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RESEARCH ARTICLE

An update on Haiman's conjectures

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Abstract
We revisit Haiman's conjecture on the relations between characters of Kazdhan–Lusztig basis elements of the Hecke algebra over S_n. The conjecture asserts that, for purposes of character evaluation, any Kazhdan–Lusztig basis element is reducible to a sum of the simplest possible ones (those associated to so-called codominant permutations). When the basis element is associated to a smooth permutation, we are able to give a geometric proof of this conjecture. On the other hand, if the permutation is singular, we provide a counterexample.

1. Introduction

The group algebra $\mathbb{C}[S_n]$ admits a q-deformation called the *Hecke algebra* H_n, constructed as follows. Since every $w \in S_n$ can be written as a product of simple transpositions $(i, i+1)$, the group algebra $\mathbb{C}[S_n]$ can be described as the $\mathbb{C}$-algebra generated by $\{T_s\}$, where s runs through all simple transpositions, with the relations

$$\begin{aligned} T_s^2 &= 1 && \text{for every simple transposition } s,\\ T_sT_{s'} &= T_{s'}T_s && \text{for every } s=(i,i+1) \text{ and } s'=(j,j+1) \text{ such that } |i-j|>1,\\ T_sT_{s'}T_s &= T_{s'}T_sT_{s'} && \text{for every } s=(i,i+1) \text{ and } s'=(j,j+1) \text{ such that } |i-j|=1. \end{aligned}$$

The algebra H_n has the same generators as $\mathbb{C}[S_n]$ but with slightly different relations, although we abuse the notation and still write T_s for these generators. Namely, H_n is the $\mathbb{C}(q^{\frac{1}{2}})$-algebra[1] generated by $\{T_s\}$, with the relations

$$\begin{aligned} T_s^2 &= (q-1)T_s + q && \text{for every simple transposition } s,\\ T_sT_{s'} &= T_{s'}T_s && \text{for every } s=(i,i+1) \text{ and } s'=(j,j+1) \text{ such that } |i-j|>1,\\ T_sT_{s'}T_s &= T_{s'}T_sT_{s'} && \text{for every } s=(i,i+1) \text{ and } s'=(j,j+1) \text{ such that } |i-j|=1. \end{aligned}$$

[1]Usually, the definition is over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.

When $q = 1$, we recover the group algebra $\mathbb{C}[S_n]$. Since each $w \in S_n$ has a (nonunique) reduced expression $w = s_1 s_2 \dots s_{\ell(w)}$ in terms of simple transpositions, the product

$$T_w := T_{s_1} T_{s_2} \dots T_{s_{\ell(w)}},$$

is well defined, independent of the choice of reduced expression for w. Then as a $\mathbb{C}(q^{\frac{1}{2}})$-vector space, $\{T_w\}_{w \in S_n}$ is a basis of H_n.

To introduce the Kazhdan–Lusztig basis, we first define the *Bruhat order* of S_n: The *length* $\ell(w)$ of w is the number of inversions of w and given $z, w \in S_n$, we say that $z \leq w$ if for some (equivalently, for every) reduced expression $w = s_1 \dots s_{\ell(w)}$ there exist $1 \leq i_1 < i_2 < \dots < i_k \leq \ell(w)$ such that $z = s_{i_1} \dots s_{i_k}$. Then letting ι denote the involution of H_n given by

$$\begin{aligned} \iota \colon H_n &\to H_n \\ q^{\frac{1}{2}} &\mapsto q^{-\frac{1}{2}} \\ T_w &\mapsto T_{w^{-1}}^{-1}, \end{aligned}$$

the *Kazhdan–Lusztig basis* $\{C'_w\}_{w \in S_n}$ of H_n is defined by the following properties:

$$\begin{aligned} \iota(C'_w) &= C'_w, \\ q^{\frac{\ell(w)}{2}} C'_w &= \sum_{z \leq w} P_{z,w}(q) T_z, \end{aligned} \tag{1.1}$$

where $P_{z,w}(q) \in \mathbb{Z}[q]$, $P_{w,w}(q) = 1$ and $\deg(P_{z,w}) < \frac{\ell(w)-\ell(z)}{2}$ for every $z \neq w$. The existence of such a basis is proved in [KL79] and the polynomials $P_{z,w}(q)$ are called *Kazhdan–Lusztig polynomials*.

The Kazdhan–Lusztig elements and polynomials are closely related to the geometry of Schubert varieties in the flag variety. The flag variety $\mathcal{B}$ is the projective variety parametrizing flags of vector subspaces of $\mathbb{C}^n$, that is,

$$\mathcal{B} = \{V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n; \dim_{\mathbb{C}}(V_i) = i\}.$$

We often abbreviate and write $V_\bullet$ to denote $V_1 \subset \dots \subset V_n$. For each permutation w, the relative Schubert variety Ω_w and its open cell Ω_w° are defined as

$$\begin{aligned} \Omega_w &:= \{(F_\bullet, V_\bullet); \dim V_i \cap F_j \geq r_{i,j}(w) \text{ for } i, j = 1, \dots, n\} \subset \mathcal{B} \times \mathcal{B}, \\ \Omega_w^\circ &:= \{(F_\bullet, V_\bullet); \dim V_i \cap F_j = r_{i,j}(w) \text{ for } i, j = 1, \dots, n\} \subset \mathcal{B} \times \mathcal{B}, \end{aligned} \tag{1.2}$$

where

$$r_{i,j}(w). := |\{k; k \leq i, w(k) \leq j\}|.$$

Then $\Omega_w = \bigsqcup_{z \leq w} \Omega_z^\circ$, where the disjoint union is taken over all permutations smaller than w in the Bruhat order of S_n.

The Kazdhan–Lusztig polynomial $P_{z,w}(q)$ measures the singularity of Ω_w at Ω_z°, in the sense that $P_{z,w}(q) = \sum_i \dim H^i((IC_{\Omega_w})_p) q^{\frac{i}{2}}$, where IC_{Ω_w} is the intersection homology complex of Ω_w and p is a point in Ω_z°.

Note that not all conditions in Equation (1.2) defining Ω_w are necessary: The *coessential set* $\mathrm{Coess}(w)$ of w is the smallest set of pairs (i, j) such that

$$\Omega_w = \{(F_\bullet, V_\bullet); \dim V_i \cap F_j \geq r_{i,j}(w)\}.$$

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Figure 1. *The graphical representation of the Dyck path associated to the Hessenberg function* $\mathbf{m} = (2,4,5,5,6,6)$ *and of the codominant permutation* $w_{\mathbf{m}} = 245361$. *To find the coessential set of w, we remove every square that is below or to the left of a dot (greyed out in the picture). The coessential set is then the set of squares that are in the upper-right corner of the connected components of the remaining figure, the squares marked with a circle,* $\mathrm{Coess}(w) = \{(1,2),(2,4),(4,5),(6,6)\}$.

Equivalently, we have

$$\mathrm{Coess}(w) := \{(i,j); w(i) \leq j < w(i+1),\ w^{-1}(j) \leq i < w^{-1}(j+1)\}.$$

See [Ful92] for more details, specially [Ful92, Equation 3.8]. Also, note there is a slight duality between the essential set and the coessential set.

If a permutation w satisfies $r_{i,j}(w) = \min(i,j)$ for every $(i,j) \in \mathrm{Coess}(w)$, we say that Ω_w is *defined by inclusions*. Indeed, the condition $\dim V_i \cap F_j = r_{i,j}(w)$ is equivalent to either $V_i \subset F_j$ or $F_j \subset V_i$. If Ω_w is defined by inclusions and for every $(i_0, j_0), (i_1, j_1) \in \mathrm{Coess}(w)$ with $i_0 \leq j_0$ and $j_1 \leq i_1$ we have that either $j_0 \leq j_1$ or $i_1 \leq i_0$, then we say that Ω_w is *defined by noncrossing inclusions*.

Given $w \in S_n$, the following conditions are equivalent (see [GR02, Theorem 1.1]):

1. $P_{e,w}(q) = 1$,
2. Ω_w is smooth,
3. Ω_w is defined by noncrossing inclusions,
4. w avoids the patterns 3412 and 4231.

