depend on the sample size, and the results of the full period 2000–2019 are between two subsamples.

To deliver the results visually in the return-variance space, Figure 2 plots the sample hedged boundary and compares the insignificant hedged boundary from the hedge ratio test with boundaries from the hedged variance as well as return tests. We also add the conventional 5% significance level lines in Figure 2, in which the lower parts of the sample hedged boundary below the 5% significance line represent the statistical equivalence between MV and safety-first hedges corresponding to different tests. Figure 2(A) plots the hedged return in the 10^{-3} scale that indicates the small differences of the 5% significance lines between tests. Thus, there are wide regions such that the α bounds (ranging from 47.464% to 49.782%) of the safety-first hedge could be statistically equivalent to the MV hedge. In Figure 2(B), the return magnitude has been transformed to 10^{-2}. This straightens out the graph and yields α bounds ranging between 9.022% and 42.331%. There are large differences in the three upper

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Table 3. *Upper bounds of insignificant collapse probability.*

Period	Number	Hedge ratio test	Hedged variance test	Hedged return test
2000–2009	522	47.464%	49.375%	49.782%
		(−0.0314)	(−0.0069)	(−0.0003)
		[0.1259]	[0.0292]	[0.0007]
2010–2019	522	9.022%	37.064%	42.331%
		(−0.4205)	(−0.0945)	(−0.0456)
		[4.1956]	[0.9731]	[0.4891]
2000–2019	1044	23.894%	44.156%	47.892%
		(−0.2950)	(−0.0559)	(−0.0096)
		[1.6821]	[0.3338]	[0.0728]

This table estimates the parameters given the 95% confidence level ($\delta = 5\%$). The upper bound of the collapse probability using the hedge ratio, hedged variance, and hedged return tests are computed according to Eqs. (24), (26), and (28). The corresponding threshold returns and risk-aversion coefficients are reported in parentheses and brackets, respectively. Taking the 2000–2009 hedge ratio test as an example, column 2 reports that $P(r_p \leq -0.0314) = 47.464\%$ with a risk-aversion coefficient $\gamma = 0.1259$. In all cases, the hedge ratio test yields the narrowest region in which the safety-first is statistically equivalent to the MV hedge. Then, the hedged variance test gives a wider region, and the hedged return test produces the widest insignificant region.

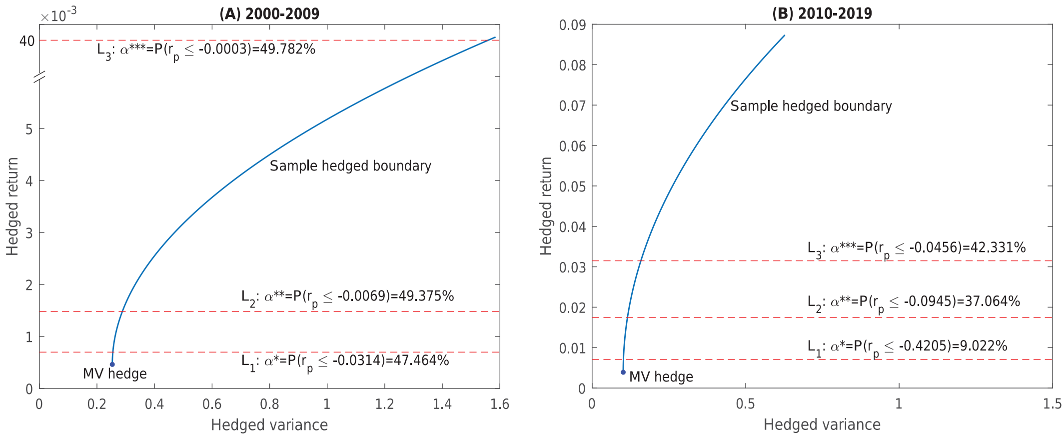

Figure 2. *Insignificant α bounds of safety-first hedges. This figure depicts the VaR-based hedges that are statistically equivalent to the MV hedge based on various significance tests. The 5% significance level lines indicate that the lower parts of the sample hedged boundary below the line represent the statistical equivalence between the MV and VaR-based hedges. (A) reports the small differences of the 5% significance lines ranging from* $\alpha^* = 47.464\%$ *to* $\alpha^{***} = 49.782\%$. *In (B), the return magnitude has been transformed to* 10^{-2}. *This yields α bounds ranging between* $\alpha^* = 9.022\%$ *and* $\alpha^{***} = 42.331\%$. *Note that the difference is primarily driven by the squared information ratio of the futures return.*

bounds of α and γ. Note that the upper bounds of α and γ using the hedge ratio test change drastically. This difference is primarily driven by the squared information ratio of the futures return.

In sum, we have two observations. First, the hedge ratio test yields the narrowest region in which the safety-first hedge is statistically equivalent to the MV hedge among all the cases. Then, the hedged variance test gives a wider region and the hedged return test produces the widest insignificant region of α. Second, we observe that the upper bounds of insignificant collapse probabilities depend critically on

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the squared information ratio of the futures return. All the tests indicate that the safety-first hedge with a broad range of collapse probabilities could be insignificantly different from the MV hedge.

