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Long-time dynamics and semi-wave of a delayed nonlocal epidemic model with free boundaries

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This paper is concerned with a nonlocal reaction–diffusion system with double free boundaries and two time delays. The free boundary problem describes the evolution of faecally–orally transmitted diseases. We first show the well-posedness of global solution, and then establish the monotonicity and asymptotic property of basic reproduction number for the epidemic model without delays, which is defined by spectral radius of the next infection operator. By introducing the generalized principal eigenvalue defined in general domain, we obtain an upper bound of the limit value of basic reproduction number. We discuss the spreading and vanishing phenomena in terms of the basic production number. By employing the perturbed approximation method and monotone iteration method, we establish the existence, uniqueness and monotonicity of solution to semi-wave problem. When spreading occurs, we determine the asymptotic spreading speeds of free boundaries by constructing suitable upper and lower solutions from the semi-wave solutions. Moreover, spreading speeds for partially degenerate diffusion case are provided in a similar way.

Keywords: Eepidemic model; free boundary; time delay; basic reproduction number; semi-wave

2020 *Mathematics Subject Classification:* 35K57; 35K61; 35R35; 34K30

1. Introduction

In this paper, to study the evolution of faecally–orally transmitted diseases, such as hand, foot and mouth diseases, cholera and so on, we consider the following nonlocal reaction–diffusion system with double free boundaries and two delays:

$$\begin{cases} u_t = d_1 u_{xx} - a_1 u + h\left(\displaystyle\int_{-\infty}^{+\infty} J_1(x-y)v(t-\tau_1, y)\mathrm{d}y\right), \\ \qquad t > 0, s_1(t) < x < s_2(t), \\ v_t = d_2 v_{xx} - a_2 v + g\left(\int_{-\infty}^{+\infty} J_2(x-y)u(t-\tau_2, y)\mathrm{d}y\right), \\ \qquad t > 0, s_1(t) < x < s_2(t), \\ u(t,x) = v(t,x) = 0, \quad t > 0,\ x \leqslant s_1(t) \text{ or } x \geqslant s_2(t), \\ s_1'(t) = -\mu[u_x(t, s_1(t)) + \rho v_x(t, s_1(t))], \quad t > 0, \\ s_2'(t) = -\mu[u_x(t, s_2(t)) + \rho v_x(t, s_2(t))], \quad t > 0, \\ s_1(0) = -s_0,\ s_2(0) = s_0, \\ u(\theta, x) = u_0(\theta, x), \quad -\tau_2 \leqslant \theta \leqslant 0,\ s_1(\theta) \leqslant x \leqslant s_2(\theta), \\ v(\theta, x) = v_0(\theta, x), \quad -\tau_1 \leqslant \theta \leqslant 0,\ s_1(\theta) \leqslant x \leqslant s_2(\theta), \end{cases} \tag{1.1}$$

where $u(t, x)$ and $v(t, x)$ represent the density of bacteria in the environment and infective human population, respectively; d_1 and d_2 are the diffusion coefficients; a_1 and a_2 are the natural death rate of the bacteria and the fatality rate of the infective human population, respectively; the nonlocal term $h(\int_{-\infty}^{+\infty} J_1(x-y)v(t-\tau_1, y)\mathrm{d}y)$ is the contribution of the infective human population in a neighbourhood of x to the density of bacteria, $g(\int_{-\infty}^{+\infty} J_2(x-y)u(t-\tau_2, y)\mathrm{d}y)$ gives the 'force of infection' on human due to the concentration of bacteria, J_1 and J_2 are transfer kernels, τ_1 and τ_2 describe the delays-in-time of positive feedback interaction between the bacteria and infective human; $(s_1(t), s_2(t))$ is the infected area at time t, and its boundary fronts $s_1(t)$ and $s_2(t)$, depending on time t, are called free boundaries. We assume that the expanding rate of the infected area is proportional to a linear combination of the spatial gradients of bacteria and infective human population at the fronts, i.e., $s_1(t)$ and $s_2(t)$ satisfy the Stefan conditions. All the parameters are positive constants. Since the infected area may vary over time during the evolution of faecally–orally transmitted diseases, the fixed boundary problem is not suitable to be applied to understand how the bacteria spread spatially to larger area from the initial infected area, which motivates us to consider the free boundary problem (1.1).

The epidemic model in problem (1.1)

$$\begin{cases} u_t = d_1 u_{xx} - a_1 u + h\left(\int_{-\infty}^{+\infty} J_1(x-y)v(t-\tau_1, y)\mathrm{d}y\right), \\ v_t = d_2 v_{xx} - a_2 v + g\left(\int_{-\infty}^{+\infty} J_2(x-y)u(t-\tau_2, y)\mathrm{d}y\right) \end{cases} \tag{1.2}$$

was studied in [38]. The authors investigated the global attractivity of the equilibria, the spreading speed of a general system without quasi-monotone conditions, and travelling wave solutions for (1.2) in whole space. System (1.2) is a generalization of the epidemic models proposed by Capasso-Maddalena [8, 9] (without delay), Thieme-Zhao [27] (with a time delay) and Wu-Hsu [37] (with two time delays). A basic feature in these models is the positive feedback interaction between the infective human and the bacteria in the environment.

Some simplified forms of (1.1) without time delays, including the partially degenerate diffusion case ($d_2 = 0$) [1, 19, 40] and non-degenerate diffusion case ($d_2 > 0$) [30], have been recently studied. Moreover, the corresponding nonlocal diffusion models were considered in [17, 31, 32, 41]. The authors established the spreading–vanishing dichotomy, discussed the influence of different parameters on the spreading and vanishing, and determined the asymptotic spreading speeds of the free boundaries. These results are extensions of the work of Du and Lin [15], in which they proposed a free boundary problem for homogeneous logistic equation to model the species invasion. Except the above-mentioned works, the results in [15] have also been extended to other population models and epidemic models, for example, time-periodic case [13, 28, 29], nonlocal case [6, 7, 14, 18] and general nonlinearities case [16].

We mention in particular that, based on [15], free boundary problems for time-delayed biological models have also been studied in very recent years, but still quite few. To model the biological invasion of an age-structured species, Sun and Fang [24] first derived a local free boundary problem for Fisher-KPP equation with time delay. Tang *et al.* [25] subsequently extended some results of [24] to a two-species weak competition model with time delays. By considering the diffusion rate of the immature population, Du *et al.* [12] further derived a nonlocal free boundary problem with time delay. For the epidemic model (1.2) with $J_1 = J_2 = \delta$ (Dirac delta function), the corresponding free boundary problems with a time delay ($\tau_1 = d_2 = 0, \tau_2 > 0$) and two time delays ($\tau_1, \tau_2, d_2 > 0$) were also considered in [10, 11], respectively.

The purpose of this paper is to establish the long-time dynamical behaviours of (1.1), and determine the asymptotic spreading speeds when spreading happens. Throughout this paper, we define

$$[a, b] \times [s_1, s_2] := \left\{ (t, x) :\ t \in [a, b],\ x \in [s_1(t), s_2(t)] \right\}.$$

The sets $(a, b] \times [s_1, s_2]$, $(a, b) \times (s_1, s_2)$, etc., are defined similarly. We always assume that the initial functions in (1.1) satisfy

$$\begin{cases} u_0(\theta, x) \in C^{1,2}([-\tau_2, 0] \times [s_1, s_2]),\ v_0(\theta, x) \in C^{1,2}([-\tau_1, 0] \times [s_1, s_2]), \\ u_0(\theta, x) \begin{cases} > 0 & \text{for } \theta \in [-\tau_2, 0],\ x \in (s_1(\theta), s_2(\theta)), \\ \equiv 0 & \text{for } \theta \in [-\tau_2, 0],\ x \notin (s_1(\theta), s_2(\theta)), \end{cases} \\ v_0(\theta, x) \begin{cases} > 0 & \text{for } \theta \in [-\tau_1, 0],\ x \in (s_1(\theta), s_2(\theta)), \\ \equiv 0 & \text{for } \theta \in [-\tau_1, 0],\ x \notin (s_1(\theta), s_2(\theta)), \end{cases} \end{cases} \tag{1.3}$$

as well as the compatible condition

$$[s_1(\theta), s_2(\theta)] \subset [-s_0, s_0] \quad \text{for } \theta \in [-\max\{\tau_1, \tau_2\}, 0]. \tag{1.4}$$

The kernel functions $J_i(\cdot)$ $(i = 1,\, 2)$ and nonlinearities $g(\cdot)$, $h(\cdot)$ satisfy the following assumptions:

(**J**): $J_i \in C(\mathbb{R}),\ J_i(0) > 0,\ J_i(-x) = J_i(x) \geqslant 0$ for $x \in \mathbb{R}$, $\int_{-\infty}^{+\infty} J_i(y)\mathrm{d}y = 1$, and $\int_{-\infty}^{+\infty} J_i(y)e^{-\lambda y}\mathrm{d}y < +\infty$ for any $\lambda > 0$;

(**H**): $h \in C^2([0,\, +\infty)),\ g \in (C^2 \cap L^\infty)([0,\, +\infty)),\ h(0) = g(0) = 0$, and $h'(z)$, $g'(z) > 0$ for any $z \in [0,\, +\infty)$, $h''(z) \leqslant 0$, $g''(z) < 0$ for all $z > 0$.

A typical example is $J_i(x) = \frac{1}{\sqrt{4\pi\varrho}}e^{-\frac{x^2}{4\varrho}}$, $h(x) = ax$ and $g(x) = \frac{px}{1+qx}$ with some $\varrho, a, p, q > 0$.

For (1.1), the nonlocal reaction terms and time delays cause several difficulties which require different treatment from earlier works. Firstly, for our nonlocal epidemic model without delays, the basic reproduction number has no explicit expression as in [**22**], and its monotonicity and asymptotic property with respect to the domain are not easy to obtain. We define the basic reproduction number by spectral radius of the next infection operator, and pay much effort to establish its monotonicity and asymptotic property, especially provide an upper bound of the limit value by introducing the generalized principal eigenvalue defined in general domain. Secondly, to overcome the effects of nonlocal terms on spreading and vanishing, we need to construct the upper and lower solutions from the principal eigenfunctions of perturbed nonlocal eigenvalue problems, instead of the unperturbed ones as in [**11**, **30**]. Thirdly, the delayed nonlocal semi-wave problem we considered is different from the previous works. It is difficult to get the critical value c_τ^* of speed c for semi-wave by discussing the distribution of real roots of transcendental equation as in [**10**, **11**, **24**]. Motivated by the works [**14**, **17**] for nonlocal diffusion models, we first study the corresponding perturbed semi-wave problem, and apply the iteration monotone method to cope with the existence and monotonicity of perturbed semi-wave solution. Then we build a dichotomy between monotone travelling wave and monotone semi-wave, which ensures that the critical values of their speeds are equal. Finally, we determine the spreading speeds for partially degenerate diffusion case without delays, which was considered in [**19**]. The upper bounds of spreading speeds were provided in [**19**], but their precise values are still unknown due to the effect of nonlocal term. We give a complete answer to the problem in this paper.

Let us now describe the results of this paper more precisely.

For the following epidemic model without delays

$$\begin{cases} \phi_t = d_1\phi_{xx} - a_1\phi + h'(0)\int_{-l}^{l} J_1(x-y)\varphi(t,y)\mathrm{d}y,\ t > 0,\ x \in (-l,l), \\ \varphi_t = d_2\varphi_{xx} - a_2\varphi + g'(0)\int_{-l}^{l} J_2(x-y)\phi(t,y)\mathrm{d}y,\ t > 0,\ x \in (-l,l), \\ (\phi(t,\pm l), \varphi(t,\pm l)) = (0,0),\ t > 0, \\ (\phi(0,x), \varphi(0,x)) = (\phi_0(x), \varphi_0(x)),\ x \in [-l,l], \end{cases} \tag{1.5}$$

we define the basic reproduction number $\mathcal{R}_0^l$ by spectral radius of the next infection operator.

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THEOREM 1.1 Basic reproduction number. *(i) $\mathcal{R}_0^l - 1$ has the same sign as λ_1, where λ_1 is the principle eigenvalue of the following eigenvalue problem*

$$\begin{cases} d_1\phi_{xx} - a_1\phi + h'(0)\int_{-l}^{l} J_1(x-y)\varphi(y)\mathrm{d}y = \lambda\phi, \ x \in (-l, l), \\ d_2\varphi_{xx} - a_2\varphi + g'(0)\int_{-l}^{l} J_2(x-y)\phi(y)\mathrm{d}y = \lambda\varphi, \ x \in (-l, l), \\ (\phi(\pm l), \varphi(\pm l)) = (0, 0). \end{cases} \tag{1.6}$$

(ii) $\mathcal{R}_0^l = \frac{1}{\mu_0^l}$, where μ_0^l is the unique positive principle eigenvalue of the following eigenvalue problem

$$\begin{cases} -d_1\phi_{xx} + a_1\phi = \mu h'(0)\int_{-l}^{l} J_1(x-y)\varphi(y)\mathrm{d}y, \ x \in (-l, l), \\ -d_2\varphi_{xx} + a_2\varphi = \mu g'(0)\int_{-l}^{l} J_2(x-y)\phi(y)\mathrm{d}y, \ x \in (-l, l), \\ (\phi(\pm l), \varphi(\pm l)) = (0, 0) \end{cases} \tag{1.7}$$

with a positive eigenfunction $\Phi_{\mu_0}^l(x) = (\phi_{\mu_0}^l(x), \varphi_{\mu_0}^l(x))$.

(iii) $\mathcal{R}_0^l$ is increasing in $l > 0$, and

$$\mathcal{R}_0^l \to \mathcal{R}^* \leqslant \mathcal{R}_0 := \sqrt{\frac{h'(0)g'(0)}{a_1 a_2}}$$

as $l \to +\infty$, where $\mathcal{R}_0$ is the basic reproduction number of the corresponding ordinary differential equations.

Denote $s_{i,\infty} = \lim_{t\to+\infty} s_i(t)$ for $i = 1, 2$. We call that the bacteria are *spreading* if $s_{2,\infty} - s_{1,\infty} = +\infty$ and $\limsup_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])}) > 0$, and the bacteria are *vanishing* if $s_{2,\infty} - s_{1,\infty} < +\infty$ and $\lim_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])}) = 0$. In terms of the basic reproduction number, we can discuss the spreading and vanishing phenomena.

THEOREM 1.2 Spreading and vanishing. *(i) If $0 < \mathcal{R}_0 \leqslant 1$, then the solution of (1.1) satisfies $\lim_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])}) = 0$.*

(ii) Assume that $\mathcal{R}^* > 1$. Then there exists $\mu^* \in [0, +\infty)$ such that $s_{2,\infty} - s_{1,\infty} = +\infty$ for $\mu > \mu^*$, and $s_{2,\infty} - s_{1,\infty} < +\infty$ for $0 < \mu \leqslant \mu^*$.

Assume that $\mathcal{R}^* > 1$, we consider the delayed nonlocal semi-wave problem

$$\begin{cases} c\phi'(\xi) = d_1\phi''(\xi) - a_1\phi(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\xi - y - c\tau_1)\mathrm{d}y\right), \ \xi > 0, \\ c\varphi'(\xi) = d_2\varphi''(\xi) - a_2\varphi(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(\xi - y - c\tau_2)\mathrm{d}y\right), \ \xi > 0, \\ (\phi(\xi), \varphi(\xi)) = (0, 0), \ \xi \leqslant 0, \\ (\phi(+\infty), \varphi(+\infty)) = (u^*, v^*), \end{cases} \tag{1.8}$$

where (u^*, v^*) is the unique positive equilibrium of the equations, which is guaranteed by the condition $\mathcal{R}_0 \geqslant \mathcal{R}^* > 1$. By employing the perturbed approximation

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method and monotone iteration method, we can establish the (non)existence of semi-wave solution to (1.8). The critical value of speed c for semi-wave is c_τ^*.