Definition 1.1. A permutation satisfying any of the conditions above is called *smooth*, otherwise it is called *singular*.

If the inclusions defining Ω_w are all of the form $V_i \subset F_j$, that is, if $i \leq j$ for every $(i,j) \in \mathrm{Coess}(w)$, we say that w is *codominant*. Codominant permutations are precisely the 312-avoiding permutations, and there is a natural bijection between codominant permutations and Hessenberg functions (or Dyck paths), that is, nondecreasing functions $\mathbf{m}\colon [n] \to [n]$ satisfying $\mathbf{m}(i) \geq i$ for $i = 1, \ldots, n$. The codominant permutation $w_{\mathbf{m}}$ associated to $\mathbf{m}$ is the lexicographically greatest permutation satisfying $w_{\mathbf{m}}(i) \leq \mathbf{m}(i)$ for all $i \in [n]$ (see Figure 1).

For codominant permutations $w_{\mathbf{m}}$, the Schubert varieties are characterized by

$$\Omega_{w_{\mathbf{m}}} = \{(V_\bullet, F_\bullet); V_i \subset F_{\mathbf{m}(i)}\}.$$

The bijection between codominant permutations and Hessenberg functions can be extended to map from the set of smooth permutations to the set of Hessenberg functions. Indeed, for every smooth permutation w, we can define a Hessenberg function $\mathbf{m}_w$ as follows. Let $I \subset [n]$ be the subset of indices i such that there exists $j \geq i$ with either $(i,j) \in \mathrm{Coess}(w)$ or $(j,i) \in \mathrm{Coess}(w)$. We define $\mathbf{m}_w$ by the conditions $\mathbf{m}_w(i) = \mathbf{m}_w(i+1)$ if $i \notin I$ and $\mathbf{m}_w(i) = j$ if $i \in I$ and j is such that either (i,j) or (j,i) is in $\mathrm{Coess}(w)$. The noncrossing condition implies that $\mathbf{m}_w$ is indeed an Hessenberg function and, if we enrich the set of Hessenberg functions with some extra datum (the datum where the inclusions change from $V_i \subset F_j$ to $F_i \subset V_j$) we can achieve a bijection; see [GL20].

We now turn our attention to characters of the Hecke algebra. Each irreducible $\mathbb{C}$-representation of S_n lifts to an irreducible $\mathbb{C}(q^{\frac{1}{2}})$-representation of H_n (see [GP00, Theorem 8.1.7]). Hence, if χ^λ is the irreducible character of S_n associated to the partition $\lambda \vdash n$ and, abusing notation, χ^λ is the

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corresponding character of H_n, we can define the *(dual) Frobenius character* of an element $a \in H_n$ by

$$\mathrm{ch}(a) := \sum_{\lambda \vdash n} \chi^\lambda(a) s_\lambda(x) \in \mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda,$$

where Λ is the algebra of symmetric functions in the variables

$$x = (x_1, \ldots, x_m, \ldots)$$

and $s_\lambda(x)$ is the Schur symmetric function associated to the partition λ. For a graded S_n-module L, we also write $\mathrm{ch}(L)$ for its *(graded) Frobenius character*.

In [Hai93, Lemma 1.1], Haiman proved that $\chi^\lambda(q^{\frac{\ell(w)}{2}} C'_w)$ is a symmetric unimodal polynomial in q with nonnegative integer coefficients. We note that [Hai93, Lemma 1.1] implies that $\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w)$ is Schur-positive, in the sense that its coefficients in the Schur-basis are polynomials in q with nonnegative integer coefficients.

Haiman also made some conjectures regarding positivity of the characters $\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w)$ and relations between them. A symmetric function in $\mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda$ is called h-positive if its coefficients in the complete homogeneous basis $\{h_\lambda\}$ are polynomials in q with nonnegative coefficients.

Conjecture 1.2 (Haiman). *For any $w \in S_n$, the (dual Frobenius) character $\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w)$ of the Kazhdan–Lusztig element C'_w is h-positive.*

For a Hessenberg funtion $\mathbf{m}\colon [n] \to [n]$, there is an associated graph $G_{\mathbf{m}}$, called an indifference graph. It is constructed as follows, its set of vertices is $[n]$ and there is an edge between i and j if $i < j \leq \mathbf{m}(i)$. These graphs are precisely the unit interval order graphs, also the incomparability graphs of $3+1$ and $2+2$ free (finite) posets. There is a close relation between the character $\mathrm{ch}(q^{\frac{\ell(w_{\mathbf{m}})}{2}} C'_{w_{\mathbf{m}}})$ and the indifference graph $G_{\mathbf{m}}$, which we now make explicit.

The chromatic quasisymmetric function of a graph with vertex set $[n]$, as introduced by Shareshian–Wachs in [SW16], is defined as follows

$$\mathrm{csf}_q(G) := \sum_{\kappa\colon [n] \to \mathbb{Z}_{\geq 0}} q^{\mathrm{asc}_G(\kappa)} \prod_{v \in [n]} x_{\kappa(v)},$$

where the sum runs through all proper colorings κ (that is, $\kappa(i) \neq \kappa(j)$ if $\{i, j\}$ is an edge of G) and

$$\mathrm{asc}_{\mathbf{m}}(\kappa) := |\{\{i, j\}; i < j, \kappa(i) < \kappa(j), \{i, j\} \text{ is a an edge of } G\}|.$$

For indifference graphs, the chromatic quasisymmetric function is actually a symmetric function, and we write $\mathrm{csf}_q(\mathbf{m}) := \mathrm{csf}_q(G)$.

By [CHSS16] (see also Corollary 3.6 below), we have that the character $\mathrm{ch}(q^{\frac{\ell(w_{\mathbf{m}})}{2}} C'_{w_{\mathbf{m}}})$ is the omega-dual of the chromatic quasisymmetric function of $G_{\mathbf{m}}$. That is:

$$\mathrm{ch}(q^{\frac{\ell(w_{\mathbf{m}})}{2}} C'_{w_{\mathbf{m}}}) = \omega(\mathrm{csf}_q(\mathbf{m})). \tag{1.3}$$

In particular, Conjecture 1.2 implies the Stanley–Stembridge conjecture on e-positivity of the chromatic symmetric function of indifference graphs of $3+1$ free posets (via results of Guay–Paquet, [GP13]) and the Shareshian–Wachs generalization of the Stanley–Stembridge conjecture on e-positivity of the chromatic quasisymmetric function of indifference graphs.

Haiman also made a conjecture about the relations between the characters $\mathrm{ch}(C'_w)$, namely, that every character $\mathrm{ch}(C'_w)$ is a sum of characters of Kazhdan–Lusztig elements of codominant permutations.

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Conjecture 1.3 [Hai93, Conjecture 3.1]. *For any $w \in S_n$, there exist codominant permutations $w_1, \dots, w_k$ such that*

$$\mathrm{ch}(C'_w) = \mathrm{ch}(C'_{w_1}) + \mathrm{ch}(C'_{w_2}) + \cdots + \mathrm{ch}(C'_{w_k})$$

and [2]

$$P_{e,w}(q) = \sum_{1 \le i \le k} q^{\frac{\ell(w)-\ell(w_i)}{2}}.$$

Conjecture 1.3 restricts to the following statement when w is smooth.

Conjecture 1.4. *If w is a smooth permutation, there exists a single codominant permutation w' such that*

$$\mathrm{ch}(C'_w) = \mathrm{ch}(C'_{w'}).$$

Haiman pointed out in [Hai93] that Conjectures 1.4 and 1.3 should 'reflect aspects of the geometry of the flag variety that cannot yet be understood using available geometric machinery'. Conjecture 1.4 was first proved combinatorially by Clearman–Hyatt–Shelton–Skandera in [CHSS16]. The purpose of this article is to provide a geometric proof of the same result, as well as a counterexample to Conjecture 1.3.