Remark 1. We also use the world's major indices to examine the consistency of the significance tests proposed. These market indices with different monthly data (from January 2014 to December 2018) contain France, Germany, Japan, and the UK from the G7 and Brazil, China, India, and Russia from the BRICS. The empirical results present similar representativeness. For brevity, these results are not reported here but are available on request.

6. Conclusion

Mean-variance hedging (MVH) with a single risk-aversion coefficient has long been of interest to traders and researchers. While it is acknowledged that the application of MVH requires an accurate risk-aversion coefficient, these limitations make MV hedges popular in reality. An equivalence is constructed between the VaR-based hedge and the MVH hedge. In the VaR-based hedge, the MVH investor's subjective risk-aversion coefficient is replaced with the estimated and manageable VaR measure. In other words, we show that the higher the investor's risk-aversion coefficient, the lower is the collapse probability given a threshold return, or the lower is the threshold return given a collapse probability.

Because the MVH hedge with a large risk-aversion coefficient is approximately equal to the MV hedge, we construct the collapse probability intervals to identify the insignificant safety-first hedges at various significance levels. The simulated evidence shows that the range of the collapse probability interval is positively related to the squared information ratio of futures. Overall, the hedge ratio test is more significant at rejecting the equivalence between hedges than the hedged variance. Our results are similar to Bodnar and Okhrin's conclusion that the MV hedge is statistically justified for the safety-first hedge with an extensive range of collapse probabilities.

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References

[1] Aït-Sahalia, Y. & Brandt, M.W. (2001). Variable selection for portfolio choice. *Journal of Finance* 56: 1297–1351.
[2] Bliss, R.R. & Panigirtzoglou, N. (2004). Option-implied risk aversion estiamtes. *Journal of Finance* 68: 407–446.
[3] Bodnar, T. & Okhrin, Y. (2013). Boundaries of the risk-aversion coefficient: Should we invest in the global minimum variance portfolio? *Applied Mathematics and Computation* 219: 5440–5448.
[4] Bodnar, T., Parolya, N., & Schmid, W. (2013). On the equivalence of quadratic optimization problems commonly used in portfolio theory. *European Journal of Operational Research* 229: 637–644.
[5] Černý, A. & Kallsen, J. (2007). On the structure of general mean-variance hedging strategies. *Annals of Probability* 35: 1479–1531.
[6] Chiu, W.Y. (2021). Mean-variance hedging in the presence of estimation risk. *Review of Derivatives Research* 24: 221–241.
[7] Chiu, W.Y. (2021). Safety-first portfolio selection. *Mathematics and Financial Economics* 15: 657–674.
[8] Cont, R. (2010). *Encyclopedia of quantitative finance*. New York: John Wiley & Sons.
[9] Cui, Y. & Feng, Y. (2020). Composite hedge and utility maximization for optimal futures hedging. *International Review of Economics and Finance* 68: 15–32.
[10] Das, S., Markowitz, H., Scheid, J., & Statman, M. (2010). Portfolio optimization with mental accounts. *Journal of Financial and Quantitative Analysis* 45: 311–334.
[11] DeMiguel, V., Garlappi, L., & Uppal, R. (2009). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? *Review of Financial Studies* 22: 1915–1953.
[12] Draper, N. & Smith, H. (1981). *Applied regression analysis*, 2nd ed. New York: John Wiley & Sons.
[13] Duffie, D. & Richardson, H.R. (1991). Mean-variance hedging in continuous time. *Annals of Applied Probability* 1: 1–15.
[14] Eberlein, E. & Kallsen, J. (2019). *Mathematical finance*. Switzerland: Springer.
[15] Ederington, L.H. (1979). The hedging performance of the new futures markets. *Journal of Finance* 43: 157–170.
[16] Hsin, C.W., Kuo, J., & Lee, C.F. (1994). A new measure to compare the hedging effectiveness of foreign currency futures versus options. *Journal of Futures Markets* 14: 685–707.
[17] Khoury, M.T. & Martel, J.-M. (1985). Optimal futures hedging in the presence of asymmetric information. *Journal of Futures Market* 5: 595–605.

[18] Lence, S.H. (1996). Relaxing the assumptions of minimum variance hedging. *Journal of Agricultural and Resource Economics* 21: 39–55.
[19] Lien, D. (2007). Optimal futures hedging: Quadratic versus exponential utility functions. *Journal of Futures markets* 28: 208–211.
[20] Pham, H. (2000). On quadratic hedging in continuous time. *Mathematical Methods of Operational Research* 51: 315–339.
[21] Pindyck, R.S. (1988). Risk-aversion and determinants of stock market behavior. *The Review of Economics and Statistics* 70: 183–190.
[22] Roy, A.D. (1952). Safety-first and holding of assets. *Econometrica* 20: 431–449.
[23] Schweizer, M. (1992). Mean-variance hedging for general claims. *Annals of Applied Probability* 2: 171–179.
[24] Schweizer, M. (1996). Approximation pricing and the variance-optimal martingle meansure. *Annals of Probability* 64: 206–236.
[25] Seber, G.A.F. (1977). *Linear regression analysis*. New York: John Wiley & Sons.
[26] Shefrin, H. & Statman, M. (2000). Behavioral portfolio theory. *Journal of Financial and Quantitative Analysis* 35: 127–151.
[27] Telser, L. (1956). Safety first and hedging. *Review of Economic Studies* 23: 1–16.

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