THEOREM 1.3 Semi-wave solution. *The semi-wave problem* (1.8) *admits an increasing solution for* $0 < c < c_\tau^*$, *but has no increasing solution for* $c \geqslant c_\tau^*$.

For any fixed μ, $\rho > 0$, it is shown that there exists a unique $c_\tau = c_\tau^{\mu,\rho} \in (0, c_\tau^*)$ such that $\mu[(\phi_\tau^{c_\tau})'_+(0) + \rho(\varphi_\tau^{c_\tau})'_+(0)] = c_\tau$, where $(\phi_\tau^{c_\tau}, \varphi_\tau^{c_\tau})$ is the semi-wave solution of (1.8) with $c = c_\tau$. By constructing suitable upper and lower solutions from the semi-wave, we can determine the asymptotic spreading speeds of free boundaries when spreading happens.

THEOREM 1.4 Spreading speed. *Assume that* $\mathcal{R}^* > 1$. *If spreading happens to* (1.1), *then* $-\lim_{t\to+\infty} \frac{s_1(t)}{t} = \lim_{t\to+\infty} \frac{s_2(t)}{t} = c_\tau$.

In a similar way, we can provide the spreading speeds for partially degenerate diffusion case in [**19**]. More details are provided in § 6.

The rest of this paper is organized as follows. In § 2, we first establish the well-posedness of the solutions to (1.1) and two comparison principles, and then give the proof of theorem 1.1 related to the basic reproduction number. In § 3, we discuss the spreading and vanishing. The existence and monotonicity of solutions to a delayed nonlocal semi-wave problem are investigated in § 4. The spreading speeds of free boundaries in (1.1) and partially degenerate diffusion case are determined in §5 and 6, respectively. The results of generalized principal eigenvalue are presented in Appendix.

2. Preliminaries

In this section, we first exhibit the well-posedness and comparison principles for the free boundary problem (1.1), and then discuss the basic reproduction number of (1.5).

2.1. Well-posedness

THEOREM 2.1. (*i*) *For any* $\gamma \in (0, 1)$, *there exists a* $T > 0$ *such that problem* (1.1) *with the initial date* $(u_0(\theta, x), v_0(\theta, x); s_1(\theta), s_2(\theta))$ *satisfying* (1.3) *and* (1.4), *admits a unique solution* $(u(t, x), v(t, x); s_1(t), s_2(t))$ *with* $u, v \in C^{\frac{1+\gamma}{2}, 1+\gamma}(D_T)$, $s_1, s_2 \in C^{1+\frac{\gamma}{2}}([0, T])$, *where* $D_T = [0, T] \times [s_1, s_2]$.

(*ii*) *For the local solution* $(u, v; s_1, s_2)$ *obtained in* (*i*), *there exist positive constants* M_1, M_2 *and* M_3 *independent of* T, *such that* $0 < u(t, x) \leqslant M_1$, $0 < v(t, x) \leqslant M_2$ *and* $0 < -s_1'(t), s_2'(t) \leqslant M_3$ *for any* $0 < t \leqslant T$ *and* $s_1(t) < x < s_2(t)$.

(*iii*) *The solution* $(u, v; s_1, s_2)$ *of* (1.1) *exists and is unique for all* $t \in (0, +\infty)$.

Proof. We only prove that $u(t, x) \leqslant M_1$ and $v(t, x) \leqslant M_2$ in (*ii*), since the remaining part can be obtained by similar arguments as in the proof of Theorems 2.4–2.5 in [**12**] and Theorem 2.1 in [**30**].

For any $z > 0$, by Taylor's formula and the concavity of h, we have $h(z) = h(z) - h(0) = h'(0)z + \frac{1}{2}h''(\xi)z^2 \leqslant h'(0)z$ for some $\xi \in (0, z)$, which implies $\frac{h(z)}{z} \leqslant h'(0)$.

Since g is bounded, we can choose M_i $(i = 1, 2)$ sufficiently large such that

$$M_2 \geqslant \frac{\|g\|_{L^\infty}}{a_2}, \quad M_1 \geqslant h'(0)\frac{M_2}{a_1}.$$

It follows that $\frac{g(M_1)}{M_2} \leqslant \frac{\|g\|_{L^\infty}}{M_2} \leqslant a_2$, $\frac{h(M_2)}{M_1} = \frac{h(M_2)}{M_2} \cdot \frac{M_2}{M_1} \leqslant h'(0)\frac{M_2}{M_1} \leqslant a_1$. We may assume that

$$\begin{aligned} u_0(\theta, x) \leqslant M_1 \quad &\text{for } (\theta, x) \in [-\tau_2, 0] \times [-s_0, s_0], \\ v_0(\theta, x) \leqslant M_2 \quad &\text{for } (\theta, x) \in [-\tau_1, 0] \times [-s_0, s_0]. \end{aligned}$$

Define $(U(t, x), V(t, x)) := e^{-kt}(M_1 - u(t, x), M_2 - v(t, x))$, where $k > 0$ is a constant to be determined. Then, for $t > 0$ and $s_1(t) < x < s_2(t)$, (U, V) satisfies

$$\begin{aligned} U_t &= d_1 U_{xx} - (a_1 + k)U + e^{-kt}\left[a_1 M_1 - h\left(\int_{-\infty}^{+\infty} J_1(x - y)v(t - \tau_1, y)\mathrm{d}y\right)\right] \\ &= d_1 U_{xx} - (a_1 + k)U + e^{-kt}(a_1 M_1 - h(M_2)) \\ &\quad + e^{-k\tau_1} h'(\xi) \int_{-\infty}^{+\infty} J_1(x - y)V(t - \tau_1, y)\mathrm{d}y \\ &\geqslant d_1 U_{xx} - (a_1 + k)U + e^{-k\tau_1} h'(\xi) \int_{-\infty}^{+\infty} J_1(x - y)V(t - \tau_1, y)\mathrm{d}y, \\ V_t &= d_2 V_{xx} - (a_2 + k)V + e^{-kt}\left[a_2 M_2 - g\left(\int_{-\infty}^{+\infty} J_2(x - y)u(t - \tau_2, y)\mathrm{d}y\right)\right] \\ &= d_2 V_{xx} - (a_2 + k)V + e^{-kt}(a_2 M_2 - g(M_1)) \\ &\quad + e^{-k\tau_2} g'(\eta) \int_{-\infty}^{+\infty} J_2(x - y)U(t - \tau_2, y)\mathrm{d}y \\ &\geqslant d_2 V_{xx} - (a_2 + k)V + e^{-k\tau_2} g'(\eta) \int_{-\infty}^{+\infty} J_2(x - y)U(t - \tau_2, y)\mathrm{d}y, \end{aligned} \tag{2.1}$$

where ξ lies between M_2 and $\int_{-\infty}^{+\infty} J_1(x - y)v(t - \tau_1, y)\mathrm{d}y$, η lies between M_1 and $\int_{-\infty}^{+\infty} J_2(x - y)u(t - \tau_2, y)\mathrm{d}y$.

We claim that $U(t, x), V(t, x) \geqslant 0$ in $(0, +\infty) \times (s_1, s_2)$. Assume by contraction that there exist some $T_0 > 0$ and $(t_0, x_0) \in (0, T_0] \times (s_1, s_2)$ such that

$$\min\{U(t_0, x_0), V(t_0, x_0)\} = \min_{(t,x)\in[0,T_0]\times[s_1,s_2]} \min\{U(t, x), V(t, x)\} < 0.$$

If $U(t_0, x_0) = \min\{U(t_0, x_0), V(t_0, x_0)\} < 0$, then $U_t(t_0, x_0) \leqslant 0$ and $U_{xx}(t_0, x_0) \geqslant 0$. On the other hand, if $t_0 \leqslant \tau_1$, then $V(t_0 - \tau_1, x) \geqslant 0 > U(t_0, x_0)$ for any $x \in \mathbb{R}$; if $t_0 > \tau_1$, then $V(t_0 - \tau_1, x) \geqslant U(t_0, x_0)$ for $x \in [s_1(t_0 - \tau_1), s_2(t_0 - \tau_1)]$ and $V(t_0 - \tau_1, x) \geqslant 0 > U(t_0, x_0)$ for $x \in \mathbb{R} \setminus [s_1(t_0 - \tau_1), s_2(t_0 - \tau_1)]$. Thus, $V(t_0 - \tau_1, x) \geqslant$

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$U(t_0, x_0)$ holds for all $x \in \mathbb{R}$. It follows that

$$
\begin{aligned}
& -(a_1+k)U(t_0,x_0) + e^{-k\tau_1}h'(\xi)\int_{-\infty}^{+\infty} J_1(x_0-y)V(t_0-\tau_1,y)\mathrm{d}y \\
& \quad \geqslant -(a_1+k)U(t_0,x_0) + e^{-k\tau_1}h'(\xi)\int_{-\infty}^{+\infty} J_1(x_0-y)U(t_0,x_0)\mathrm{d}y \\
& \quad = (-a_1-k+e^{-k\tau_1}h'(\xi))U(t_0,x_0) \geqslant (-a_1-k+h'(\xi))U(t_0,x_0).
\end{aligned}
$$

Choose

$$
k = \max\left\{ \|h'\|_{L^\infty([0,\max\{M_2,K_2\}])}, \|g'\|_{L^\infty([0,\max\{M_1,K_1\}])} \right\}
$$

with $K_1 = \|u\|_{L^\infty([-\tau_2,\, T_0-\tau_2]\times[s_1,\, s_2])}$ and $K_2 = \|v\|_{L^\infty([-\tau_1,\, T_0-\tau_1]\times[s_1,\, s_2])}$. Thus,

$$
-(a_1+k)U(t_0,x_0) + e^{-k\tau_1}h'(\xi)\int_{-\infty}^{+\infty} J_1(x_0-y)V(t_0-\tau_1,y)\mathrm{d}y > 0,
$$

which contradicts with the first equation in (2.1). If $V(t_0, x_0) = \min\{U(t_0, x_0), V(t_0, x_0)\} < 0$, we can prove the claim in a similar way. This completes the proof. □

We introduce two comparison principles for the free boundary problem (1.1), which can be proved similarly as the proof of Lemma 2.5 in [**1**].

Lemma 2.2. *Let* $T \in (0, +\infty)$*,* $\bar{s}_1, \bar{s}_2 \in C([-\max\{\tau_1, \tau_2\}, T]) \cap C^1((0, T])$*,* $\bar{u}_0(\theta, x) \in C^{1,2}([-\tau_2, 0] \times [\bar{s}_1, \bar{s}_2])$*,* $\bar{v}_0(\theta, x) \in C^{1,2}([-\tau_1, 0] \times [\bar{s}_1, \bar{s}_2])$*,* $\bar{u}(t, x), \bar{v}(t, x) \in C^{1,2}((0, T] \times [\bar{s}_1, \bar{s}_2])$*, and*

$$
\begin{cases}
\bar{u}_t \geqslant d_1\bar{u}_{xx} - a_1\bar{u} + h\left(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1,y)\mathrm{d}y \right), \\
\qquad 0 < t \leqslant T, \bar{s}_1(t) < x < \bar{s}_2(t), \\
\bar{v}_t \geqslant d_2\bar{v}_{xx} - a_2\bar{v} + g\left(\int_{-\infty}^{+\infty} J_2(x-y)\bar{u}(t-\tau_2,y)\mathrm{d}y \right), \\
\qquad 0 < t \leqslant T, \bar{s}_1(t) < x < \bar{s}_2(t), \\
\bar{u}(t,x) = \bar{v}(t,x) = 0,\ 0 < t \leqslant T,\ x \leqslant \bar{s}_1(t) \text{ or } x \geqslant \bar{s}_2(t), \\
\bar{s}_1'(t) \leqslant -\mu[\bar{u}_x(t,\bar{s}_1(t)) + \rho\bar{v}_x(t,\bar{s}_1(t))],\ 0 < t \leqslant T, \\
\bar{s}_2'(t) \geqslant -\mu[\bar{u}_x(t,\bar{s}_2(t)) + \rho\bar{v}_x(t,\bar{s}_2(t))],\ 0 < t \leqslant T, \\
\bar{u}(\theta,x) = \bar{u}_0(\theta,x),\ \ -\tau_2 \leqslant \theta \leqslant 0,\ \bar{s}_1(\theta) \leqslant x \leqslant \bar{s}_2(\theta), \\
\bar{v}(\theta,x) = \bar{v}_0(\theta,x),\ \ -\tau_1 \leqslant \theta \leqslant 0,\ \bar{s}_1(\theta) \leqslant x \leqslant \bar{s}_2(\theta).
\end{cases}
$$

If $(u, v; s_1, s_2)$ *is a solution of* (1.1) *with* $[s_1(\theta), s_2(\theta)] \subset [\bar{s}_1(\theta), \bar{s}_2(\theta)]$ *for* $\theta \in [-\max\{\tau_1, \tau_2\}, 0]$*,* $u_0(\theta, x) \leqslant \bar{u}_0(\theta, x)$ *for* $(\theta, x) \in [-\tau_2, 0] \times [s_1, s_2]$ *and* $v_0(\theta, x) \leqslant \bar{v}_0(\theta, x)$ *for* $(\theta, x) \in [-\tau_1, 0] \times [s_1, s_2]$*, then* $[s_1(t), s_2(t)] \subset [\bar{s}_1(t), \bar{s}_2(t)]$ *and* $(u(t, x), v(t, x)) \leqslant (\bar{u}(t, x), \bar{v}(t, x))$ *for* $t \in (0, T]$*,* $x \in (s_1(t), s_2(t))$*.*

Lemma 2.3. *Let* $T \in (0, +\infty)$*,* $\bar{s}_2 \in C([-\max\{\tau_1, \tau_2\}, T]) \cap C^1((0, T])$*,* $\bar{u}_0(\theta, x) \in C([-\tau_2, 0] \times (-\infty, \bar{s}_2]) \cap C^{1,2}([-\tau_2, 0]\times(0, \bar{s}_2])$*,* $\bar{v}_0(\theta, x) \in C([-\tau_1, 0]\times(-\infty, \bar{s}_2])$

$\cap C^{1,2}([-\tau_1, 0] \times (0, \bar{s}_2])$, $\bar{u}(t, x)$, $\bar{v}(t, x) \in C([0, T] \times (-\infty, \bar{s}_2]) \cap C^{1,2}((0, T] \times (0, \bar{s}_2])$, *and*

$$\begin{cases} \bar{u}_t \geqslant d_1\bar{u}_{xx} - a_1\bar{u} + h\left(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\right), \ 0 < t \leqslant T, \ 0 < x < \bar{s}_2(t), \\ \bar{v}_t \geqslant d_2\bar{v}_{xx} - a_2\bar{v} + g\left(\int_{-\infty}^{+\infty} J_2(x-y)\bar{u}(t-\tau_2, y)\mathrm{d}y\right), \ 0 < t \leqslant T, \ 0 < x < \bar{s}_2(t), \\ \bar{u}(t,x) = \bar{v}(t,x) = 0, \ 0 < t \leqslant T, \ x \geqslant \bar{s}_2(t), \\ \bar{u}(t,x) \geqslant u(t,x), \ \bar{v}(t,x) \geqslant v(t,x), \ 0 < t \leqslant T, \ x \leqslant 0, \\ \bar{s}_2'(t) \geqslant -\mu[\bar{u}_x(t, \bar{s}_2(t)) + \rho\bar{v}_x(t, \bar{s}_2(t))], \ 0 < t \leqslant T, \\ \bar{u}(\theta, x) = \bar{u}_0(\theta, x), \ -\tau_2 \leqslant \theta \leqslant 0, \ -\infty < x \leqslant \bar{s}_2(\theta), \\ \bar{v}(\theta, x) = \bar{v}_0(\theta, x), \ -\tau_1 \leqslant \theta \leqslant 0, \ -\infty < x \leqslant \bar{s}_2(\theta). \end{cases}$$

If the solution $(u, v; s_1, s_2)$ *of* (1.1) *satisfies that* $s_2(\theta) \leqslant \bar{s}_2(\theta)$ *for* $\theta \in [-\max\{\tau_1, \tau_2\}, 0]$, $u_0(\theta, x) \leqslant \bar{u}_0(\theta, x)$ *for* $(\theta, x) \in [-\tau_2, 0] \times (-\infty, s_2]$ *and* $v_0(\theta, x) \leqslant \bar{v}_0(\theta, x)$ *for* $(\theta, x) \in [-\tau_1, 0] \times (-\infty, s_2]$, *then* $s_2(t) \leqslant \bar{s}_2(t)$ *and* $(u(t, x), v(t, x)) \leqslant (\bar{u}(t, x), \bar{v}(t, x))$ *for* $t \in (0, T]$, $x \in (0, s_2(t))$.