1.1. Results

Let X be an $n \times n$ matrix and w be a permutation. The *Lusztig variety* associated to X and w is the subvariety of the flag variety defined by

$$\mathcal{Y}_w(X) := \{V_\bullet; XV_i \cap V_j \ge r_{i,j}(w) \text{ for } i, j = 1, \dots, n\}. \tag{1.4}$$

When X is regular semisimple (has distinct eigenvalues), the intersection homology $IH^*(\mathcal{Y}_w(X))$ has a natural graded S_n-module structure induced by the monodromy action of $\pi_1(GL_n^{rs}, X)$ on $IH^*(\mathcal{Y}_w(X))$. For w a smooth permutation, so that $\mathcal{Y}_w(X)$ is also smooth, this action can be explicitly characterized by a *dot action* on $H^*(\mathcal{Y}_w(X))$ (as in [Tym08]). We have the following result due to Lusztig [Lus86], (see also [AN22]).

Theorem 1.5 (Lusztig). *For any $w \in S_n$, we have*

$$\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w) = \mathrm{ch}(IH^*(\mathcal{Y}_w(X))).$$

In Section 2, we will prove the following:

Theorem 1.6. *Let $X \in SL_n(\mathbb{C})$ be regular semisimple and $w \in S_n$ smooth. Then there exists a codominant permutation w' such that $H^*(\mathcal{Y}_w(X))$ and $H^*(\mathcal{Y}_{w'}(X))$ are isomorphic as S_n-modules. In particular,* $\mathrm{ch}(C'_w) = \mathrm{ch}(C'_{w'})$.

The main idea is to see that both $\mathcal{Y}_w(X)$ and $\mathcal{Y}_{w'}(X)$ are smooth GKM spaces, and hence their cohomologies are described by their moment graphs. Since the moment graph of $\mathcal{Y}_w(X)$ only depends on the transpositions which are smaller than w in the Bruhat order, it suffices to see that there exists a codominant permutation whose set of smaller transpositions is equal to that of w. In fact, these transpositions are precisely the transpositions (i, j) such that $i < j \le \mathbf{m}_w(i)$ (see, for example, [GL20]).

If w and w' are Coxeter elements, a stronger result holds, and we actually have that $\mathcal{Y}_w(X)$ is isomorphic to $\mathcal{Y}_{w'}(X)$ whenever X is regular semisimple (see [AN22, Example 1.23]). Although for Coxeter elements, Conjecture 1.4 is a consequence of [Hai93, Proposition 4.2]. We note that our proof of Theorem 1.6 only proves the isomorphisms of cohomology groups and not of varieties (see Conjecture 3.9).

[2] The condition on the Kazhdan–Lusztig polynomials is a consequence of the character equality.

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Concerning singular permutations, we have the following theorems.

Theorem 1.7. *Let $w \in S_n$ be a singular permutation and s a simple transposition such that ws is smooth and $sws < w$. Then*

$$\operatorname{ch}(C'_w) = (q^{-1/2} + q^{1/2}) \operatorname{ch}(C'_{ws}).$$

The analogous equality holds if sw is smooth. Geometrically, if w and s satisfy the above conditions and X is regular semisimple, then $\mathcal{Y}_w(X)$ and $\mathcal{Y}_{ws}(X)$ fit into the following diagram

$$\begin{tikzcd} & \mathcal{Y}_{ws}(X) \arrow[d, "g"] \\ \mathcal{Y}_w(X) \arrow[r, "f"] & \mathcal{Z}, \end{tikzcd}$$

where f is a $\mathbb{P}^1$-bundle and g is small.

Theorem 1.7 is a direct consequence of Corollary 3.2, Lemma 3.3 and Proposition 3.4. These results also apply when w is smooth, in which case we recover the so-called *modular law* for the chromatic quasisymmetric function of indifference graphs (see [AN21a]) and provide a geometric interpretation of it in Example 3.5 (see also [DCLP88] and [PS22]). The modular law also appears in other symmetric functions associated to indifference graphs, such as the LLT-polynomials ([Lee20]) and the symmetric function of increasing forests ([AN21b]).

Theorem 1.8 (Counterexample to Conjecture 1.3). *Let $w = 62754381 \in S_8$. Then $P_{e,w}(q) = 1 + q$ and there do not exist codominant permutations w_0, w_2 such that*

$$\operatorname{ch}(C'_w) = \operatorname{ch}(C'_{w_0}) + \operatorname{ch}(C'_{w_2}).$$

Proof. Set $s = (1, 2)$. Then $sws = 16754382 < w$. Moreover, $ws = 26754381 = w_{\mathbf{m}_1}$, where $\mathbf{m}_1 = (2, 6, 7, 7, 7, 7, 8, 8)$ is a Hessenberg function. In particular, ws is codominant, hence smooth, so that $P_{e,w}(q) = 1 + q$. By Theorem 1.7, we have that $\operatorname{ch}(C'_w) = (q^{-\frac{1}{2}} + q^{\frac{1}{2}}) \operatorname{ch}(C'_{ws})$. Assume that there exist codominant permutations w_0 and w_2 such that

$$(q^{-\frac{1}{2}} + q^{\frac{1}{2}}) \operatorname{ch}(C'_{ws}) = \operatorname{ch}(C'_{w_0}) + \operatorname{ch}(C'_{w_2}).$$

By the equality in Conjecture 1.3, we have that $\ell(w_0) = 15$ and $\ell(w_2) = 17$ (note that $\ell(w_{\mathbf{m}_1}) = 16$ and $\ell(w) = 17$). By Equation (1.3), there exist Hessenberg functions $\mathbf{m}_0$ and $\mathbf{m}_2$ such that (recalling $w_{\mathbf{m}_1} = ws$)

$$(1 + q) \operatorname{csf}_q(G_{\mathbf{m}_1}) = \operatorname{csf}_q(G_{\mathbf{m}_2}) + q \operatorname{csf}_q(G_{m_0}). \tag{1.5}$$

There are 63 Hessenberg functions $\mathbf{m}_0$ with $\ell(w_{\mathbf{m}_0}) = 15$ and 42 Hessenberg functions $\mathbf{m}_2$ with $\ell(w_{\mathbf{m}_2}) = 17$. Computing $\operatorname{csf}_q(\mathbf{m}_1)$ and all the possible values $\operatorname{csf}_q(\mathbf{m}_0)$ and $\operatorname{csf}_q(\mathbf{m}_2)$ (for instance, using the algorithm in [AN21a]), we can check that there do not exist $\mathbf{m}_0$ and $\mathbf{m}_2$ satisfying the condition,

$$(1 + q) \operatorname{csf}_q(G_{\mathbf{m}_1}) = \operatorname{csf}_q(G_{\mathbf{m}_2}) + q \operatorname{csf}_q(G_{m_0}).$$

This finishes the proof. ◻

In view of Theorems 1.7 and 1.8, we propose a weaker version of Conjecture 1.3:

Conjecture 1.9. *For each permutation $w \in S_n$, there exists codominant permutations $w_1, \ldots, w_k \in S_n$ such that $\operatorname{ch}(q^{\frac{\ell(w)}{2}} C'_w)$ is a combination of $\operatorname{ch}(q^{\frac{\ell(w_i)}{2}} C'_{w_i})$ with coefficients in $\mathbb{N}[q]$.*

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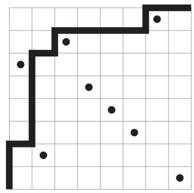

Figure 2. *The graphical representation of the Dyck path associated to the Hessenberg function* $\mathbf{m}_1 = (2, 4, 5, 5, 6, 6)$ *and of the permutation* $w = 62754381$.

2. Proof of Theorem 1.6

We begin by recalling some properties of GKM-spaces (see [GKM98]). A *GKM-space*, is a smooth projective variety $\mathcal{X}$ with an action of a torus T such that the number of fixed points and the number of one-dimensional orbits are finite. The equivariant cohomology $H_T^*(\mathcal{X})$ is then encoded in a combinatorial object called the *moment graph* of $\mathcal{X}$. The vertices of the moment graph are the fixed points, while the edges are the one-dimensional orbits, each of which has exactly two fixed points on its closure. More precisely, the moment graph describes the image of the inclusion map $H_T^*(X) \hookrightarrow H_T^*(X^T)$.