REMARK 2.4. $(\bar{u}, \bar{v}; \bar{s}_1, \bar{s}_2)$ in lemma 2.2 is called an upper solution of (1.1), and $(\bar{u}, \bar{v}; \bar{s}_2)$ in lemma 2.3 is that of one-side case. Lower solutions of (1.1) can be defined analogously by reversing all the inequalities.

2.2. Basic reproduction number

In epidemiology, the basic reproduction number is an important index of measuring the transmission potential of a disease. In order to discuss the spreading and vanishing phenomenon for the free boundary problem (1.1), we need to study the basic reproduction number of the epidemic model without delays (1.5). In theorem 1.1, we establish the relationship between the basic reproduction number and principle eigenvalues, and provide the monotonicity and asymptotic properties of the basic reproduction number with respect to the domain.

In this section, we first introduce the definition of the basic reproduction number of (1.5), and then give the proof of theorem 1.1.

For any $l > 0$, let $T_i(t)$ be the solution semigroup on $C_0([-l, l], \mathbb{R})$ associated with the following linear parabolic equation

$$\begin{cases} w_t = d_i w_{xx} - a_i w, \quad t > 0, -l < x < l, \\ w(t, \pm l) = 0, \quad t > 0. \end{cases}$$

Let

$$T(t)\Phi := \left(T_1(t)\phi, T_2(t)\varphi\right), \quad \forall \Phi = (\phi, \varphi) \in X := C_0([-l, l], \mathbb{R}^2), \ t \geqslant 0.$$

It is clear that $T(t)$ is a positive C_0-semigroup on X. We further define a positive linear operator $\mathcal{G}$ from X to $Y := C([-l, l], \mathbb{R}^2)$ by

$$\mathcal{G}(\Phi)(x) := (\mathcal{G}_1(\Phi)(x), \mathcal{G}_2(\Phi)(x)), \quad \forall \Phi = (\phi, \varphi) \in X,$$

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where

$$\mathcal{G}_1(\Phi)(x) = h'(0)\int_{-l}^{l} J_1(x-y)\varphi(y)\mathrm{d}y,$$
$$\mathcal{G}_2(\Phi)(x) = g'(0)\int_{-l}^{l} J_2(x-y)\phi(y)\mathrm{d}y.$$

Then the distribution of total new infection of human is

$$\int_0^{+\infty} h'(0)\int_{-l}^{l} J_1(x-y)(T_2(t)\varphi)(y)\mathrm{d}y\mathrm{d}t = \int_0^{+\infty} \mathcal{G}_1(T_1(t)\phi, T_2(t)\varphi)(x)\mathrm{d}t,$$

and the distribution of total new infection of bacteria is

$$\int_0^{+\infty} g'(0)\int_{-l}^{l} J_2(x-y)(T_1(t)\phi)(y)\mathrm{d}y\mathrm{d}t = \int_0^{+\infty} \mathcal{G}_2(T_1(t)\phi, T_2(t)\varphi)(x)\mathrm{d}t.$$

It follows that

$$\mathcal{L}(\Phi) := \int_0^{+\infty} \mathcal{G}(T(t)\Phi)\mathrm{d}t = \mathcal{G}\int_0^{+\infty} (T(t)\Phi)\mathrm{d}t$$

is the next infection operator, which maps the initial distribution Φ of infectious bacteria and humans to the distribution of the total infective bacteria and humans produced during the infection period.

We define the *basic reproduction number* of the epidemic model (1.5)

$$\mathcal{R}_0^l := r(\mathcal{L}),$$

where $r(\mathcal{L})$ is the spectral radius of $\mathcal{L}$. Here, we use the notation $\mathcal{R}_0^l$ to emphasis the dependence of the basic reproduction number on the domain $(-l,\, l)$.

Proof of theorem 1.1. (i) The corresponding linear evolution system (1.5) generates a compact, strongly positive semigroup on $Z_+ := Z \cap X_+$, where $Z := C_0^1([-l,\, l], \mathbb{R}^2)$, and $X_+ := C_0([-l,\, l], \mathbb{R}_+^2)$ is the cone of nonnegative functions in X. Therefore, by standard arguments in Theorem 6.1 of [**23**], we deduce that the elliptic problem (1.6) has a principal eigenvalue λ_1 with a strictly positive eigenvector.

Similar as the proof of Lemma 2.2 in [**33**], we can prove that $\mathcal{R}_0^l - 1$ has the same sign as λ_1. We also refer the readers to [**20**] (Theorem 3.7).

(ii) We first prove that (1.7) admits a unique positive principle eigenvalue with a positive eigenvector.

Define $L_i = -d_i\partial_{xx} + a_i$, $i = 1,\, 2$, and let L_i also denote the realization of L_i in $C([-l,\, l], \mathbb{R})$ subject to Dirichlet boundary condition. Let

$$L\Phi = (L_1\phi, L_2\varphi) = (-d_1\partial_{xx}\phi + a_1\phi, -d_2\partial_{xx}\varphi + a_2\varphi),\ \forall\Phi = (\phi,\varphi) \in \mathrm{dom}(L) \subset Z.$$

The operator $L : Z \supset \mathrm{dom}(L) \to Y := C([-l,\, l], \mathbb{R}^2)$ is invertible, with compact inverse. It follows that the problem (1.7) is equivalent to the equation

$$\Phi = \mu L^{-1}\mathcal{G}(\Phi).$$

Define $\mathcal{A}_\mu := \mu L^{-1}\mathcal{G}$, and let $r(\mathcal{A}_\mu)$ be its spectral radius. Note that $\mathcal{G}$ is a bounded linear operator from Z to Y, and L^{-1} is a compact linear operator from

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Y to Z. Then for any fixed $\mu > 0$, $\mathcal{A}_\mu : Z \to Z$ is a compact linear operator, and strongly positive with respect to the solid cone Z_+, i.e., $\mathcal{A}_\mu(Z_+ \setminus \{\mathbf{0}\}) \subset \text{Int } Z_+ \neq \emptyset$. By the Krein–Rutman theorem (strong form), $r(\mathcal{A}_\mu) > 0$ and there exists $\Phi_\mu \in \text{Int } Z_+$ such that $\mathcal{A}_\mu \Phi_\mu = r(\mathcal{A}_\mu)\Phi_\mu$. Moreover, $r(\mathcal{A}_\mu)$ is a geometrically simple eigenvalue.

By the Gelfand's formula, $r(\mathcal{A}_\mu) = \lim_{n\to\infty} \|\mathcal{A}_\mu^n\|_Z^{\frac{1}{n}} = \mu \lim_{n\to\infty} \|(L^{-1}\mathcal{G})^n\|_Z^{\frac{1}{n}} = \mu r(\mathcal{A}_1)$, $\forall \mu > 0$. Since $r(\mathcal{A}_1) > 0$, there exists a unique $\mu_0^l > 0$ such that $r(\mathcal{A}_{\mu_0^l}) = 1$. In fact, $\mu_0^l = \frac{1}{r(\mathcal{A}_1)}$. Then, we have $\Phi_{\mu_0^l} = \mu_0^l L^{-1}\mathcal{G}(\Phi_{\mu_0^l})$, which implies that (1.7) admits a unique positive principle eigenvalue μ_0^l with a positive eigenvector $\Phi_{\mu_0^l}$.

The equality $\mathcal{R}_0^l = \frac{1}{\mu_0^l}$ can be proved by similar arguments as in the proof of Theorem 3.2 in [**34**]. We also refer the readers to [**20**] (Theorem 3.8).

(iii) To stress the dependence of $\mathcal{A}_1$, L^{-1} and $\mathcal{G}$ on l, here we use the notations $\mathcal{A}_1^l$, $(L^l)^{-1}$ and $\mathcal{G}^l$. Obviously, for any $l_2 > l_1 > 0$ and $\Phi = (\phi, \varphi) \in Z_+$,

$$\begin{aligned}
\mathcal{G}^{l_2}(\Phi)(x) &= (\mathcal{G}_1^{l_2}(\Phi)(x), \mathcal{G}_2^{l_2}(\Phi)(x)) \\
&= \left(h'(0)\int_{-l_2}^{l_2} J_1(x-y)\varphi(y)\mathrm{d}y, g'(0)\int_{-l_2}^{l_2} J_2(x-y)\phi(y)\mathrm{d}y\right) \\
&\geqslant \left(h'(0)\int_{-l_1}^{l_1} J_1(x-y)\varphi(y)\mathrm{d}y, g'(0)\int_{-l_1}^{l_1} J_2(x-y)\phi(y)\mathrm{d}y\right) \\
&= (\mathcal{G}_1^{l_1}(\Phi)(x), \mathcal{G}_2^{l_1}(\Phi)(x)) = \mathcal{G}^{l_1}(\Phi)(x).
\end{aligned}$$

Moreover, by the maximum principle for elliptic equations, we know that

$$(L^l)^{-1}(\Phi_2) \geqslant (L^l)^{-1}(\Phi_1) \quad \text{for any } l > 0 \text{ and } \Phi_1, \Phi_2 \in Y_+ \text{ with } \Phi_2 \geqslant \Phi_1,$$

and

$$(L^{l_2})^{-1}(\Phi) \geqslant (L^{l_1})^{-1}(\Phi) \quad \text{for any } l_2 > l_1 > 0 \ \text{ and } \ \Phi \in \tilde{Y}_+,$$

where $Y_+ := C([-l, l], \mathbb{R}_+^2)$ and $\tilde{Y}_+ := C([-l_2, l_2], \mathbb{R}_+^2)$. Thus, for any $l_2 > l_1$ and $\Phi \in Z_+$,

$$(L^{l_2})^{-1}\mathcal{G}^{l_2}(\Phi) \geqslant (L^{l_2})^{-1}\mathcal{G}^{l_1}(\Phi) \geqslant (L^{l_1})^{-1}\mathcal{G}^{l_1}(\Phi).$$

Since each $(L^l)^{-1}\mathcal{G}^l$ is a positive and bounded linear operator on Z, by Theorem 1.1 in [**5**] we know that $r(\mathcal{A}_1^l) = r((L^l)^{-1}\mathcal{G}^l)$ is an increasing function of l. It follows that $\mu_0^l = \frac{1}{r(\mathcal{A}_1^l)}$ is decreasing in l, and $\mathcal{R}_0^l = \frac{1}{\mu_0^l}$ (by (ii)) is increasing in l.

Note that $(L_1^l)^{-1}(f)(x) = \int_{-l}^{l} G(x, \xi)f(\xi)\mathrm{d}\xi$, where $G(x, \xi)$ is the Green's function defined as

$$G(x,\xi) = \begin{cases} \dfrac{(e^{\lambda(l-x)} - e^{-\lambda(l-x)})(e^{\lambda(l+\xi)} - e^{-\lambda(l+\xi)})}{2d_1\lambda(e^{2\lambda l} - e^{-2\lambda l})}, & -l \leqslant \xi \leqslant x, \\ \dfrac{(e^{\lambda(l+x)} - e^{-\lambda(l+x)})(e^{\lambda(l-\xi)} - e^{-\lambda(l-\xi)})}{2d_1\lambda(e^{2\lambda l} - e^{-2\lambda l})}, & x \leqslant \xi \leqslant l \end{cases}$$

with $\lambda=\sqrt{\frac{a_1}{d_1}}$. It is easy to check that $G(x,\xi)>0$ and

$$\begin{aligned}0<\int_{-l}^{l}G(x,\xi)\mathrm{d}\xi&=\frac{1}{d_1\lambda^2}-\frac{(e^{\lambda(l-x)}-e^{-\lambda(l-x)})+(e^{\lambda(l+x)}-e^{-\lambda(l+x)})}{d_1\lambda^2(e^{2\lambda l}-e^{-2\lambda l})}\\&\leqslant\frac{1}{d_1\lambda^2}=\frac{1}{a_1}.\end{aligned}$$

Then, we have

$$\|(L_1^l)^{-1}(f)\|_\infty=\left\|\int_{-l}^{l}G(\cdot,\xi)f(\xi)\mathrm{d}\xi\right\|_\infty\leqslant\left\|\int_{-l}^{l}G(\cdot,\xi)\mathrm{d}\xi\right\|_\infty\|f\|_\infty\leqslant\frac{1}{a_1}\|f\|_\infty. \tag{2.2}$$

Moreover,

$$\partial_xG(x,\xi)=\begin{cases}\dfrac{-\lambda(e^{\lambda(l-x)}+e^{-\lambda(l-x)})(e^{\lambda(l+\xi)}-e^{-\lambda(l+\xi)})}{2d_1\lambda(e^{2\lambda l}-e^{-2\lambda l})}, & -l\leqslant\xi<x,\\ \dfrac{\lambda(e^{\lambda(l+x)}+e^{-\lambda(l+x)})(e^{\lambda(l-\xi)}-e^{-\lambda(l-\xi)})}{2d_1\lambda(e^{2\lambda l}-e^{-2\lambda l})}, & x<\xi\leqslant l.\end{cases}$$

Direct calculations yield

$$\begin{aligned}0<\int_{-l}^{l}|\partial_xG(x,\xi)|\mathrm{d}\xi&=\frac{1}{d_1\lambda}-\frac{e^{-\lambda x}(e^{\lambda l}-e^{-\lambda x})+e^{\lambda x}(e^{\lambda l}-e^{\lambda x})}{d_1\lambda(e^{2\lambda l}-e^{-2\lambda l})}\\&\quad-\frac{e^{-\lambda l}(e^{\lambda x}-e^{-\lambda l})+e^{-\lambda l}(e^{-\lambda x}-e^{-\lambda l})}{d_1\lambda(e^{2\lambda l}-e^{-2\lambda l})}\leqslant\frac{1}{d_1\lambda}=\frac{1}{\sqrt{a_1d_1}}.\end{aligned}$$

Then, we have

$$\begin{aligned}\|\nabla(L_1^l)^{-1}(f)\|_\infty&=\left\|\int_{-l}^{l}\partial_xG(\cdot,\xi)f(\xi)\mathrm{d}\xi\right\|_\infty\\&\leqslant\left\|\int_{-l}^{l}|\partial_xG(\cdot,\xi)|\mathrm{d}\xi\right\|_\infty\|f\|_\infty\leqslant\frac{1}{\sqrt{a_1d_1}}\|f\|_\infty,\end{aligned}$$

which together with (2.2) imply

$$\|(L_1^l)^{-1}(f)\|_\infty+\|\nabla(L_1^l)^{-1}(f)\|_\infty\leqslant\left(\frac{1}{a_1}+\frac{1}{\sqrt{a_1d_1}}\right)\|f\|_\infty.$$

In a similar way, we can prove $\|(L_2^l)^{-1}(f)\|_\infty+\|\nabla(L_2^l)^{-1}(f)\|_\infty\leqslant(\frac{1}{a_2}+\frac{1}{\sqrt{a_2d_2}})\|f\|_\infty$. Thus,

$$\|(L^l)^{-1}\|_{Y\to Z}\leqslant\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{\sqrt{a_1d_1}}+\frac{1}{\sqrt{a_2d_2}}. \tag{2.3}$$

On the other hand,

$$\|\mathcal{G}_1^l(\Phi)\|_\infty = \left\| h'(0)\int_{-l}^{l} J_1(\cdot - y)\varphi(y)\mathrm{d}y \right\|_\infty \leqslant h'(0)\left\| \int_{-l}^{l} J_1(\cdot - y)\mathrm{d}y \right\|_\infty \|\varphi\|_\infty \leqslant h'(0)\|\Phi\|_\infty.$$

Similarly, $\|\mathcal{G}_2^l(\Phi)\|_\infty \leqslant g'(0)\|\Phi\|_\infty$. Thus,

$$\|\mathcal{G}^l\|_{Z\to Y} \leqslant h'(0) + g'(0). \tag{2.4}$$

By (2.3) and (2.4), we have

$$r(\mathcal{A}_1^l) \leqslant \|\mathcal{A}_1^l\|_Z = \|(L^l)^{-1}\mathcal{G}^l\|_Z \leqslant \|(L^l)^{-1}\|_{Y\to Z}\|\mathcal{G}^l\|_{Z\to Y} \leqslant \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{\sqrt{a_1 d_1}} + \frac{1}{\sqrt{a_2 d_2}}\right)(h'(0) + g'(0)) =: M.$$

It follows that μ_0^l has a positive lower bound independent of l, i.e.,

$$\mu_0^l = \frac{1}{r(\mathcal{A}_1^l)} \geqslant \frac{1}{M}, \quad \forall l > 0,$$

which together with the fact that $\mu_0^l = \frac{1}{r(\mathcal{A}_1^l)}$ is decreasing in l imply that $\mu^* := \lim_{l\to+\infty} \mu_0^l$ exists and satisfies $\mu^* \geqslant \frac{1}{M} > 0$. Then $\mathcal{R}^* := \lim_{l\to+\infty} \frac{1}{\mu_0^l} = \frac{1}{\mu^*}$ is well-defined and satisfies $0 < \mathcal{R}^* \leqslant M$.

Now, we provide a more accurate upper bound of $\mathcal{R}^*$, i.e., $\mathcal{R}^* \leqslant \mathcal{R}_0 := \sqrt{\frac{h'(0)g'(0)}{a_1 a_2}}$. It is sufficient to show that $\mu^* \geqslant \sqrt{\frac{a_1 a_2}{h'(0)g'(0)}}$.

Recall that $r(\mathcal{A}_\mu^l)$ is a geometrically simple eigenvalue of $\mathcal{A}_\mu^l$ by the Krein–Rutman theorem. We may assume that the corresponding positive eigenvector $\Phi_{\mu_0^l}^l = (\phi_{\mu_0^l}^l, \varphi_{\mu_0^l}^l)$ of (1.7) satisfies $\|\Phi_{\mu_0^l}^l\|_\infty = 1$. Thus, there exist a sequence $\{l_n\}$ and positive function Φ^* satisfying $\|\Phi^*\|_\infty = 1$, such that $\Phi_{\mu_0^{l_n}}^{l_n} \to \Phi^*$ in $C_{loc}^2(\mathbb{R})$ as $n \to \infty$. Then, (μ^*, Φ^*) solves

$$\begin{cases} -d_1\phi_{xx}^* + a_1\phi^* = \mu^* h'(0)\int_{-\infty}^{+\infty} J_1(x-y)\varphi^*(y)\mathrm{d}y, & x \in (-\infty, +\infty), \\ -d_2\varphi_{xx}^* + a_2\varphi^* = \mu^* g'(0)\int_{-\infty}^{+\infty} J_2(x-y)\phi^*(y)\mathrm{d}y, & x \in (-\infty, +\infty). \end{cases} \tag{2.5}$$

As in [3, 4], we define the generalized principal eigenvalue in a (possibly unbounded) domain $\Omega \subset \mathbb{R}$ as follows

$$\begin{aligned} \mu_1(\Omega) &:= \sup E^\Omega \\ &:= \sup\Big\{\mu \in \mathbb{R}:\ \exists(\phi,\varphi) \in C^2(\Omega,\mathbb{R}^2) \cap C_{loc}^1(\overline{\Omega},\mathbb{R}^2),\ (\phi,\varphi) > \mathbf{0} \text{ in } \Omega, \\ &\qquad \text{and } -d_1\phi_{xx} + a_1\phi \geqslant \mu h'(0)\int_\Omega J_1(x-y)\varphi(y)\mathrm{d}y, \\ &\qquad -d_2\varphi_{xx} + a_2\varphi \geqslant \mu g'(0)\int_\Omega J_2(x-y)\phi(y)\mathrm{d}y \quad \text{for } x \in \Omega\Big\}. \end{aligned} \tag{2.6}$$

Here, $C^1_{loc}(\overline{\Omega}, \mathbb{R}^2)$ denotes the set of functions $(\phi, \varphi) \in C^1(\Omega, \mathbb{R}^2)$ for which (ϕ, φ) and (ϕ_x, φ_x) can be extended by continuity on $\partial\Omega$, but which are not necessarily bounded. From (ii) and (2.5), we know $E^{(-l,l)}, E^{\mathbb{R}} \neq \emptyset$ for any $l > 0$. We claim that (i) $\mu_1((-l, l)) = \mu_0^l$ for any $l > 0$, where μ_0^l is the principal eigenvalue of (1.7); (ii) $\mu_1((-l, l)) \to \mu_1(\mathbb{R})$ as $l \to +\infty$, and then $\mu^* = \mu_1(\mathbb{R})$. The claim can be proved by similar arguments as in the proofs of Proposition 4.4 and Theorem 2.2 in [**2**]. For the convenience of the reader, we provide the details in proposition A of Appendix.

Assume that $(\tilde{\mu}, \tilde{\Phi}(x)) = (\tilde{\mu}, c_1, c_2)$ is a solution of (2.5) with $\|\tilde{\Phi}\|_\infty = 1$, where c_1, c_2 are positive constants. Due to $\int_{-\infty}^{+\infty} J_i(x)\mathrm{d}x = 1$, $i = 1, 2$, we have

$$\begin{cases} a_1c_1 = \tilde{\mu}h'(0)c_2, \\ a_2c_2 = \tilde{\mu}g'(0)c_1, \\ c_1 + c_2 = 1. \end{cases}$$

By simple calculations,

$$(\tilde{\mu}, c_1, c_1) = \left(\sqrt{\frac{a_1a_2}{h'(0)g'(0)}}, 1 - \frac{a_1}{a_1 + h'(0)\sqrt{\frac{a_1a_2}{h'(0)g'(0)}}}, \frac{a_1}{a_1 + h'(0)\sqrt{\frac{a_1a_2}{h'(0)g'(0)}}}\right).$$

Then $\tilde{\mu} = \sqrt{\frac{a_1a_2}{h'(0)g'(0)}} \in E^{\mathbb{R}}$, which implies that $\mu_1(\mathbb{R}) \geqslant \tilde{\mu} = \sqrt{\frac{a_1a_2}{h'(0)g'(0)}}$. It follows that $\mathcal{R}^* = \frac{1}{\mu^*} = \frac{1}{\mu_1(\mathbb{R})} \leqslant \mathcal{R}_0 := \sqrt{\frac{h'(0)g'(0)}{a_1a_2}}$, which completes the proof. □

Remark 2.5. (1) We remark that $\mathcal{R}^*$ may be not equal to $\mathcal{R}_0$. Here, we give two cases to illustrate the result $\mathcal{R}^* \leqslant \mathcal{R}_0$.

Case I. If $J_1 = J_2 = \delta$ (Dirac delta function), then

$$\mathcal{R}_0^l = \sqrt{\frac{h'(0)g'(0)}{[d_1(\frac{\pi}{2l})^2 + a_1][d_2(\frac{\pi}{2l})^2 + a_2]}}.$$

As $l \to \infty$, we have $\mathcal{R}_0^l \to \sqrt{\frac{h'(0)g'(0)}{a_1a_2}}$. Therefore $\mathcal{R}^* = \mathcal{R}_0$. More details can be seen in [**22**].

Case II. If $d_i = d$, $a_i = a$, $J_i = J$ $(i = 1, 2)$ and $h = g$, then, by taking $\psi = \phi + \varphi$, (1.7) reduces to the following single equation

$$\begin{cases} -d\psi_{xx} + a\psi = \mu h'(0) \int_{-l}^{l} J(x - y)\psi(y)\mathrm{d}y, \ x \in (-l, l), \\ \psi(\pm l) = 0. \end{cases}$$

From the variational characterization of the principal eigenvalue, we have

$$\mu_0^l = \inf_{\substack{\psi \in H_0^1((-l,l)) \\ \|\psi\|_{L^2=1}}} \left\{ \frac{\int_{-l}^{l} d|\nabla\psi|^2\mathrm{d}x + a}{h'(0)\int_{-l}^{l}\int_{-l}^{l} J(x - y)\psi(y)\psi(x)\mathrm{d}x\mathrm{d}y} \right\},$$

and then

$$\mathcal{R}_0^l = \sup_{\substack{\psi\in H_0^1((-l,l))\\ \|\psi\|_{L^2}=1}} \left\{ \frac{h'(0)\int_{-l}^{l}\int_{-l}^{l} J(x-y)\psi(y)\psi(x)\mathrm{d}x\mathrm{d}y}{\int_{-l}^{l} d|\nabla\psi|^2\mathrm{d}x + a} \right\}.$$

Thus,

$$\mathcal{R}^* \leqslant \frac{h'(0)}{a} \sup_{\substack{\psi\in H^1(\mathbb{R})\\ \|\psi\|_{L^2}=1}} \left\{ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} J(x-y)\psi(y)\psi(x)\mathrm{d}x\mathrm{d}y \right\}.$$

Note that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} J(x-y)\psi(y)\psi(x)\mathrm{d}x\mathrm{d}y \leqslant \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} J(x-y)\frac{\psi(y)^2+\psi(x)^2}{2}\mathrm{d}x\mathrm{d}y = 1,$$

and the first equality holds if and only if ψ is a constant function, which contradicts with $\psi \in H^1(\mathbb{R})$. Therefore $\mathcal{R}^* < \frac{h'(0)}{a} = \mathcal{R}_0$.

(2) If the interval $(-l, l)$ is replaced by (a, b) with $-\infty < a < 0 < b < +\infty$, then the conclusions in theorem 1.1 are still valid.

3. Spreading and vanishing

In this section, we discuss the spreading and vanishing phenomenon of the bacteria in terms of the basic reproduction number, and then provide the sharp criteria for spreading and vanishing.

Denote $s_{1,\infty} = \lim_{t\to+\infty} s_1(t)$ and $s_{2,\infty} = \lim_{t\to+\infty} s_2(t)$. Then, we have the following results.

LEMMA 3.1. *(i) If $s_{2,\infty} - s_{1,\infty} = +\infty$, then $s_{2,\infty} = -s_{1,\infty} = +\infty$.*
(ii) If $s_{2,\infty} - s_{1,\infty} < +\infty$, then $s_0 < -s_{1,\infty}, s_{2,\infty} < +\infty$, and

$$\lim_{t\to+\infty}\left(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])} \right) = 0.$$

(iii) If $\mathcal{R}^* > 1$ and $s_{2,\infty} - s_{1,\infty} = +\infty$, then $\lim_{t\to+\infty}(u(t, x), v(t, x)) = (u^*, v^*)$ locally uniformly for $x \in \mathbb{R}$, where (u^*, v^*) is the unique equilibrium of (1.1).

Proof. All the conclusions can be proved by similar arguments as Lemmas 3.1–3.2 and Theorem 4.5 in [**1**] with minor modifications (see also Theorem 2.3 in [**11**]), here we omit the details. We remark that the condition $\mathcal{R}^* > 1$ in (*iii*) is assumed for applying theorem 1.1 in the proof. □

THEOREM 3.2. *If $0 < \mathcal{R}_0 \leqslant 1$, then the solution $(u, v; s_1, s_2)$ of (1.1) satisfies*

$$\lim_{t\to+\infty}\left(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])} \right) = 0.$$

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Proof. Let $(w_1(t), w_2(t))$ be the unique solution of

$$\begin{cases} w_1' = -a_1 w_1 + h(w_2(t-\tau_1)),\ t > 0, \\ w_2' = -a_2 w_2 + g(w_1(t-\tau_2)),\ t > 0, \\ w_1(\theta) = \|u_0\|_{C([-\tau_2,0]\times[s_1,s_2])},\ \theta \in [-\tau_2, 0], \\ w_2(\theta) = \|v_0\|_{C([-\tau_1,0]\times[s_1,s_2])},\ \theta \in [-\tau_1, 0]. \end{cases} \tag{3.1}$$

From the comparison principle, we know that $(u(t, x), v(t, x)) \leqslant (w_1(t), w_2(t))$ in $[0, +\infty) \times [s_1, s_2]$.

Since $h''(z) \leqslant 0$, $g''(z) < 0$ for all $z > 0$, we have

$$\frac{h(\lambda z)}{\lambda z} = \frac{h(\lambda z) - h(0)}{\lambda z} \geqslant \frac{h(z) - h(0)}{z} = \frac{h(z)}{z},$$
$$\frac{g(\lambda z)}{\lambda z} = \frac{g(\lambda z) - g(0)}{\lambda z} > \frac{g(z) - g(0)}{z} = \frac{g(z)}{z}$$

for $z > 0$ and $\lambda \in (0, 1)$. That is, h, g are subhomogeneous. From Theorem 3.2 in [**42**], we know that $(0, 0)$ is globally asymptotically stable for (3.1). That is, (w_1, w_2) satisfies $\lim_{t\to+\infty}(w_1(t), w_2(t)) = (0, 0)$, which implies $\lim_{t\to+\infty}(\|u(t, \cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t, \cdot)\|_{C([s_1(t),s_2(t)])}) = 0$. More details can be seen in [**10**, **11**]. □

REMARK 3.3. Due to the effects of delays, we can not show $s_{2,\infty} - s_{1,\infty} < +\infty$ as in [**1**], even for the local, partially degenerate case with one delay considered in [**10**]. We leave it for further consideration.

Next, we discuss the spreading and vanishing phenomenon of (1.1) for $\mathcal{R}^* > 1$ in terms of $\mathcal{R}_0^{s_0}$. The selected forms of upper and lower solutions in the proof of next two theorems have been used in many related works. However, to overcome the effects of nonlocal terms, in this paper we construct upper and lower solutions from the principle eigenfunctions of perturbed eigenvalue problems. This idea is inspired by the work of Huang and Wang [**18**].