If X is an $n \times n$ diagonal regular semisimple matrix, the torus $T \cong (\mathbb{C}^*)^n$ of diagonal matrices acts on the variety $\mathcal{Y}_w(X)$. When w is smooth, this variety is a GKM-space because the action is a restriction of that of T on the whole flag variety, where the number of fixed points and one-dimensional orbits are indeed finite.

Since $\mathcal{Y}_w(X)$ is a T-invariant subvariety of $\mathcal{B}$, we have that the moment graph of $\mathcal{Y}_w(X)$ is a subgraph of the moment graph of the flag variety $\mathcal{B}$. We briefly recall the moment graph of $\mathcal{B}$ (see [Car94] and [Tym08, Proposition 2.1]). The fixed points in $\mathcal{B}$ are indexed by permutations $w \in S_n$ (in fact, they are equal to $\mathcal{Y}_e(X)$ for X a regular semisimple diagonal matrix). To see this, it is enough to see that a flag $V_\bullet$ is fixed by T if and only if each V_i is generated by eigenvectors of T. However, the eigenvectors of T are precisely the canonical basis vectors $e_1, \dots, e_n$, so there exists $w \in S_n$ such that $V_i = \langle e_{w(1)}, \dots, e_{w(n)} \rangle$.

The one-dimensional orbits are associated to tuples (w_1, w_2, t), where w_1, w_2 are permutations in S_n (corresponding to fixed points) with $\ell(w_1) < \ell(w_2)$ and t is a transposition satisfying $w_1 = w_2 t$. Then the orbit can be described as follows: Write $t = (ij)$ with $i < j$, and define $v_i = e_{w_2(i)} + ce : w_2(j)$ for $c \in \mathbb{C}^*$. When varying $c \in \mathbb{C}^*$, the flags $V_\bullet^c$ given by $V_k^c = \langle e_{w_2(1)} \dots e_{w_2(i-1)}, v_i, e_{w_2(i+1)}, \dots, e_{w_2(k)} \rangle$ determine the one-dimensional orbit given by (w_1, w_2, t). In fact, when c goes to 0, the limit of $V_\bullet^c$ is the flag induced by w_2, while when c goes to infinity, the limit of $V_\bullet^c$ is V_{w_1}. So the one-dimensional orbit associated to (w_1, w_2, t) connects the fixed points corresponding to w_1 and w_2.

To describe the moment graph of $\mathcal{Y}_w(X)$, it is enough to see which fixed points and one-dimensional orbits are contained in $\mathcal{Y}_w(X)$. Since $\mathcal{Y}_e(X) \subset \mathcal{Y}_w(X)$, we have that all fixed points of $\mathcal{B}$ belong in $\mathcal{Y}_w(X)$. We claim the following.

Lemma 2.1. *The one-dimensional orbit associated to* (w_1, w_2, t) *is contained in* $\mathcal{Y}_w(X)$ *if and only if the transposition* t *is smaller than* w *in the Bruhat order of* S_n.

Proof. Consider the flag $V_\bullet^c$ in the one-dimensional orbit (w_1, w_2, t). An easy computation shows that $XV_\ell^c \cap V_k^c = r_{\ell,k}(t)$. In particular, $V_\bullet^c \in \mathcal{Y}_t(X)^\circ$. Since $\mathcal{Y}_w(X) = \bigsqcup_{z \le w} \mathcal{Y}_z(X)^\circ$, we have that $V_\bullet^c \in \mathcal{Y}_w(X)$ if and only if $t \le w$. □

Moreover, the moment graph also encodes the action of S_n on the equivariant cohomology group $H_T^*(\mathcal{Y}_w(X))$; see [Tym08] and [BC18, Section 9]. This follows from the fact that $H_T^*(\mathcal{Y}_w(X))$ is contained $H_T^*(\mathcal{Y}_e(X))$. The latter admits a natural action of S_n, constructed as follows: The variety $\mathcal{Y}_e(X)$ consists of $n!$ points p_w and $H_T^*(p_w) = \mathbb{C}[t_1, \dots, t_n]$. For a permutation $\sigma \in S_n$, it acts on the tuple

$$(f_w(t_1, \dots, t_n))_{w \in S_n} \in \bigoplus_{w \in S_n} \mathbb{C}[t_1, \dots, t_n] = H_T^*(\mathcal{Y}_e(X)),$$

by

$$\sigma \cdot (f_w(t_1, \dots, t_n))_{w \in S_n} = (g_w(t_1, \dots, t_n))_{w \in S_n},$$

where $g_w = f_{\sigma^{-1}w}(t_{\sigma(1)}, \dots, t_{\sigma(n)})$. This action restricts to an action on $H_T^*(\mathcal{Y}_w(X))$. Since the moment graph describes the image of the inclusion $H_T^*(\mathcal{Y}_w(X)) \hookrightarrow H_T^*(\mathcal{Y}_e(X))$, we have that it also describes the S_n action on $H_T^*(\mathcal{Y}_w(X))$. In particular, if w and w' are smooth permutations and $\mathcal{Y}_w(X)$ and $\mathcal{Y}_{w'}(X)$ have the same moment graph, then

$$\operatorname{ch}(H^*(\mathcal{Y}_w(X))) = \operatorname{ch}(H^*(\mathcal{Y}_{w'}(X))).$$

Lemma 2.2. *Let w be a smooth permutations and $\mathbf{m}$ its associated Hessenberg function. A transposition $t = (ij)$ with $i < j$ is smaller that or equal to w in the Bruhat order of S_n if and only if $j \leq \mathbf{m}(i)$.*

Proof. This is contained in [GL20, Theorem 5.1]. One can see this geometrically from the characterization of smooth Schubert varieties. Consider the pair $(V_\bullet, F_\bullet)$, where $V_\bullet$ is induced by the matrix

$$(e_1, \dots, e_{i-1}, e_j, e_{i+1}, \dots, e_{j-1}, e_i, e_{j+1}, \dots, e_n)$$

and $F_\bullet$ is induced by the identity matrix $(e_1, \dots, e_n)$. Then we have $V_i \subset F_j$ and $F_i \subset V_j$, but $V_i \not\subset F_{j-1}$ and $F_i \not\subset V_{j-1}$. In particular, we have that $(V, F) \in \Omega_w$ if and only if $j \leq \mathbf{m}(i)$. Since $(V, F) \in \Omega_t^\circ$, the result holds. □

Proof of Theorem 1.6. Let w' be the codominant permutation associated to the Hessenberg function $\mathbf{m}$ associated to w. By Lemmas 2.1 and 2.2, the moment graphs of $\mathcal{Y}_w(X)$ and $\mathcal{Y}_{w'}(X)$ are equal and since the dot action only depends on the moment graph, $\operatorname{ch}(H^*(\mathcal{Y}_w(X))) = \operatorname{ch}(H^*(\mathcal{Y}_{w'}(X)))$. By Theorem 1.5, we have the result. □

3. Proof of Theorem 1.7

To prove Theorem 1.7, we need a few algebraic results about Hecke algebras and singular permutations. Let $w \in S_n$ be a permutation and s a simple transposition. Assume that $sw < w < ws$. Then by the multiplication rule of Kazhdan–Lusztig elements of the Hecke algebra (see [Hai93, Equation 8.8]), we have

$$\begin{aligned} C'_w C'_s &= C'_{ws} + \sum_{\substack{z \leq w \\ zs < z}} \mu(z, w) C'_z, \\ C'_s C'_w &= (q^{-\frac{1}{2}} + q^{\frac{1}{2}}) C'_w, \end{aligned}$$

where $\mu(z, w)$ is the coefficient of $q^{\frac{\ell(w)-\ell(z)-1}{2}}$ in the Kazhdan–Lusztig polynomial $P_{z,w}(q)$. Since $\chi^\lambda(C'_w C'_s) = \chi^\lambda(C'_s C'_w)$ for every partition $\lambda \vdash n$, we have that

$$\operatorname{ch}((q^{-\frac{1}{2}} + q^{\frac{1}{2}}) C'_w) = \operatorname{ch}(C'_{ws}) + \sum_{\substack{z \leq w \\ zs < z}} \mu(z, w) \operatorname{ch}(C'_z). \tag{3.1}$$

If w is smooth, then $\mu(z, w) = 0$ except for the permutations z such that $z \leq w$ and $\ell(z) = \ell(w) - 1$, and in this case $\mu(z, w) = 1$. To simplify notation, we will write $z \lessdot w$ to mean that $z \leq w$ and $\ell(z) = \ell(w) - 1$. We will see below that if w is smooth and satisfies $sw < w < ws$ for some simple reflection s, then there exists at most one permutation z satisfying $z \lessdot w$ and $zs < z$.