THEOREM 3.4. *If* $\mathcal{R}_0^{s_0} \geqslant 1$*, then* $s_{2,\infty} - s_{1,\infty} = +\infty$*.*

Proof. By theorem 1.1 (iii), we know that the basic reproduction number is strictly increasing with respect to the domain. Note that $(-s_0, s_0) \subset (s_1(t_0), s_2(t_0))$ for any fixed $t_0 > 0$. If $\mathcal{R}_0^{s_0} = 1$, then the basic reproduction number of (1.5) with $(-l, l)$ replaced by $(s_1(t_0), s_2(t_0))$ is larger than 1. In such a case, we can choose some $t_0 > 0$ as initial time. Therefore it suffices to consider the case $\mathcal{R}_0^{s_0} > 1$.

From theorem 1.1 (i), we have sign $(\mathcal{R}_0^{s_0} - 1) =$ sign $\lambda_1 > 0$. Then there exists a constant $0 < \delta^* \ll 1$ such that $\lambda_1^\delta > 0$ for all $0 < \delta < \delta^*$, where λ_1^δ is the principle eigenvalue of the following perturbed eigenvalue problem

$$\begin{cases} d_1\phi_{xx} - a_1\phi + (h'(0) - \delta)\int_{-s_0}^{s_0} J_1(x-y)\varphi(y)\mathrm{d}y = \lambda\phi,\ x \in (-s_0, s_0), \\ d_2\varphi_{xx} - a_2\varphi + (g'(0) - \delta)\int_{-s_0}^{s_0} J_2(x-y)\phi(y)\mathrm{d}y = \lambda\varphi,\ x \in (-s_0, s_0), \\ (\phi(\pm s_0), \varphi(\pm s_0)) = (0, 0). \end{cases} \tag{3.2}$$

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Let $\Phi^\delta(x) = (\phi^\delta(x), \varphi^\delta(x))$ be the positive eigenfunction of (3.2) associated with λ_1^δ.

Define $\underline{u}(t, x) = \varepsilon\phi^\delta(x)$, $\underline{v}(t, x) = \varepsilon\varphi^\delta(x)$ for $(t, x) \in [\max\{\tau_1, \tau_2\}, +\infty) \times [-s_0, s_0]$, where ε is a small positive constant to be determined. Direct calculations yield

$$\begin{aligned}
&\underline{u}_t - d_1\underline{u}_{xx} + a_1\underline{u} - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y\right) \\
&\quad = -\varepsilon\lambda_1^\delta\phi^\delta + \varepsilon(h'(0)-\delta)\int_{-s_0}^{s_0} J_1(x-y)\varphi^\delta(y)\mathrm{d}y - h\left(\int_{-s_0}^{s_0} \varepsilon J_1(x-y)\varphi^\delta(y)\mathrm{d}y\right) \\
&\quad = -\varepsilon\left[\lambda_1^\delta\phi^\delta + \left(\delta + h'(\eta_1^\varepsilon(x)) - h'(0)\right)\int_{-s_0}^{s_0} J_1(x-y)\varphi^\delta(y)\mathrm{d}y\right]
\end{aligned}$$

and

$$\begin{aligned}
&\underline{v}_t - d_2\underline{v}_{xx} + a_2\underline{v} - g\left(\int_{-\infty}^{+\infty} J_2(x-y)\underline{u}(t-\tau_2, y)\mathrm{d}y\right) \\
&\quad = -\varepsilon\left[\lambda_1^\delta\varphi^\delta + \left(\delta + g'(\eta_2^\varepsilon(x)) - g'(0)\right)\int_{-s_0}^{s_0} J_2(x-y)\phi^\delta(y)\mathrm{d}y\right],
\end{aligned}$$

where $\eta_1^\varepsilon(x) \in (0, \varepsilon\int_{-s_0}^{s_0} J_1(x-y)\varphi^\delta(y)\mathrm{d}y)$ and $\eta_2^\varepsilon(x) \in (0, \varepsilon\int_{-s_0}^{s_0} J_2(x-y)\phi^\delta(y)\mathrm{d}y)$.

Note that $\eta_1^\varepsilon(x), \eta_2^\varepsilon(x) \to 0$ as $\varepsilon \to 0$. We can choose $\varepsilon > 0$ sufficiently small such that $\delta + h'(\eta_1^\varepsilon(x)) - h'(0) > 0$ and $\delta + g'(\eta_2^\varepsilon(x)) - g'(0) > 0$, which imply that

$$\begin{cases}
\underline{u}_t - d_1\underline{u}_{xx} + a_1\underline{u} - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y\right) \leqslant 0, \\
\underline{v}_t - d_2\underline{v}_{xx} + a_2\underline{v} - g\left(\int_{-\infty}^{+\infty} J_2(x-y)\underline{u}(t-\tau_2, y)\mathrm{d}y\right) \leqslant 0,
\end{cases}$$

for any $t > \max\{\tau_1, \tau_2\}$ and $-s_0 < x < s_0$. We can also assume $\varepsilon > 0$ small such that

$$\begin{aligned}
&\underline{u}(\theta, x) = \varepsilon\phi^\delta(x) \leqslant u(\theta, x), \ \max\{\tau_1, \tau_2\} - \tau_2 \leqslant \theta \leqslant \max\{\tau_1, \tau_2\}, -s_0 \leqslant x \leqslant s_0, \\
&\underline{v}(\theta, x) = \varepsilon\varphi^\delta(x) \leqslant v(\theta, x), \ \max\{\tau_1, \tau_2\} - \tau_1 \leqslant \theta \leqslant \max\{\tau_1, \tau_2\}, -s_0 \leqslant x \leqslant s_0.
\end{aligned}$$

Moreover, it is easy to deduce that

$$\begin{aligned}
&\underline{u}(t, x) = \underline{v}(t, x) = 0, \ t \geqslant \max\{\tau_1, \tau_2\}, \ x \leqslant -s_0 \text{ or } x \geqslant s_0, \\
&0 = s_0' \leqslant -\mu[\underline{u}_x(t, s_0) + \rho\underline{v}_x(t, s_0)], \ t > \max\{\tau_1, \tau_2\}, \\
&0 = -s_0' \geqslant -\mu[\underline{u}_x(t, -s_0) + \rho\underline{v}_x(t, -s_0)], \ t > \max\{\tau_1, \tau_2\}, \\
&[-s_0, s_0] \subseteq [s_1(\theta), s_2(\theta)], \ t > \max\{\tau_1, \tau_2\}.
\end{aligned}$$

Therefore, $(\underline{u}, \underline{v}; -s_0, s_0)$ is a lower solution of (1.1). By the comparison principle,

$$\liminf_{t\to+\infty}\left(\|u(t,\cdot)\|_{C([s_1(t), s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t), s_2(t)])}\right) \geqslant \varepsilon(\phi^\delta(0) + \varphi^\delta(0)) > 0,$$

which implies $s_{2,\infty} - s_{1,\infty} = +\infty$. □

THEOREM 3.5. *If* $\mathcal{R}_0^{s_0} < 1$, *then* $s_{2,\infty} - s_{1,\infty} < +\infty$ *provided that* μ *is sufficiently small.*

Proof. From theorem 1.1 (i), we have sign $(\mathcal{R}_0^{s_0} - 1) = \text{sign } \lambda_1 < 0$. Then there exists a constant $0 < \delta^* \ll 1$ such that $\lambda_1^\delta < 0$ for all $0 < \delta < \delta^*$, where λ_1^δ is the principle eigenvalue of the following perturbed eigenvalue problem

$$\begin{cases} d_1\phi_{xx} - a_1\phi + (h'(0) + \delta)\int_{-s_0}^{s_0} J_1(x-y)\varphi(y)\mathrm{d}y = \lambda\phi, \ x \in (-s_0, s_0), \\ d_2\varphi_{xx} - a_2\varphi + (g'(0) + \delta)\int_{-s_0}^{s_0} J_2(x-y)\phi(y)\mathrm{d}y = \lambda\varphi, \ x \in (-s_0, s_0), \\ (\phi(\pm s_0), \varphi(\pm s_0)) = (0, 0). \end{cases} \tag{3.3}$$

Let $\Phi^\delta(x) = (\phi^\delta(x), \varphi^\delta(x))$ be the positive eigenfunction of (3.3) associated with λ_1^δ.

We define

$$\begin{aligned} k(t) &= s_0(1 + \sigma - \frac{\sigma}{2}e^{-\sigma t}), \quad t > 0, \\ \bar{u}(t,x) &= \alpha e^{-\sigma t}\phi^\delta\left(\frac{s_0 x}{k(t)}\right), \quad t > 0, \ x \in [-k(t), k(t)], \\ \bar{v}(t,x) &= \alpha e^{-\sigma t}\varphi^\delta\left(\frac{s_0 x}{k(t)}\right), \quad t > 0, \ x \in [-k(t), k(t)], \\ k(\theta) &\equiv k(0) = s_0(1 + \frac{\sigma}{2}), \quad \theta \in [-\max\{\tau_1, \tau_2\}, 0], \\ \bar{u}(\theta, x) &= \bar{u}(0, x) = \alpha\phi^\delta\left(\frac{2x}{2+\sigma}\right), \quad \theta \in [-\tau_2, 0], \ x \in [-k(\theta), k(\theta)], \\ \bar{v}(\theta, x) &= \bar{v}(0, x) = \alpha\varphi^\delta\left(\frac{2x}{2+\sigma}\right), \quad \theta \in [-\tau_1, 0], \ x \in [-k(\theta), k(\theta)] \end{aligned}$$

and extend $\bar{u}(t, x)$ (resp. $\bar{v}(t, x)$) by 0 for $t \in [-\tau_2, +\infty)$, $x \in (-\infty, -k(t)) \cup (k(t), +\infty)$ (resp. $t \in [-\tau_1, +\infty)$, $x \in (-\infty, -k(t)) \cup (k(t), +\infty)$).

Direct calculations show that

$$\begin{aligned} &\bar{u}_t - d_1\bar{u}_{xx} + a_1\bar{u} - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\right) \\ &= -\sigma\bar{u} - \frac{s_0 x k'(t)}{k^2(t)}\alpha e^{-\sigma t}(\phi^\delta)'\left(\frac{s_0 x}{k(t)}\right) - d_1\alpha e^{-\sigma t}\left(\frac{s_0}{k(t)}\right)^2(\phi^\delta)''\left(\frac{s_0 x}{k(t)}\right) \\ &\quad + a_1\bar{u} - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\right) \\ &\geqslant -\sigma\bar{u} - \frac{s_0 x k'(t)}{k^2(t)}\alpha e^{-\sigma t}(\phi^\delta)'\left(\frac{s_0 x}{k(t)}\right) \end{aligned}$$

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$$
\begin{aligned}
&-\alpha e^{-\sigma t}\left(\frac{s_0}{k(t)}\right)^2\left[(a_1+\lambda_1^\delta)\phi^\delta\left(\frac{s_0x}{k(t)}\right)\right.\\
&\left.-(h'(0)+\delta)\int_{-s_0}^{s_0}J_1\left(\frac{s_0x}{k(t)}-y\right)\varphi^\delta(y)\mathrm{d}y\right]\\
&+a_1\bar{u}-h'(0)\int_{-\infty}^{+\infty}J_1(x-y)\bar{v}(t-\tau_1,y)\mathrm{d}y\\
=&-\frac{\alpha\sigma^2e^{-2\sigma t}s_0}{2k(t)}z(\phi^\delta)'(z)+\left[a_1-\sigma-(a_1+\lambda_1^\delta)\left(\frac{s_0}{k(t)}\right)^2\right]\alpha e^{-\sigma t}\phi^\delta(z)\\
&+\left[\alpha(h'(0)+\delta)e^{-\sigma t}\left(\frac{s_0}{k(t)}\right)^2\int_{-s_0}^{s_0}J_1(z-y)\varphi^\delta(y)\mathrm{d}y\right.\\
&\left.-h'(0)\int_{-\infty}^{+\infty}J_1\left(\frac{k(t)z}{s_0}-y\right)\bar{v}(t-\tau_1,y)\right)\mathrm{d}y\Bigg]\\
=:&\ I+II+III,
\end{aligned}
$$

where $z:=\frac{s_0x}{k(t)}\in(-s_0,\,s_0)$.

Since $-z(\phi^\delta)'(z)|_{z=\pm s_0}>0$ by the Hopf boundary lemma, we have $(I+II)|_{z=\pm s_0}>0$. By the continuity, we know that $I+II>0$ in some neighbourhood $\mathcal{O}\subseteq[-s_0,\,s_0]$ of $z=\pm s_0$. For $z\in[-s_0,\,s_0]\setminus\mathcal{O}$, $\phi^\delta(z)\geqslant c$ with some positive constant c, and then $II\to-\alpha\lambda_1^\delta\phi^\delta(z)\geqslant-\alpha\lambda_1^\delta c>0$ as $\sigma\to0$. Note that $\lim_{\sigma\to0}I=0$. We can choose σ sufficiently small such that $I+II>0$ on $[-s_0,\,s_0]\setminus\mathcal{O}$. Therefore, $I+II$ is positive on $[-s_0,\,s_0]$ for small σ.

Now we consider the third term. Since J_1 is a nonnegative, continuous function satisfying $J_1(0)>0$, we have $\int_{-s_0}^{s_0}J_1(z-y)\varphi^\delta(y)\mathrm{d}y>0$ for any $z\in[-s_0,\,s_0]$. As $\sigma\to0$,

$$
\begin{aligned}
III&\to\alpha\delta\textstyle\int_{-s_0}^{s_0}J_1(z-y)\varphi^\delta(y)\mathrm{d}y\\
&\geqslant\alpha\delta\min_{z\in[-s_0,s_0]}\textstyle\int_{-s_0}^{s_0}J_1(z-y)\varphi^\delta(y)\mathrm{d}y=:\alpha\delta c_1>0.
\end{aligned}
$$

By choosing σ sufficiently small, we can get $III>0$.

In summary, for $(t,\,x)\in(0,\,+\infty)\times(-k,\,k)$,

$$
\bar{u}_t-d_1\bar{u}_{xx}+a_1\bar{u}-h\left(\int_{-\infty}^{+\infty}J_1(x-y)\bar{v}(t-\tau_1,y)\mathrm{d}y\right)\geqslant0.
$$

In a similar way, we can prove that

$$
\bar{v}_t-d_2\bar{v}_{xx}+a_2\bar{v}-g\left(\int_{-\infty}^{+\infty}J_2(x-y)\bar{u}(t-\tau_2,y)\mathrm{d}y\right)\geqslant0.
$$

Moreover, choose α large enough such that

$$
\bar{u}_0(\theta,x)=\alpha\phi^\delta\left(\frac{2x}{2+\sigma}\right)\geqslant\|u_0\|_{L^\infty([-\tau_2,0]\times[s_1,s_2])}\geqslant u_0(\theta,x)
$$

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for $(\theta, x) \in [-\tau_2, 0] \times [s_1, s_2]$, and

$$\bar{v}_0(\theta, x) = \alpha\varphi^\delta\left(\frac{2x}{2+\sigma}\right) \geqslant \|v_0\|_{L^\infty([-\tau_1,0]\times[s_1,s_2])} \geqslant v_0(\theta, x)$$

for $(\theta, x) \in [-\tau_1, 0] \times [s_1, s_2]$. Then take $\mu > 0$ sufficiently small such that, for $t > 0$,

$$k'(t) = s_0\frac{\sigma^2}{2}e^{-\sigma t} \geqslant -\mu[\bar{u}_x(t, k(t)) + \rho\bar{v}_x(t, k(t))],$$
$$-k'(t) = -s_0\frac{\sigma^2}{2}e^{-\sigma t} \leqslant -\mu[\bar{u}_x(t, -k(t)) + \rho\bar{v}_x(t, -k(t))].$$

Besides, it is easy to check that

$$[-k(\theta), k(\theta)] = \left[-s_0\left(1+\frac{\sigma}{2}\right), s_0\left(1+\frac{\sigma}{2}\right)\right] \supset [-s_0, s_0] \supset [s_1(\theta), s_2(\theta)]$$

for $\theta \in [-\max\{\tau_1, \tau_2\}, 0]$.

Therefore, $(\bar{u}, \bar{v}; -k(t), k(t))$ is an upper solution of (1.1) and we have

$$-s_{1,\infty}, s_{2,\infty} \leqslant \lim_{t\to+\infty} k(t) = s_0(1+\sigma),$$

which completes the proof. □

By using similar arguments as the proof of Theorem 4.4 in [**26**], we can obtain the following result for the case $\mathcal{R}_0^{s_0} < 1 < \mathcal{R}^*$ with large μ.

THEOREM 3.6. *If $\mathcal{R}_0^{s_0} < 1 < \mathcal{R}^*$, then $s_{2,\infty} - s_{1,\infty} = +\infty$ provided that μ is sufficiently large.*

In what follows, we exhibit the sharp criteria of (1.1). The proof relies on the conclusions of theorems 3.4–3.6. More details can be seen in the proofs of Theorem 3.9 in [**15**] and Theorem 4.5 in [**26**], here we omit the details.

THEOREM 3.7. *Assume that $\mathcal{R}^* > 1$. Then there exists $\mu^* \in [0, +\infty)$ such that $s_{2,\infty} - s_{1,\infty} = +\infty$ for $\mu > \mu^*$, and $s_{2,\infty} - s_{1,\infty} < +\infty$ for $0 < \mu \leqslant \mu^*$.*

REMARK 3.8. In theorems 3.2 and 3.7, we discuss the long-time behaviour of solution for $\mathcal{R}_0 \leqslant 1$ and $\mathcal{R}^* > 1$, respectively. However, the case $\mathcal{R}^* \leqslant 1 < \mathcal{R}_0$ is still unknown.

4. Nonlocal semi-wave problem with delays

In this section, we consider the delayed nonlocal semi-wave problem (1.8). The semi-wave solution of (1.8) will play an important role in determining the precise asymptotic spreading speed of (1.1) when spreading occurs. We always assume $\mathcal{R}^* > 1$ in this section.

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4.1. Perturbed semi-wave problem

To establish the existence of semi-wave solutions to (1.8), we first consider the corresponding perturbed problem:

$$\begin{cases} c\phi'(\xi) = d_1\phi''(\xi) - a_1\phi(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\xi - y - c\tau_1)\mathrm{d}y\right), & \xi > 0,\\ c\varphi'(\xi) = d_2\varphi''(\xi) - a_2\varphi(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(\xi - y - c\tau_2)\mathrm{d}y\right), & \xi > 0,\\ (\phi(\xi), \varphi(\xi)) = (\delta u^*, \delta v^*), \quad \xi \leqslant 0,\\ (\phi(+\infty), \varphi(+\infty)) = (u^*, v^*), \end{cases} \tag{4.1}$$

where $\delta \in (0, \frac{1}{2})$ is a small parameter. Then the desired semi-wave solution $(\phi(\xi), \varphi(\xi))$ of (1.8) can be obtained from the solutions $(\phi^\delta(\xi), \varphi^\delta(\xi))$ of (4.1) by taking $\delta \to 0$.

For convenient, we denote $\beta_{i1} = \frac{c-\sqrt{c^2+4a_id_i}}{2d_i}$, $\beta_{i2} = \frac{c+\sqrt{c^2+4a_id_i}}{2d_i}$, $i = 1, 2$. Let $\Phi = (\phi, \varphi)$, we define the operators $(\mathcal{F}_1, \mathcal{F}_2) : C(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$\mathcal{F}_1(\Phi)(\xi) = \begin{cases} \delta u^* e^{\beta_{11}\xi} + \dfrac{1}{d_1(\beta_{12} - \beta_{11})}\Bigg[\int_0^\xi (e^{\beta_{11}(\xi - s)} - e^{\beta_{11}\xi - \beta_{12}s}) \\ \qquad \times h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(s - y - c\tau_1)\mathrm{d}y\right)\mathrm{d}s \\ \qquad + \int_\xi^{+\infty} (e^{\beta_{12}(\xi - s)} - e^{\beta_{11}\xi - \beta_{12}s}) \\ \qquad \times h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(s - y - c\tau_1)\mathrm{d}y\right)\mathrm{d}s\Bigg], \ \xi > 0,\\ \delta u^*, \ \xi \leqslant 0, \end{cases}$$

and

$$\mathcal{F}_2(\Phi)(\xi) = \begin{cases} \delta v^* e^{\beta_{21}\xi} + \dfrac{1}{d_2(\beta_{22} - \beta_{21})}\Bigg[\int_0^\xi (e^{\beta_{21}(\xi - s)} - e^{\beta_{21}\xi - \beta_{22}s}) \\ \qquad \times g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(s - y - c\tau_2)\mathrm{d}y\right)\mathrm{d}s \\ \qquad + \int_\xi^{+\infty} (e^{\beta_{22}(\xi - s)} - e^{\beta_{21}\xi - \beta_{22}s}) \\ \qquad \times g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(s - y - c\tau_2)\mathrm{d}y\right)\mathrm{d}s\Bigg], \ \xi > 0,\\ \delta v^*, \ \xi \leqslant 0. \end{cases}$$

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It is easy to show that the operators $\mathcal{F}_i$ $(i = 1, 2)$ are well-defined and satisfy

$$\begin{cases} c(\mathcal{F}_1(\Phi))'(\xi) = d_1(\mathcal{F}_1(\Phi))''(\xi) - a_1\mathcal{F}_1(\Phi)(\xi) \\ \qquad + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\xi - y - c\tau_1)\mathrm{d}y\right), \ \xi > 0, \\ c(\mathcal{F}_2(\Phi))'(\xi) = d_2(\mathcal{F}_2(\Phi)''(\xi) - a_2\mathcal{F}_2(\Phi)(\xi) \\ \qquad + g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(\xi - y - c\tau_2)\mathrm{d}y\right), \ \xi > 0, \\ (\mathcal{F}_1(\Phi)(\xi), \mathcal{F}_2(\Phi)(\xi)) = (\delta u^*, \delta v^*), \ \xi \leqslant 0. \end{cases}$$

Thus, (ϕ, φ) is a fixed point of $(\mathcal{F}_1, \mathcal{F}_2)$ in $C(\mathbb{R}, \mathbb{R}^2)$ if and only if it is a solution of (4.1) in $C(\mathbb{R}, \mathbb{R}^2)$.

We define the set Γ as follows:

$$\Gamma = \Bigg\{(\phi, \varphi) \in C(\mathbb{R}, \mathbb{R}^2) : \ (i) \ \phi(\xi), \varphi(\xi) \text{ are increasing in } \xi \in \mathbb{R}_+,$$
$$(ii) \ (\phi(\xi), \varphi(\xi)) = (\delta u^*, \delta v^*) \text{ for } \xi \leqslant 0, \ (iii) \ (\phi(+\infty), \varphi(+\infty)) = (u^*, v^*)\Bigg\}.$$

LEMMA 4.1. *For any* $\Phi = (\phi, \varphi) \in \Gamma$, *we have*

(i) $(\mathcal{F}_1(\Phi)(\xi), \mathcal{F}_2(\Phi)(\xi)) \geqslant (0, 0)$ *for any* $\xi \in \mathbb{R}$;

(ii) $(\mathcal{F}_1(\Phi)(\xi), \mathcal{F}_2(\Phi)(\xi))$ *are increasing in* $\xi \in \mathbb{R}$;

(iii) *if* $\Phi_i = (\phi_i, \varphi_i) \in \Gamma$ $(i = 1, 2)$ *satisfy* $\Phi_1 \leqslant \Phi_2$, *then* $\mathcal{F}_i(\Phi_1)(\xi) \leqslant \mathcal{F}_i(\Phi_2)(\xi)$ *for any* $\xi \in \mathbb{R}$.

Proof. Since $\beta_{i2} > \beta_{i1}$ $(i = 1, 2)$, we can easily check that (i) and (iii) hold. Now we prove (ii).

By the definitions of $\mathcal{F}_i$, it is sufficient to consider the case $\xi > 0$. Note that $\beta_{11} < 0$ and φ is a positive increasing function. For $\xi > 0$, we have

$$\begin{aligned}(\mathcal{F}_1(\Phi))'(\xi) &= \delta u^*\beta_{11}e^{\beta_{11}\xi} + \frac{1}{d_1(\beta_{12} - \beta_{11})} \\ &\times \Bigg[\beta_{11}\int_0^{\xi}(e^{\beta_{11}(\xi - s)} - e^{\beta_{11}\xi - \beta_{12}s})h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(s - y - c\tau_1)\mathrm{d}y\right)\mathrm{d}s \\ &+ \int_{\xi}^{+\infty}(\beta_{12}e^{\beta_{12}(\xi - s)} - \beta_{11}e^{\beta_{11}\xi - \beta_{12}s})h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(s - y - c\tau_1)\mathrm{d}y\right)\mathrm{d}s\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\geqslant \delta u^*\beta_{11}e^{\beta_{11}\xi}+\frac{1}{d_1(\beta_{12}-\beta_{11})}h\left(\int_{-\infty}^{+\infty}J_1(y)\varphi(\xi-y-c\tau_1)\mathrm{d}y\right)\\
&\quad\times\left[\beta_{11}\int_0^{\xi}(e^{\beta_{11}(\xi-s)}-e^{\beta_{11}\xi-\beta_{12}s})\mathrm{d}s+\int_{\xi}^{+\infty}(\beta_{12}e^{\beta_{12}(\xi-s)}-\beta_{11}e^{\beta_{11}\xi-\beta_{12}s})\mathrm{d}s\right]\\
&=\delta u^*\beta_{11}e^{\beta_{11}\xi}+\frac{e^{\beta_{11}\xi}}{d_1\beta_{12}}h\left(\int_{-\infty}^{+\infty}J_1(y)\varphi(\xi-y-c\tau_1)\mathrm{d}y\right)\\
&\geqslant \delta u^*\beta_{11}e^{\beta_{11}\xi}+\frac{e^{\beta_{11}\xi}}{d_1\beta_{12}}h\left(\int_{-\infty}^{+\infty}\delta v^*J_1(y)\mathrm{d}y\right).
\end{aligned}
$$

In view of $\int_{-\infty}^{+\infty}J_1(y)\mathrm{d}y=1$ and h is subhomogeneous (see the proof of theorem 3.2),

$$
\begin{aligned}
(\mathcal{F}_1(\Phi))'(\xi)&=\delta u^*\beta_{11}e^{\beta_{11}\xi}+\frac{h(\delta v^*)e^{\beta_{11}\xi}}{d_1\beta_{12}}\geqslant \delta u^*\beta_{11}e^{\beta_{11}\xi}+\frac{\delta h(v^*)e^{\beta_{11}\xi}}{d_1\beta_{12}}\\
&=\delta u^*\beta_{11}e^{\beta_{11}\xi}+\frac{\delta a_1u^*e^{\beta_{11}\xi}}{d_1\beta_{12}}\geqslant 0.
\end{aligned}
$$

Similarly, we can deduce that $(\mathcal{F}_2(\Phi))'(\xi)\geqslant 0$ for $\xi>0$. □

Next, we give the definitions of upper and lower solutions for (4.1).

DEFINITION 4.2. *Assume that* $(\bar{\phi},\bar{\varphi}),(\underline{\phi},\underline{\varphi})$ *are continuous function pairs from* $\mathbb{R}$ *into* $[\delta u^*,u^*]\times[\delta v^*,v^*]$. *We call that* $(\bar{\phi},\bar{\varphi}),(\underline{\phi},\underline{\varphi})$ *are respectively an upper solution and a lower solution of* (4.1), *if* $\bar{\phi},\underline{\phi}$ *are twice continuously differentiable on* $\mathbb{R}_+\setminus\{\xi_i\}_{i=1}^m$, $\bar{\varphi},\underline{\varphi}$ *are twice continuously differentiable on* $\mathbb{R}_+\setminus\{\eta_j\}_{j=1}^k$, *and they satisfy*

$$
\begin{cases}
c\bar{\phi}'(\xi)\geqslant d_1\bar{\phi}''(\xi)-a_1\bar{\phi}(\xi)+h\left(\int_{-\infty}^{+\infty}J_1(y)\bar{\varphi}(\xi-y-c\tau_1)\mathrm{d}y\right),\ \xi>0,\ \xi\notin\{\xi_i\}_{i=1}^m,\\
c\bar{\varphi}'(\xi)\geqslant d_2\bar{\varphi}''(\xi)-a_2\bar{\varphi}(\xi)+g\left(\int_{-\infty}^{+\infty}J_2(y)\bar{\phi}(\xi-y-c\tau_2)\mathrm{d}y\right),\ \xi>0,\ \xi\notin\{\eta_j\}_{j=1}^k,\\
\bar{\phi}'_+(\xi_i)\leqslant\bar{\phi}'_-(\xi_i),\quad i=1,\cdots,m,\\
\bar{\varphi}'_+(\eta_j)\leqslant\bar{\varphi}'_-(\eta_j),\quad j=1,\cdots,k,\\
(\bar{\phi}(\xi),\bar{\varphi}(\xi))=(\delta u^*,\delta v^*),\quad \xi\leqslant 0,\\
(\bar{\phi}(+\infty),\bar{\varphi}(+\infty))=(u^*,v^*)
\end{cases}
$$

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and

$$\begin{cases} c\underline{\phi}'(\xi)\leqslant d_1\underline{\phi}''(\xi)-a_1\underline{\phi}(\xi)+h\left(\int_{-\infty}^{+\infty} J_1(y)\underline{\varphi}(\xi-y-c\tau_1)\mathrm{d}y\right),\ \xi>0,\ \xi\notin\{\xi_i\}_{i=1}^m,\\ c\underline{\varphi}'(\xi)\leqslant d_2\underline{\varphi}''(\xi)-a_2\underline{\varphi}(\xi)+g\left(\int_{-\infty}^{+\infty} J_2(y)\underline{\phi}(\xi-y-c\tau_2)\mathrm{d}y\right),\ \xi>0,\ \xi\notin\{\eta_j\}_{j=1}^k,\\ \underline{\phi}'_+(\xi_i)\geqslant \underline{\phi}'_-(\xi_i), \quad i=1,\cdots,m,\\ \underline{\varphi}'_+(\eta_j)\geqslant \underline{\varphi}'_-(\eta_j), \quad j=1,\cdots,k,\\ (\underline{\phi}(\xi),\underline{\varphi}(\xi))=(\delta u^*,\delta v^*), \quad \xi\leqslant 0,\\ (\underline{\phi}(+\infty),\underline{\varphi}(+\infty))\leqslant (u^*,v^*). \end{cases}$$

Now we establish the existence of solution to the perturbed semi-wave problem (4.1) by applying monotone iteration method, which is an efficient method for travelling wave solutions, see [**36**].