Proposition 3.1. *Let $w \in S_n$ be a smooth permutation and s a simple reflection such that $sw < w < ws$. Then one of the following holds:*

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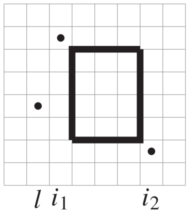

Figure 3. *The relative position of $w(l)$, $w(i_1)$ and $w(i_2)$ given by Equation (3.2). Note that we can not have any dots inside the box.*

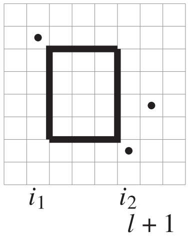

Figure 4. *The relative position of $w(l+1)$, $w(i_1)$ and $w(i_2)$ given by Equation (3.3). Note that we can not have any dots inside the box.*

1. *The permutation ws is smooth and there exists precisely one $z \lessdot w$ such that $zs < z$. Moreover, z is smooth.*
2. *The permutation ws is singular and there does not exists any $z \lessdot w$ such that $zs < z$.*

Proof. We first prove that there exists at most one $z \lessdot w$ such that $zs < z$. Write $s = (l, l+1)$, and assume that $z \in S_n$ is a permutation satisfying $z \lessdot w$ and $zs < s$. Since $z \lessdot w$ (which means that $\ell(z) = \ell(w) - 1$), we have that there exist i_1, i_2 such that

- $1 \leq i_1 < i_2 \leq n$,
- $z(j) = w(j)$ for every $j \in [n] \setminus \{i_1, i_2\}$,
- $z(i_k) = w(i_{3-k})$,
- $w(i_1) > w(i_2)$,
- for every $i_1 < j < i_2$ we have that either $w(j) < w(i_2)$ or $w(j) > w(i_1)$.

Since $ws > w$ and $zs < z$, we have $w(l) < w(l+1)$ and $z(l) > z(l+1)$. Hence, either $i_1 = l+1$ or $i_2 = l$.

If $i_1 = l+1$, we have

$$\begin{aligned} &w(j) < w(i_2) \text{ or } w(j) > w(i_1) = w(l+1) \text{ for every } i_1 < j < i_2, \\ &w(l+1) = w(i_1) > w(l) > w(i_2). \end{aligned} \tag{3.2}$$

On the other hand, if $i_2 = l$, we have

$$\begin{aligned} &w(j) < w(i_2) = w(l) \text{ or } w(j) > w(i_1) \text{ for every } i_1 < j < i_2, \\ &w(i_1) > w(l+1) > w(i_2) = w(l). \end{aligned} \tag{3.3}$$

See Figures 3 and 4 below for a depiction of these conditions.

Assume that there exist two distinct permutations z, z' satisfying the conditions above, and let i_1, i_2 and i'_1, i'_2 be as above for z and z', respectively. We now compare the relative position of i_1, i_2, i'_1, i'_2.

- Case 1. Assume that $i_2 = i'_2 = l$ and $i_1 < i'_1$ (the case $i_1 < i'_1$ being analogous). By Equation (3.3), we have that $w(i_1) > w(l+1) > w(l)$, $w(i'_1) > w(l+1) > w(l)$. Since $i_1 < i'_i < i_2$ and $w(i'_1) > w(l)$,

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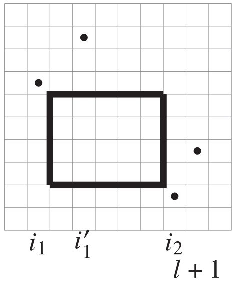

Figure 5. *The relative position of $w(i_1)$, $w(i'_1)$, $w(i_2)$ and $w(l+1)$. Note that $w(i'_1)$ must be outside the box, and hence $w(i'_1) > w(i_1)$, we have a 3412 pattern on w.*

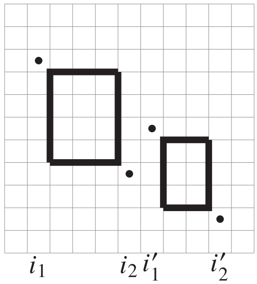

Figure 6. *The relative position of $w(i_1)$, $w(i_2)$, $w(i'_1)$ and $w(i_2)$. Note that we have a 4231 pattern on w.*

we have $w(i'_1) > w(i_1)$ (again, by Equation (3.3)). Hence, $w(i'_1) > w(i_1) > w(\ell+1) > w(\ell)$ and this is a 3412 pattern on w, which is a contradiction with the smoothness of w. See Figure 5.

- Case 2. Assume that $i_1 = i'_1 = l+1$. This case is analogous to the previous one (just replace Equation (3.3) with Equation (3.2)).
- Case 3. Assume that $i_2 = l$ and $i'_1 = l+1$. In this case, we have that $i_1 < i_2 = l < i'_1 = l+1 < i'_2$. By Equations (3.3) and (3.2), $w(l+1) > w(l) > w(i'_2)$ and $w(i_1) > w(l+1) > w(l)$, so $w(i_1) > w(l+1) > w(l) > w(i'_2)$, which is a 4231 pattern on w, contradicting the smoothness of w. See Figure 6.

Similar considerations also prove that if z exists, it must be smooth.

We now prove that if ws is singular, there exists no $z \lessdot w$ with $zs < z$. Since ws is singular, there exist $j_1 < j_2 < j_3 < j_4$ forming a 4231 or 3412 pattern in ws. Since w is smooth, $\{l, l+1\} \subset \{j_1, j_2, j_3, j_4\}$. Since $w(l) < w(l+1)$, we have three cases.

- Case 1. Assume that we have a 4231 pattern in ws with $j_1 = l, j_2 = l+1$. Then j_1, j_2, j_3, j_4 induces a 2431 pattern on w with $j_1 = l, j_2 = l+1$. Let us assume that there exists $i_1 < i_2 := l = j_1$ satisfying Equation (3.3). Then $w(i_1) > w(l+1)$ and $i_1, j_1, \dots, j_4$ induces a 52431 pattern on w, which contains a 4231 pattern, and this is a contradiction. Let us assume that there exists $l+1 = j_2 =: i_1 < i_2$ satisfying Equation (3.2). Then $w(i_2) < w(l)$ and for every $l+1 < k < i_2$ we have either $w(k) > w(l+1)$ or $w(k) < w(i_2)$. Then $i_2 < j_3$ since $w(i_2) < w(l) < w(j_3) < w(l+1)$. This means that w contains either a 35241 or a 35412 pattern, but the first has a 4231 pattern, while the second has a 3412 pattern, which again contradicts the smoothness of w.
- Case 2. Assume that we have a 4231 pattern in ws with $j_3 = l, j_4 = l+1$. Then we have a 4213 pattern on w, and the argument is similar as above.
- Case 3. Assume that we have a 3412 pattern in ws with $j_2 = l, j_3 = l+1$, so that j_1, j_2, j_3, j_4 induces a 3142 pattern on w with $j_2 = \ell, j_3 = \ell+1$. Let us assume there exists $i_1 < i_2 := l = j_2$ satisfying Equation (3.3). Then $w(i_1) > w(l+1)$, and for every $i_1 < k < l$ we have either $w(k) > w(i_1)$ or $w(k) < w(l)$. Then $i_1 > j_1$ and we have a 35142 pattern on w, a contradiction. Let us assume

that there exist $l+1 = j_3 =: i_1 < i_2$ satisfying Equation (3.2). Then $w(i_2) < w(l)$ and for every $l+1 < k < i_2$ we have either $w(k) < w(i_2)$ or $w(k) > w(l+1)$ so that $i_2 < j_4$ and we have a 42513 pattern on w, also a contradiction.