THEOREM 4.3. *If there exist an upper solution* $(\bar{\phi},\,\bar{\varphi})\in\Gamma$ *and a lower solution* $(\underline{\phi},\,\underline{\varphi})$ *(which is not necessary in* Γ*) of* (4.1)*, satisfying* $(\delta u^*,\,\delta v^*)\leqslant(\underline{\phi}(\xi),\,\underline{\varphi}(\xi))\leqslant(\bar{\phi}(\xi),\,\bar{\varphi}(\xi))\leqslant(u^*,\,v^*)$ *for* $\xi\in\mathbb{R}_+$*, then the perturbed problem* (4.1) *admits an increasing solution.*

Proof. The proof is divided into the following three steps.

Step 1: For $n=1,\,2,\,\cdots$, we consider the following iteration scheme

$$\begin{cases} c\phi_n'(\xi)=d_1\phi_n''(\xi)-a_1\phi_n(\xi)+h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_{n-1}(\xi-y-c\tau_1)\mathrm{d}y\right),\ \xi>0,\\ c\varphi_n'(\xi)=d_2\varphi_n''(\xi)-a_2\varphi_n(\xi)+g\left(\int_{-\infty}^{+\infty} J_2(y)\phi_{n-1}(\xi-y-c\tau_2)\mathrm{d}y\right),\ \xi>0,\\ (\phi_n(\xi),\varphi_n(\xi))=(\delta u^*,\delta v^*),\ \xi\leqslant 0,\\ (\phi_0,\varphi_0)=(\bar{\phi},\bar{\varphi}). \end{cases}$$

Let $\Phi_n(\xi)=(\phi_n(\xi),\varphi_n(\xi))$, we have

$$\phi_n(\xi)=\mathcal{F}_1(\Phi_{n-1})(\xi),\ \varphi_n(\xi)=\mathcal{F}_2(\Phi_{n-1})(\xi). \tag{4.2}$$

Step 2: We claim that, for each $n=1,\,2,\,\cdots$, (i) $(\phi_n,\,\varphi_n)\in\Gamma$; (ii) $(\underline{\phi}(\xi),\,\underline{\varphi}(\xi))\leqslant(\phi_n(\xi),\,\varphi_n(\xi))\leqslant(\phi_{n-1}(\xi),\,\varphi_{n-1}(\xi))\leqslant(\bar{\phi}(\xi),\,\bar{\varphi}(\xi))$ on $\mathbb{R}$.

(i) Since $(\bar{\phi},\bar{\varphi})\in\Gamma$, $(\bar{\phi},\bar{\varphi})$ is increasing in $\xi\in\mathbb{R}$. From lemma 4.1 (ii), $(\phi_1,\varphi_1)=(\mathcal{F}_1(\bar{\phi})(\xi),\mathcal{F}_2(\bar{\varphi})(\xi))$ is also increasing in ξ. By repeating this process, we know that (ϕ_n,φ_n) is increasing in ξ for each $n\geqslant 1$.

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Next, we prove $(\phi_n(+\infty), \varphi_n(+\infty)) = (u^*, v^*)$. Note that $\beta_{11} < 0$ and $\beta_{12} > 0$. By the L'Hôpital's rule,

$$
\begin{aligned}
&\lim_{\xi\to+\infty} \phi_1(\xi) \\
&= \lim_{\xi\to+\infty} \delta u^* e^{\beta_{11}\xi} + \frac{1}{d_1(\beta_{12}-\beta_{11})} \\
&\quad \times \lim_{\xi\to+\infty} \Bigg[\frac{\int_0^{\xi} e^{-\beta_{11}s} h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(s-y-c\tau_1)\mathrm{d}y \right)\mathrm{d}s}{e^{-\beta_{11}\xi}} \\
&\quad + \frac{\int_{\xi}^{+\infty} e^{-\beta_{12}s} h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(s-y-c\tau_1)\mathrm{d}y \right)\mathrm{d}s}{e^{-\beta_{12}\xi}} \\
&\quad - e^{\beta_{11}\xi} \int_0^{+\infty} e^{-\beta_{12}s} h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(s-y-c\tau_1)\mathrm{d}y \right)\mathrm{d}s \Bigg] \\
&= -\frac{1}{d_1(\beta_{12}-\beta_{11})} \left(\frac{1}{\beta_{11}} - \frac{1}{\beta_{12}} \right) \lim_{\xi\to+\infty} h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(\xi-y-c\tau_1)\mathrm{d}y \right) \\
&= -\frac{1}{d_1\beta_{11}\beta_{12}} h\left(v^* \int_{-\infty}^{+\infty} J_1(y)\mathrm{d}y \right) \\
&= -\frac{h(v^*)}{d_1\beta_{11}\beta_{12}} = \frac{h(v^*)}{a_1} = u^*.
\end{aligned}
$$

Similarly, we can show $\lim_{\xi\to+\infty} \varphi_1(\xi) = v^*$.

By repeating the above process, we can obtain $\lim_{\xi\to+\infty}(\phi_n(\xi), \varphi_n(\xi)) = (u^*, v^*)$ for each $n = 2, 3, \cdots$. Thus, $(\phi_n, \varphi_n) \in \Gamma$.

(ii) We first prove $(\phi_1(\xi), \varphi_1(\xi)) \leqslant (\bar{\phi}(\xi), \bar{\varphi}(\xi))$.

Let $\xi_0 = 0$ and $\xi_{m+1} = +\infty$. Assume that $\xi \in (\xi_i, \xi_{i+1})$ for some $i \in \{0, 1, \cdots, m\}$, we have

$$
\begin{aligned}
&\phi_1(\xi) \\
&= \delta u^* e^{\beta_{11}\xi} + \frac{1}{d_1(\beta_{12}-\beta_{11})} \Bigg[\int_0^{\xi} (e^{\beta_{11}(\xi-s)} - e^{\beta_{11}\xi-\beta_{12}s}) \\
&\quad \times h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(s-y-c\tau_1)\mathrm{d}y \right)\mathrm{d}s \\
&\quad + \int_{\xi}^{+\infty} (e^{\beta_{12}(\xi-s)} - e^{\beta_{11}\xi-\beta_{12}s}) h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(s-y-c\tau_1)\mathrm{d}y \right)\mathrm{d}s \Bigg]
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \delta u^* e^{\beta_{11}\xi} + \frac{1}{d_1(\beta_{12}-\beta_{11})}\Bigg[\int_0^{\xi}(e^{\beta_{11}(\xi-s)}-e^{\beta_{11}\xi-\beta_{12}s})\\
&\qquad\times\left(c\bar{\phi}'(s)-d_1\bar{\phi}''(s)+a_1\bar{\phi}(s)\right)\mathrm{d}s\\
&\qquad+\int_{\xi}^{+\infty}(e^{\beta_{12}(\xi-s)}-e^{\beta_{11}\xi-\beta_{12}s})\left(c\bar{\phi}'(s)-d_1\bar{\phi}''(s)+a_1\bar{\phi}(s)\right)\mathrm{d}s\Bigg]\\
&=\delta u^* e^{\beta_{11}\xi}-e^{\beta_{11}\xi}\bar{\phi}(0)+\bar{\phi}(\xi)\\
&\qquad+\frac{1}{\beta_{12}-\beta_{11}}\Bigg[\sum_{k=1}^{i}(e^{\beta_{11}(\xi-\xi_k)}-e^{\beta_{11}\xi-\beta_{12}\xi_k})(\bar{\phi}'_+(\xi_k)-\bar{\phi}'_-(\xi_k))\\
&\qquad+\sum_{k=i+1}^{m}(e^{\beta_{12}(\xi-\xi_k)}-e^{\beta_{11}\xi-\beta_{12}\xi_k})(\bar{\phi}'_+(\xi_k)-\bar{\phi}'_-(\xi_k))\Bigg]\\
&\leqslant \bar{\phi}(\xi).
\end{aligned}
$$

By the continuity, we can get the same result for the endpoints ξ_i $(i=1,\cdots,m)$. In a similar way, we can prove $\varphi_1(\xi)\leqslant\bar{\varphi}(\xi)$.

By lemma 4.1 (iii), we can deduce that $(\phi_n(\xi),\,\varphi_n(\xi))$ is decreasing with respect to n. It follows that

$$
(\phi_n(\xi),\varphi_n(\xi))\leqslant(\phi_{n-1}(\xi),\varphi_{n-1}(\xi))\leqslant\cdots\leqslant(\phi_1(\xi),\varphi_1(\xi))\leqslant(\bar{\phi}(\xi),\bar{\varphi}(\xi))
$$

for each $n\geqslant 2$. Moreover, it is easy to check that $(\phi_n(\xi),\,\varphi_n(\xi))\geqslant(\underline{\phi}(\xi),\,\underline{\varphi}(\xi))$ for each $n\geqslant 1$.

Step 3: We claim that (4.1) has an increasing solution $(\phi,\,\varphi)$.

According to Step 2 (ii), $(\phi(\xi),\,\varphi(\xi))=\lim_{n\to+\infty}(\phi_n(\xi),\,\varphi_n(\xi))$ exists for $\xi\in\mathbb{R}$, and satisfies $(\underline{\phi}(\xi),\,\underline{\varphi}(\xi))\leqslant(\phi(\xi),\,\varphi(\xi))\leqslant(\bar{\phi}(\xi),\,\bar{\varphi}(\xi))$. Moreover, $(\phi(\xi),\,\varphi(\xi))$ is increasing in $\xi\in\mathbb{R}$, $(\phi(\xi),\,\varphi(\xi))=(\delta u^*,\,\delta v^*)$ for $\xi\leqslant 0$, and $(\phi(+\infty),\,\varphi(+\infty))=(u^*,v^*)$.

Direct calculations yield that $(\phi(\xi),\,\varphi(\xi))=\lim_{n\to+\infty}(\phi_n(\xi),\,\varphi_n(\xi))$ satisfies the equations in (4.1), which completes the proof. □

Next, we construct a pair of upper and lower solutions of (4.1).

For any fixed $c>0$, we choose $m>0$ sufficiently large such that

$$
0<\frac{1}{m}<\min\left\{\frac{c}{a_1},\frac{c}{a_2},c\tau_1,c\tau_2\right\}.
$$

Define

$$\bar{\phi}(\xi)=\begin{cases}\delta u^*,\ \xi\leqslant 0,\\ u^*+u^*(\delta-1)(m\xi-1)^2,\ 0<\xi\leqslant\dfrac{1}{m},\\ u^*,\ \xi>\dfrac{1}{m},\end{cases}\tag{4.3}$$

$$\bar{\varphi}(\xi)=\begin{cases}\delta v^*,\ \xi\leqslant 0,\\ v^*+v^*(\delta-1)(m\xi-1)^2,\ 0<\xi\leqslant\dfrac{1}{m},\\ v^*,\ \xi>\dfrac{1}{m},\end{cases}\tag{4.4}$$

and

$$(\underline{\phi}(\xi),\underline{\varphi}(\xi))=(\delta u^*,\delta v^*),\quad \xi\in\mathbb{R}.$$

LEMMA 4.4. *$(\bar{\phi}(\xi),\ \bar{\varphi}(\xi))$ and $(\underline{\phi}(\xi),\ \underline{\varphi}(\xi))$ as defined above are respectively an upper solution and a lower solution of (4.1). Moreover, $(\bar{\phi},\ \bar{\varphi})\in\Gamma$.*

Proof. It is easy to check that $(\bar{\phi},\ \bar{\varphi})\in\Gamma$.

(i) For $0<\xi<\frac{1}{m}$, we have $\delta v^*\leqslant\bar{\varphi}(\xi)\leqslant v^*$, $\xi\in\mathbb{R}$. By simple calculations,

$$\begin{aligned}
&c\bar{\phi}'(\xi)-d_1\bar{\phi}''(\xi)+a_1\bar{\phi}(\xi)-h\left(\int_{-\infty}^{+\infty}J_1(y)\bar{\varphi}(\xi-y-c\tau_1)\mathrm{d}y\right)\\
&=u^*(1-\delta)\left[-2mc(m\xi-1)+2d_1m^2-a_1(m\xi-1)^2\right]+a_1u^*\\
&\quad-h\left(\int_{-\infty}^{+\infty}J_1(y)\bar{\varphi}(\xi-y-c\tau_1)\mathrm{d}y\right)\\
&\geqslant u^*(1-\delta)\left[-2mc(m\xi-1)+2d_1m^2-a_1(m\xi-1)^2\right]\\
&\quad+a_1u^*-h\left(\int_{-\infty}^{+\infty}J_1(y)v^*\mathrm{d}y\right)\\
&=u^*(1-\delta)\left[-2mc(m\xi-1)+2d_1m^2-a_1(m\xi-1)^2\right]+a_1u^*-h(v^*)\\
&=u^*m^2(1-\delta)\left[-2c\left(\xi-\frac{1}{m}\right)+2d_1-a_1(\xi-\frac{1}{m})^2\right].
\end{aligned}$$

Due to $-\frac{c}{a_1}\leqslant-\frac{1}{m}\leqslant s-\frac{1}{m}<0$, we have

$$-2c\left(\xi-\frac{1}{m}\right)+2d_1-a_1\left(\xi-\frac{1}{m}\right)^2>0.$$

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It follows that

$$c\bar{\phi}'(\xi) - d_1\bar{\phi}''(\xi) + a_1\bar{\phi}(\xi) - h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(\xi - y - c\tau_1)\mathrm{d}y\right) \geqslant 0.$$

(*ii*) For $\xi > \frac{1}{m}$, we have $\bar{\phi}(\xi) = u^*$ and $\delta v^* \leqslant \bar{\varphi}(\xi) \leqslant v^*$ for $\xi \in \mathbb{R}_+$. Then

$$\begin{aligned} &c\bar{\phi}'(\xi) - d_1\bar{\phi}''(\xi) + a_1\bar{\phi}(\xi) - h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(\xi - y - c\tau_1)\mathrm{d}y\right) \\ &\geqslant a_1u^* - h\left(\int_{-\infty}^{+\infty} J_1(y)v^*\mathrm{d}y\right) = a_1u^* - h(v^*) = 0. \end{aligned}$$

In summary,

$$\begin{aligned} &c\bar{\phi}'(\xi) - d_1\bar{\phi}''(\xi) + a_1\bar{\phi}(\xi) - h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{\varphi}(\xi - y - c\tau_1)\mathrm{d}y\right) \geqslant 0, \\ &\forall \xi \in \mathbb{R}_+ \backslash \left\{\frac{1}{m}\right\}. \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} &c\bar{\varphi}'(\xi) - d_2\bar{\varphi}''(\xi) + a_2\bar{\varphi}(\xi) - g\left(\int_{-\infty}^{+\infty} J_2(y)\bar{\phi}(\xi - y - c\tau_2)\mathrm{d}y\right) \geqslant 0, \\ &\forall \xi \in \mathbb{R}_+ \backslash \left\{\frac{1}{m}\right\}. \end{aligned}$$

Moreover, $\bar{\phi}'_+(\frac{1}{m}) = \bar{\phi}'_-(\frac{1}{m}) = 0$ and $\bar{\varphi}'_+(\frac{1}{m}) = \bar{\varphi}'_-(\frac{1}{m}) = 0$. Thus, $(\bar{\phi}, \bar{\varphi}) \in \Gamma$ is an upper solution of (4.1).

Next, we prove that $(\underline{\phi}(\xi), \underline{\varphi}(\xi)) = (\delta u^*, \delta v^*)$, $\xi \in \mathbb{R}$ is a lower solution of (4.1).