Finally, we will prove that if there is no $z \lessdot w$ with $zs < z$, then ws is singular. First, assume that there exists $i < l$ such that $w(i) > w(l+1)$ and consider the greatest possible such i. If $z = w \cdot (i,l)$, then $zs < z$ and $z < w$. This means that $z << w$, and that is equivalent to the existence of $i < j < l$ with $w(i) > w(j) > w(l)$. Since i is the greatest $i < l$ with $w(i) > w(l+1)$, we have that $w(i) > w(l+1) > w(j) > w(l)$, which implies that $i, j, l, l+1$ induces a 4213 pattern on w and hence a 4231 pattern on ws. If there exists $i > l+1$ with $w(i) < w(l)$, the argument is the same.

Therefore, let us assume that $w(i) < w(l+1)$ for every $i < l$ and $w(i) > w(l)$ for every $i > l+1$. In particular, we have that $w^{-1}(j) < l$ for every $j < w(l)$. Let k be the maximum of $\{w(i)\}_{i\le l}$, and note that $w(l) \le k < w(\ell+1)$. Assume that there exists $j < k$ with $w^{-1}(j) > l+1$. By the argument above, we have that $j > w(l)$ (and hence $k > w(l)$), so $w^{-1}(k) < l < l+1 < w^{-1}(j)$ and $w(l+1) > k > j > w(l)$, which implies that $w^{-1}(k), l, l+1, w^{-1}(j)$ induces a 3142 pattern on w, and hence a 4231 pattern on ws. On the other hand, if $w^{-1}(j) \le l$ for every $j \le k$, then $\{w(1), \dots, w(l)\} = \{1, \dots, k\}$, and in particular $k = l$. But then $(l, l+1)w > w$, a contradiction since $sw < w$ by hypothesis. This finishes the proof. □

We have the following direct corollary.

Corollary 3.2. *Let w be a smooth permutation and s a simple transposition such that $ws > w > sw$.*

1. *If ws is smooth and z is the only permutation $z \lessdot w$ with $zs < z$, then* $(q^{-\frac{1}{2}} + q^{\frac{1}{2}})\operatorname{ch}(C'_w) = \operatorname{ch}(C'_{ws}) + \operatorname{ch}(C'_z)$.
2. *If ws is singular, then* $(q^{-\frac{1}{2}} + q^{\frac{1}{2}})\operatorname{ch}(C'_w) = \operatorname{ch}(C'_{ws})$.

Proof. It follows directly from Equation (3.1) and Proposition 3.1. □

Note that Corollary 3.2 proves the combinatorial statement of Theorem 1.7. We now prove the geometric statement, which also gives an alternative proof of the combinatorial statement.

Let w and s be as in Corollary 3.2, and let $\mathcal{P}_s$ be the partial flag variety associated to s, that is, if $s = (l, l+1)$ then

$$\mathcal{P}_s = \{V_1 \subset V_2 \subset \dots V_{l-1} \subset V_{l+1} \subset \dots \subset V_n = \mathbb{C}^n; \dim_{\mathbb{C}}(V_i) = i\}.$$

Using the algebraic group notation, we write $G = GL_n$ and B for the Borel subgroup of G of upper triangular matrices. For each permutation $w \in S_n$ let $\dot{w}$ denote the associated permutation matrix $\dot{w} \in G$. We write P_s for the parabolic subgroup associated to s, that is, $P_s = B \sqcup B\dot{s}B$ so that $\mathcal{P}_s = G/P_s$. In this notation, the Lusztig varieties are given by $\mathcal{Y}_w(X)^\circ = \{gB; g^{-1}Xg \in B\dot{w}B\}$.

Lemma 3.3. *Let $w \in S_n$ be a permutation, s a simple transposition and X a regular semisimple $n \times n$ matrix. Then*

1. *If $sw < w$ and $ws < w$, then the forgetful map $\mathcal{Y}_w(X) \to \mathcal{P}_s$ is a $\mathbb{P}^1$-bundle over its image.*
2. *If $ws \neq sw$ and either $w < ws$ or $w < sw$, then the forgetful map $\mathcal{Y}^\circ_w(X) \to \mathcal{P}_s$ is injective.*

Proof. We begin with item (1). For $s = (l, l+1)$, the hypothesis is equivalent to $w(l) > w(l+1)$ and $w^{-1}(l) > w^{-1}(l+1)$, and in particular, the coessential set of w

$$\operatorname{Coess}(w) := \{(a,b); w(a) \le b < w(a+1), w^{-1}(b) \le a < w^{-1}(b+1)\}$$

does not contains any pair (a,b) with either $a = l$ or $b = l$. This means that the conditions involving $\dim(XV_l \cap V_b)$ and $\dim(XV_a \cap V_l)$ are redundant in $\mathcal{Y}_w(X)$, hence V_l can be chosen arbitrarily.

Let us prove item (2). Since $\mathcal{Y}^\circ_w(X) = \{gB; g^{-1}Xg \in B\dot{w}B\}$, to prove that the map $\mathcal{Y}^\circ_w(X) \to \mathcal{P}_s$ is injective it suffices to prove that there do not exist g_1B and g_2B distinct such that $g_1^{-1}Xg_1 \in B\dot{w}B$, $g_2^{-1}Xg_2 \in B\dot{w}B$, and $g_1 \in g_2P_s$. Assume by way of contradiction that such a pair g_1, g_2 exists. Since

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$P_s = B \cup B\dot{s}B$ and $g_1 B \neq g_2 B$, we have that $g_1 \in g_2 B\dot{s}B$, in particular $g_1 = g_2 b_1 \dot{s} b_2$ for some $b_1, b_2 \in B$. Therefore,

$$g_2^{-1} X g_2 \in B\dot{w}B,$$
$$b_2^{-1}\dot{s}b_1^{-1}g_2^{-1} X g_2 b_1 \dot{s} b_2 \in B\dot{w}B.$$

Since $b_1, b_2 \in B$, we have

$$b_1^{-1}g_2^{-1} X g_2 b_1 \in B\dot{w}B,$$
$$b_1^{-1}g_2^{-1} X g_2 b_1 \in \dot{s}B\dot{w}B\dot{s}.$$

This means that $B\dot{w}B \cap \dot{s}B\dot{w}B\dot{s} \neq \emptyset$. Let us assume, without loss of generality, that $sw < w < ws$. Then by [MT11, proof of Lemma 11.14]

$$B\dot{w}B\dot{s} \subset B\dot{w}B \cdot B\dot{s}B = B\dot{w}\dot{s}B,$$

and by [MT11, Lemma 11.14]

$$\dot{s}B\dot{w}\dot{s}B \subset B\dot{s}B \cdot B\dot{w}\dot{s}B \subset B\dot{w}\dot{s}B \cup B\dot{s}\dot{w}\dot{s}B.$$

Since $sws \neq w$ (otherwise, $ws = sw$), we have

$$B\dot{w}B \cap (B\dot{w}\dot{s}B \cup B\dot{s}\dot{w}\dot{s}B) \neq \emptyset,$$

which is a contradiction of the Bruhat decomposition of G. □

Let X be a regular matrix, $w \in S_n$ an irreducible permutation, that is, a permutation that is not contained in any proper Young subgroup, and s a simple transposition satisfying the conditions in Corollary 3.2. Consider the forgetful map $\mathcal{Y}_{ws}(X) \to \mathcal{P}_s$, and let $\mathcal{Z}$ be the image. By [AN22, Corollary 8.6], $\mathcal{Y}_{ws}(X)$ and $\mathcal{Y}_w(X)$ are irreducible, and so $\mathcal{Z}$ is as well. By Lemma 3.3, the map $\mathcal{Y}_{ws}(X) \to \mathcal{Z}$ is a $\mathbb{P}^1$-bundle, while the map $\mathcal{Y}_w^{\circ}(X) \to \mathcal{P}_s$ is injective. Since $\mathcal{Y}_w(X) \subset \mathcal{Y}_{ws}(X)$ ($w < ws$), the image of $\mathcal{Y}_w(X)$ is contained in $\mathcal{Z}$. Since $\mathcal{Y}_w^{\circ}(X) \to \mathcal{Z}$ is injective and the dimensions agree, $\mathcal{Y}_w(X) \to \mathcal{Z}$ is birational. Let $z \in S_n$ be the permutation such that $z \lessdot w$ and $zs < z$. Then we have:

Proposition 3.4. *The map $\mathcal{Y}_w(X) \to \mathcal{Z}$ is semismall and the preimage of the relevant locus is precisely $\mathcal{Y}_z(X)$ (if z exists).*

Proof. The fact that $\mathcal{Y}_w(X) \to \mathcal{Z}$ is semismall follows from the fact that the map is birational and its fibers have dimension at most one (since they are contained in those of $\mathcal{Y}_{ws}(X) \to \mathcal{Z}$). We have that $\mathcal{Y}_w(X) = \mathcal{Y}_w^{\circ}(X) \cup \bigcup_{z' \lessdot w} \mathcal{Y}_{z'}(X)$, where $\mathcal{Y}_{z'}(X)$ has codimension one in $\mathcal{Y}_w(X)$. We claim that the images of $\mathcal{Y}_w^{\circ}(X)$ and $\mathcal{Y}_{z'}(X)$ are disjoint. Assume for contradiction that there exist $g_1 B$ and $g_2 B$ such that $g_1^{-1} X g_1 \in B\dot{w}B$, $g_2^{-1} X g_2 \in \overline{B\dot{z}'B}$ and $g_1 P_s = g_2 P_s$. Arguing as in the proof of Lemma 3.3, we have

$$\overline{B\dot{z}'B} \cap (B\dot{w}\dot{s}B \cup B\dot{s}\dot{w}\dot{s}B) \neq \emptyset. \tag{3.4}$$

However, $\ell(ws) = \ell(w) + 1$, $\ell(sws) = \ell(w)$ and $\ell(z) = \ell(w) - 1$, and $\overline{B\dot{z}'B} = \bigcup_{z'' \leq z'} B\dot{z}''B$. By the Bruhat decomposition, Equation (3.4) is a contradiction.

Moreover, since the fibers have dimension at most one, the preimage of the relevant locus has codimension one in $\mathcal{Y}_w(X)$. By the discussion above, this preimage must be a union of $\mathcal{Y}_{z'}(X)$ for some $z' \lessdot w$. By the lifiting property [Bre92, Proposition 2.2.7], either $sz' < z'$ or $z' = sw$. If $z' = sw$, then $z' = sw < sws = zs'$ and $z's = sws \neq w = sz'$, so by Lemma 3.3 $\mathcal{Y}_{z'}^{\circ}(X) \to \mathcal{Z}$ is injective, and hence $\mathcal{Y}_{z'}^{\circ}(X)$ is not contained in the preimage of the relevant locus. If $sz' < z'$ and $z' < z's$, then $sz' \neq z's$, so by Lemma 3.3 $\mathcal{Y}_{z'}^{\circ}(X) \to \mathcal{Z}$ is injective, and hence $\mathcal{Y}_{z'}^{\circ}(X)$ is not contained in the preimage of the relevant locus. Finally, if $sz' < z'$ and $z's < z'$, then $z' = z$, so by Lemma 3.3 $\mathcal{Y}_{z'}(X) \to \mathcal{Z}$ is $\mathbb{P}^1$-bundle over its image, and hence $\mathcal{Y}_{z'}(X)$ is contained in the preimage of the relevant locus. Since the preimage of the relavant locus has codimension one, it is precisely $\mathcal{Y}_{z'}(X)$. $\square$

By the decomposition theorem (we set $\mathcal{Z}_1$ as the image of $\mathcal{Y}_z(C)$ if z exsits), $IH^*(\mathcal{Y}_{ws}(X)) = IH^*(\mathcal{Z}) \otimes (\mathbb{C} \oplus \mathbb{C}[-2])$, $H^*(\mathcal{Y}_w(X)) = IH^*(\mathcal{Z}) \otimes IH^*(\mathcal{Z}_1)[-2]$ and $IH^*(\mathcal{Y}_z(X)) = IH^*(\mathcal{Z}_\infty) \otimes (\mathbb{C} \oplus \mathbb{C}[-2])$. Then

$$\begin{aligned}
\mathrm{ch}(IH^*(\mathcal{Y}_{ws}(X))) &= (1+q)\,\mathrm{ch}(IH^*(\mathcal{Z})),\\
\mathrm{ch}(H^*(\mathcal{Y}_w(X))) &= \mathrm{ch}(IH^*(\mathcal{Z})) + q\,\mathrm{ch}(IH^*(\mathcal{Z}_1)),\\
\mathrm{ch}(IH^*(\mathcal{Y}_z(X))) &= (1+q)\,\mathrm{ch}(IH^*(\mathcal{Z}_\infty)),
\end{aligned}$$

which implies

$$(1+q)\,\mathrm{ch}(H^*(\mathcal{Y}_w(X))) = \mathrm{ch}(IH^*(\mathcal{Y}_{ws}(X))) + q\,\mathrm{ch}(IH^*(\mathcal{Y}_z(X))).$$

This, in turn, is equivalent by Theorem 1.5 to

$$(1+q)\,\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w) = \mathrm{ch}(q^{\frac{\ell(w)+1}{2}} C'_{ws}) + q\,\mathrm{ch}(q^{\frac{\ell(w)-1}{2}} C'_z).$$

When w is codominant and ws is smooth, then both ws and z are codominant as well. Below, we give an example of what happens for Hessenberg varieties.

Example 3.5 (Geometric interpretation of the modular law). Let $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2$ be Hessenberg functions and $i \in [n]$ an integer such that $\mathbf{m}_0(j) = \mathbf{m}_1(j) = \mathbf{m}_2(j)$ for every $j \neq i$, $\mathbf{m}_0(i) = \mathbf{m}_1(i) - 1 = \mathbf{m}_2(i) - 2$ and $\mathbf{m}_1(\mathbf{m}_1(i)+1) = \mathbf{m}_1(\mathbf{m}_1(i))$. Set $l = m_1(1)$ and let $s = (l, l+1)$ be a simple transposition.

We claim that $w_{\mathbf{m}_1}s < w_{\mathbf{m}_1} < w_{\mathbf{m}_2} = sw_{\mathbf{m}_1}$, $w_{\mathbf{m}_0} \lessdot w_{\mathbf{m}_1}$ and $sw_{\mathbf{m}_0} < w_{\mathbf{m}_0}$, so we are in the hypothesis of Corollary 3.2. Indeed, since $\mathbf{m}_1(i) = l$ and $\mathbf{m}_1(i-1) < l$, we have that $w_{\mathbf{m}_1}(i) = l$, while $w_{\mathbf{m}_1}^{-1}(l+1) > i$. So $w_{\mathbf{m}_1}s < w_{\mathbf{m}_1} < sw_{\mathbf{m}_1}$. Since $\mathbf{m}_2(i) = l+1$ and $\mathbf{m}_2$ agrees with $\mathbf{m}_1$ everywhere else, $w_{\mathbf{m}_2} = sw_{\mathbf{m}_1}$. Finally, $w_{\mathbf{m}_0} \lessdot w_{\mathbf{m}_1}$, and since $\mathbf{m}_0(i) < l$ and $\mathbf{m}_0(i+1) > l$, we have $sw_{\mathbf{m}_0} < w_{\mathbf{m}_0}$.

Let X be a regular semisimple matrix, then the Hessenberg varieties are

$$\begin{aligned}
\mathcal{Y}_{\mathbf{m}_0} &= \{V_\bullet; XV_i \subset V_{l-1}; XV_j \subset V_{\mathbf{m}_1(j)} \text{ for } j \in [n] \setminus \{i\}\},\\
\mathcal{Y}_{\mathbf{m}_1} &= \{V_\bullet; XV_i \subset V_l; XV_j \subset V_{\mathbf{m}_1(j)} \text{ for } j \in [n] \setminus \{i\}\},\\
\mathcal{Y}_{\mathbf{m}_2} &= \{V_\bullet; XV_i \subset V_{l+1}; XV_j \subset V_{\mathbf{m}_1(j)} \text{ for } j \in [n] \setminus \{i\}\}.
\end{aligned}$$

Since $\mathbf{m}_1(l+1) = \mathbf{m}_1(l)$, the conditions $XV_l \subset V_{\mathbf{m}_1(l)}$ and $XV_{l+1} \subset V_{\mathbf{m}_1(l+1)} = V_{\mathbf{m}_1(l)}$ are redundant. In particular, there exists no condition involving V_k in $\mathcal{Y}_{\mathbf{m}_0}(X)$ and $\mathcal{Y}_{\mathbf{m}_2}(X)$. Then the forgetful maps

$$\begin{aligned}
\mathcal{Y}_{\mathbf{m}_0}(X) &\to \mathcal{P}_s\\
\mathcal{Y}_{\mathbf{m}_2}(X) &\to \mathcal{P}_s
\end{aligned}$$

are $\mathbb{P}^1$-bundles over their images, which are, respectively,

$$\begin{aligned}
\mathcal{Z}_0 &= \{\overline{V}_\bullet; X\overline{V}_i \subset \overline{V}_{l-1}, X\overline{V}_j \subset \overline{V}_{\mathbf{m}_1(j)}, \text{ for } j \in [n] \setminus \{i, l\}\},\\
\mathcal{Z}_2 &= \{\overline{V}_\bullet; X\overline{V}_i \subset \overline{V}_{l+1}, X\overline{V}_j \subset \overline{V}_{\mathbf{m}_1(j)}, \text{ for } j \in [n] \setminus \{i, l\}\},
\end{aligned}$$

where we write $\overline{V}_\bullet$ for a partial flag $\overline{V}_1 \subset \ldots \subset \overline{V}_{l-1} \subset \overline{V}_{l+1} \subset \ldots \subset \overline{V}_n$ in $\mathcal{P}_s$. The fibers of the map $f\colon \mathcal{Y}_{\mathbf{m}_1}(X) \to \mathcal{Z}_2$ can be described as

$$f^{-1}(\overline{V}_\bullet) = \{V_\bullet; V_j = \overline{V}_j \text{ for } j \in [n] \setminus \{l\}, V_{l-1} + XV_i \subset V_l \subset V_{l+1}\}.$$

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So $f^{-1}(\overline{V}_\bullet)$ is isomorphic to $\mathbb{P}^1$ if $X\overline{V}_i \subset V_{l-1}$, as in this case $\overline{V}_{l-1} + X\overline{V}_i = \overline{V}_{l-1}$ or is a single point $V_\bullet$, with $V_l = \overline{V}_{k-1} + X\overline{V}_i$. Note that $\dim \overline{V}_{l-1} + X\overline{V}_i \le l$, as $X\overline{V}_{i-1} \subset V_{m_1(i-1)} \subset V_{l-1}$. In fact, $\mathcal{Y}_{\mathbf{m}_1}(X)$ is the blowup of $\mathcal{Z}_2$ along $\mathcal{Z}_0$.

$$\begin{tikzcd}
\mathcal{Y}_{\mathbf{m}_0}(X) \arrow[r] \arrow[d, "\mathbb{P}^1\text{-bundle}"'] & \mathcal{Y}_{\mathbf{m}_1}(X) \arrow[r] \arrow[d, "\substack{\text{Isomorphism outside}\\ \mathcal{Y}_{\mathbf{m}_0}(X)}"'] & \mathcal{Y}_{\mathbf{m}_2}(X) \arrow[dl, "\mathbb{P}^1\text{-bundle}"] \\
\mathcal{Z}_0 \arrow[r] & \mathcal{Z}_2. &
\end{tikzcd}$$

This means that

$$\begin{aligned}
&\mathrm{ch}(H^*(\mathcal{Y}_{\mathbf{m}_0}(X))) = (1+q)\,\mathrm{ch}(H^*(\mathcal{Z}_0))\\
&\mathrm{ch}(H^*(\mathcal{Y}_{\mathbf{m}_1}(X))) = \mathrm{ch}(H^*(Bl_{\mathcal{Z}_0}\mathcal{Z}_2)) = \mathrm{ch}(H^*(\mathcal{Z}_2)) + q\,\mathrm{ch}(H^*(\mathcal{Z}_0))\\
&\mathrm{ch}(H^*(\mathcal{Y}_{\mathbf{m}_2}(X))) = (1+q)\,\mathrm{ch}(H^*(\mathcal{Z}_2))
\end{aligned}$$

and hence we get

$$(1+q)\,\mathrm{csf}_q(\mathbf{m}_1) = \mathrm{csf}_q(\mathbf{m}_2) + q\,\mathrm{csf}_q(\mathbf{m}_1).$$

We refer to [AN22, Example 1.24] for an example where ws is singular.

A direct consequence of Example 3.5 is that characters of Kazhdan–Lusztig elements of codominant permutations are omega-dual to chromatic quasisymmetric functions of indifference graphs, first proved in [CHSS16].

Corollary 3.6. *If* $\mathbf{m}\colon [n] \to [n]$ *is a Hessenberg function, then*

$$\mathrm{ch}(q^{\frac{\ell(w_{\mathbf{m}})}{2}} C'_{w_{\mathbf{m}}}) = \omega(\mathrm{csf}_q(G_{\mathbf{m}})).$$

Proof. If $\mathbf{m}_0$, $\mathbf{m}_1$, and $\mathbf{m}_2$ are Hessenberg functions as in Example 3.5, then applying Corollary 3.2 to $w_{\mathbf{m}_1}$, we see that $w_{\mathbf{m}_1}s = w_{\mathbf{m}_2}$ and $z = w_{\mathbf{m}_0}$. This means that the relation in item (1) is precisely the modular law (see [GP13] and [OS14]). By [AN21a, Theorem 1.1], the modular law is sufficient to characterize the values of $\mathrm{ch}(C'_w)$ for w codominant from the values $\mathrm{ch}(q^{\frac{\ell(w_\lambda)}{2}} C'_{w_\lambda})$. Since $\mathrm{ch}(q^{\frac{\ell(w_\lambda)}{2}} C'_{w_\lambda}) = \lambda!_q h_\lambda = \omega(G_{\mathbf{m}_\lambda})$, the result follows. □

Remark 3.7. We set H_n^{cod} to be the $\mathbb{C}(q^{\frac{1}{2}})$-linear subspace of H_n generated by C'_w, for w codominant. From [AN21a], the kernel of the linear map

$$\mathrm{ch}\colon H_n^{cod} \to \mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda$$

is generated by the relations in Corollary 3.2 item (1) for w codominant.

Question 3.8. Is the kernel of the linear map $\mathrm{ch}\colon H_n \to \mathbb{C}(q^{\frac{1}{2}}) \otimes \Lambda$ generated by the relations in Equation (3.1)?

3.1. The geometry of $\mathcal{Y}_w(X)$ when w is smooth

In the proof of Theorem 1.6 in Section 2, we saw that for each smooth permutation $w \in S_n$ there exists a codominant permutation w' such that the moment graphs of $\mathcal{Y}_w(X)$ and $\mathcal{Y}_{w'}(X)$ are the same and, in particular, they have isomorphic equivariant cohomology. We also saw that all the varieties $\mathcal{Y}_w(X)$ associated to Coxeter elements w are isomorphic. We make the following conjecture which is a strengthening of Theorem 1.6.

Conjecture 3.9. *Let $X \in SL_n(\mathbb{C})$ be regular semisimple and $w \in S_n$ smooth. Then there exists a codominant permutation w' such that $\mathcal{Y}_w(X)$ and $\mathcal{Y}_{w'}(X)$ are homeomorphic.*

We remark that the corresponding statement for Schubert varieties is false, for instance $\Omega_{3142,F_\bullet}$ is not homeomorphic to $\Omega_{2341,F_\bullet}$ (and this is the only Schubert variety associated with a codominant permutation with the same Poincaré polynomial as of $\Omega_{3142,F_\bullet}$). On the other hand, both 3142 and 2341 are Coxeter elements so that $\mathcal{Y}_{3142}(X)$ is isomorphic to $\mathcal{Y}_{2341}(X)$ if X is regular semisimple.

Competing interest. On behalf of all authors, the corresponding author states that there is no competing interest.

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