Obviously, for $\xi > 0$,

$$\begin{aligned} &c\underline{\phi}'(\xi) - d_1\underline{\phi}''(\xi) + a_1\underline{\phi}(\xi) - h\left(\int_{-\infty}^{+\infty} J_1(y)\underline{\varphi}(\xi - y - c\tau_1)\mathrm{d}y\right) \\ &= a_1\delta u^* - h\left(\int_{-\infty}^{+\infty} J_1(y)\underline{\varphi}(\xi - y - c\tau_1)\mathrm{d}y\right) \\ &= a_1\delta u^* - h\left(\int_{-\infty}^{+\infty} J_1(y)\delta v^*\mathrm{d}y\right) \leqslant a_1\delta u^* - \delta h(v^*) = 0. \end{aligned}$$

Similarly, we can obtain

$$c\underline{\varphi}'(\xi) - d_2\underline{\varphi}''(\xi) + a_2\underline{\varphi}(\xi) - g\left(\int_{-\infty}^{+\infty} J_2(y)\underline{\phi}(\xi - y - c\tau_2)\mathrm{d}y\right) \leqslant 0 \quad \text{for } \xi > 0,$$

which means that $(\underline{\phi}, \underline{\varphi})$ is a lower solution of (4.1). □

THEOREM 4.5. *For all $\delta \in (0,\, \frac{1}{2})$, the perturbed semi-wave problem (4.1) admits an increasing solution $(\phi^\delta(\xi),\, \varphi^\delta(\xi))$. Moreover, $(\phi^\delta(\xi),\, \varphi^\delta(\xi))$ obtained in this way is increasing with respect to $\delta \in (0,\, \frac{1}{2})$.*

Proof. From theorem 4.3 and lemma 4.4, we can establish the existence of increasing solution to (4.1).

Assume that $0 < \delta_1 < \delta_2 < \frac{1}{2}$. In view of the definitions of $(\bar{\phi}, \bar{\varphi})$ in (4.3)–(4.4), we have $(\bar{\phi}^{\delta_2}, \bar{\varphi}^{\delta_2}) > (\bar{\phi}^{\delta_1}, \bar{\varphi}^{\delta_1})$. Applying the iteration scheme (4.2) and lemma 4.1 (iii), we get

$$\begin{aligned}(\phi_1^{\delta_2}, \varphi_1^{\delta_2}) &= (\mathcal{F}_1(\bar{\phi}^{\delta_2}, \bar{\varphi}^{\delta_2})(\xi), \mathcal{F}_2(\bar{\phi}^{\delta_2}, \bar{\varphi}^{\delta_2})(\xi)) \\ &\geqslant (\mathcal{F}_1(\bar{\phi}^{\delta_1}, \bar{\varphi}^{\delta_1})(\xi), \mathcal{F}_2(\bar{\phi}^{\delta_1}, \bar{\varphi}^{\delta_1})(\xi)) = (\phi_1^{\delta_1}, \varphi_1^{\delta_1}).\end{aligned}$$

By repeating the above process, we can obtain $(\phi_n^{\delta_2}, \varphi_n^{\delta_2}) \geqslant (\phi_n^{\delta_1}, \varphi_n^{\delta_1})$ for each $n \geqslant 1$. It follows that the two limit solutions satisfy $(\phi^{\delta_2}, \varphi^{\delta_2}) \geqslant (\phi^{\delta_1}, \varphi^{\delta_1})$, which completes the proof. □

We remark that, for the perturbed semi-wave problem (4.1), the iteration monotone method is more efficient than the Schauder's fixed point method applied in [**10**, **11**, **35**], especially in proving the monotonicity of semi-wave solution with respect to the parameter δ.

4.2. Original semi-wave problem

THEOREM 4.6. *For any fixed $c > 0$, either the semi-wave problem (1.8) or the following problem*

$$\begin{cases} c\phi'(\xi) = d_1\phi''(\xi) - a_1\phi(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\xi - y - c\tau_1)\mathrm{d}y \right),\ \xi \in \mathbb{R}, \\ c\varphi'(\xi) = d_2\varphi''(\xi) - a_2\varphi(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(\xi - y - c\tau_2)\mathrm{d}y \right),\ \xi \in \mathbb{R}, \\ (\phi(-\infty), \varphi(-\infty)) = (0, 0), \quad (\phi(+\infty), \varphi(+\infty)) = (u^*, v^*) \end{cases} \tag{4.5}$$

has an increasing solution $(\phi,\, \varphi)$, but they can not occur simultaneously.

Proof. (i) Assume that $\{\delta_n\}_{n=1}^{\infty}$ is a sequence satisfying $\delta_n \in (0,\, \frac{1}{2})$ and $\delta_n \searrow 0$ as $n \to \infty$. By theorem 4.5, the perturbed problem (4.1) with $\delta = \delta_n$ has an increasing solution $(\phi^{\delta_n}, \varphi^{\delta_n})$. Define $\xi_n := \max\{\xi : \phi^{\delta_n}(\xi) = \frac{1}{2}u^*\}$. From theorem 4.5, we can deduce that ξ_n is increasing with respect to n, and then $\xi_0 := \lim_{n\to\infty} \xi_n \in (0,\, +\infty]$ is well-defined.

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Define $(\tilde{\phi}_n(\xi), \tilde{\varphi}_n(\xi)) := (\phi^{\delta_n}(\xi+\xi_n), \varphi^{\delta_n}(\xi+\xi_n)),\ \forall \xi \in \mathbb{R}$. Then $\tilde{\phi}_n(0) = \frac{1}{2}u^*$ and $(\tilde{\phi}_n(\xi), \tilde{\varphi}_n(\xi))$ satisfies

$$\begin{cases} c\tilde{\phi}_n'(\xi) = d_1\tilde{\phi}_n''(\xi) - a_1\tilde{\phi}_n(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\tilde{\varphi}_n(\xi - y - c\tau_1)\mathrm{d}y\right),\ \xi > -\xi_n, \\ c\tilde{\varphi}_n'(\xi) = d_2\tilde{\varphi}_n''(\xi) - a_2\tilde{\varphi}_n(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\tilde{\phi}_n(\xi - y - c\tau_2)\mathrm{d}y\right),\ \xi > -\xi_n, \\ (\tilde{\phi}_n(\xi), \tilde{\varphi}_n(\xi)) = (\delta_n u^*, \delta_n v^*),\ \xi \leqslant -\xi_n, \\ (\tilde{\phi}_n(+\infty), \tilde{\varphi}_n(+\infty)) = (u^*, v^*). \end{cases}$$

Note that $\mathbf{0} \leqslant (\phi^{\delta_n}, \varphi^{\delta_n}) \leqslant (u^*,\ v^*)$, i.e., $(\phi^{\delta_n}, \varphi^{\delta_n})$ are uniformly bounded with respect to n. From the integration presentations of solution $(\phi^{\delta_n}, \varphi^{\delta_n}) = (\mathcal{F}_1(\phi^{\delta_n}, \varphi^{\delta_n}), \mathcal{F}_2(\phi^{\delta_n}, \varphi^{\delta_n}))$, we can easily deduce that $(\phi^{\delta_n}, \varphi^{\delta_n})$ are uniformly bounded in $C^2(\mathbb{R}_+)$ with respect to n. By the Arzela–Ascoli theorem, there is a subsequence of $(\tilde{\phi}_n, \tilde{\varphi}_n)$, which converges to $(\tilde{\phi},\ \tilde{\varphi})$ in $C^2_{loc}(\mathbb{R})$. Obviously, $(\tilde{\phi}(\xi),\ \tilde{\varphi}(\xi))$ is increasing in ξ and satisfies $\tilde{\phi}(0) = \frac{1}{2}u^*$.

Case I: $\xi_0 = +\infty$. In such a case, $(\tilde{\phi}(\xi),\ \tilde{\varphi}(\xi))$ satisfies

$$\begin{cases} c\tilde{\phi}'(\xi) = d_1\tilde{\phi}''(\xi) - a_1\tilde{\phi}(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\tilde{\varphi}(\xi - y - c\tau_1)\mathrm{d}y\right),\ \xi \in \mathbb{R}, \\ c\tilde{\varphi}'(\xi) = d_2\tilde{\varphi}''(\xi) - a_2\tilde{\varphi}(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\tilde{\phi}(\xi - y - c\tau_2)\mathrm{d}y\right),\ \xi \in \mathbb{R}. \end{cases}$$

Since $(\tilde{\phi}(\xi),\ \tilde{\varphi}(\xi))$ is increasing and uniformly continuous on $\mathbb{R}_+$, by lemma 2.3 in [**36**] we can deduce that $\lim_{\xi\to\infty} \tilde{\phi}'(\xi) = \lim_{\xi\to\infty} \tilde{\phi}''(\xi) = 0$ and $\lim_{\xi\to\infty} \tilde{\varphi}'(\xi) = \lim_{\xi\to\infty} \tilde{\varphi}''(\xi) = 0$, which imply that $(\tilde{\phi}(\pm\infty), \tilde{\varphi}(\pm\infty)) = (0,0)$ or (u^*, v^*). In view of $\tilde{\phi}(0) = \frac{1}{2}u^*$, we know that $(\tilde{\phi}(-\infty), \tilde{\varphi}(-\infty)) = (0,0)$ and $(\tilde{\phi}(+\infty), \tilde{\varphi}(+\infty)) = (u^*, v^*)$.

Case II: $\xi_0 \in (0,\ +\infty)$. In such a case, $(\tilde{\phi}(\xi),\ \tilde{\varphi}(\xi))$ satisfies

$$\begin{cases} c\tilde{\phi}'(\xi) = d_1\tilde{\phi}''(\xi) - a_1\tilde{\phi}(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\tilde{\varphi}(\xi - y - c\tau_1)\mathrm{d}y\right),\ \xi > -\xi_0, \\ c\tilde{\varphi}'(\xi) = d_2\tilde{\varphi}''(\xi) - a_2\tilde{\varphi}(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\tilde{\phi}(\xi - y - c\tau_2)\mathrm{d}y\right),\ \xi > -\xi_0, \\ (\tilde{\phi}(\xi), \tilde{\varphi}(\xi)) = (0,0),\ \xi \leqslant -\xi_0. \end{cases}$$

Let $(\phi(\xi),\ \varphi(\xi)) = (\tilde{\phi}(\xi - \xi_0),\ \tilde{\varphi}(\xi - \xi_0))$, we can also prove $(\phi(+\infty), \varphi(+\infty)) = (u^*, v^*)$. Obviously, $(\phi(\xi), \varphi(\xi)) = (0,0)$ for $\xi \leqslant 0$. This completes the proof of the first part.

(ii) We prove that the two cases can not happen simultaneously for any fixed $c > 0$.

Suppose, to the contrary, that there exists some $c_0 > 0$ such that (4.5) and (1.8) have two increasing solutions $\Phi_1(\xi) = (\phi_1(\xi),\ \varphi_1(\xi))$ and $\Phi_2(\xi) = (\phi_2(\xi),\ \varphi_2(\xi))$, respectively. Similar as in the proof of lemma 2.10 in [17], for any $\theta \in \mathbb{R}$ and some fixed $k \in (0,\, 1)$, we define

$$\begin{aligned}
\Phi_1^\theta(\xi) &= (\phi_1^\theta(\xi), \varphi_1^\theta(\xi)) = (\phi_1(\xi+\theta), \varphi_1(\xi+\theta)),\\
\tilde{\Phi}_2(\xi) &= (\tilde{\phi}_2(\xi), \tilde{\varphi}_2(\xi)) = (k\phi_2(\xi), k\varphi_2(\xi)) = k\Phi_2(\xi),\\
\hat{\Phi}^\theta(\xi) &= (\hat{\phi}^\theta(\xi), \hat{\varphi}^\theta(\xi)) = (\phi_1^\theta(\xi) - \tilde{\phi}_2(\xi), \varphi_1^\theta(\xi) - \tilde{\varphi}_2(\xi)) = \Phi_1^\theta(\xi) - \tilde{\Phi}_2(\xi).
\end{aligned}$$

Obviously, $\hat{\Phi}^\theta$ is increasing in θ, and then $(\sigma_1(\theta), \sigma_2(\theta)) := (\inf_{\xi \geqslant 0} \hat{\phi}^\theta(\xi), \inf_{\xi \geqslant 0} \hat{\varphi}^\theta(\xi))$ is increasing, continuous in θ.

Note that $\hat{\Phi}^\theta(\xi) = \Phi_1^\theta(\xi) - \tilde{\Phi}_2(\xi) \geqslant \Phi_1^\theta(0) - \tilde{\Phi}_2(+\infty) = \Phi_1(\theta) - k(u^*, v^*)$ for any $\xi \geqslant 0$. Since $\lim_{\theta \to +\infty} \Phi_1(\theta) = (u^*, v^*)$, we have $\lim_{\theta \to +\infty} \hat{\Phi}^\theta(\xi) \geqslant (1-k)(u^*, v^*)$ uniformly on $[0,\, +\infty)$. Then there exists sufficiently large $\bar{\theta} \gg 1$ (independent of ξ) such that

$$\hat{\Phi}^\theta(\xi) > \frac{1}{2}(1-k)(u^*, v^*) > \mathbf{0} \tag{4.6}$$

on $[0,\, +\infty)$ for all $\theta > \bar{\theta}$. Moreover, as $\theta \to -\infty$,

$$\hat{\Phi}^\theta(1) = \Phi_1^\theta(1) - \tilde{\Phi}_2(1) = \Phi_1(1+\theta) - k\Phi_2(1) \to -k\Phi_2(1) < \mathbf{0}. \tag{4.7}$$

Since $(\sigma_1(\theta), \sigma_2(\theta))$ is increasing, continuous in θ, by (4.6) and (4.7), there exist $\theta_1,\, \theta_2 \in \mathbb{R}$ such that $\sigma_1(\theta) > 0$ for $\theta > \theta_1$, $\sigma_1(\theta_1) = 0$, and $\sigma_2(\theta) > 0$ for $\theta > \theta_2$, $\sigma_2(\theta_2) = 0$.

We may assume that $\theta_1 \geqslant \theta_2$. It follows that $\hat{\Phi}^{\theta_1} \geqslant \mathbf{0}$ for $\xi \geqslant 0$. It is easy to check that $(\hat{\phi}^{\theta_1}(+\infty),\, \hat{\varphi}^{\theta_2}(+\infty)) = (1-k)(u^*,\, v^*) > \mathbf{0}$, and $(\hat{\phi}^{\theta_1}(0),\, \hat{\varphi}^{\theta_2}(0)) = (\phi_1^{\theta_1}(0), \varphi_1^{\theta_2}(0)) > \mathbf{0}$. Then $\sigma_1(\theta_1) = \inf_{\xi \geqslant 0} \hat{\phi}^{\theta_1}(\xi) = 0$ is attainable at some $\xi_1 \in (0,\, +\infty)$, i.e., $\hat{\phi}^{\theta_1}(\xi_1) = 0$.

Since $k \in (0,\, 1)$ and $h,\, g$ are subhomogeneous, we have

$$\begin{aligned}
&h\bigg(\int_{-\infty}^{+\infty} J_1(y)\varphi_1(\xi+\theta_1-y-c_0\tau_1)\mathrm{d}y\bigg) - kh\bigg(\int_{-\infty}^{+\infty} J_1(y)\varphi_2(\xi-y-c_0\tau_1)\mathrm{d}y\bigg)\\
&\quad\geqslant h\bigg(\int_{-\infty}^{+\infty} J_1(y)\varphi_1(\xi+\theta_1-y-c_0\tau_1)\mathrm{d}y\bigg)\\
&\qquad - h\bigg(\int_{-\infty}^{+\infty} J_1(y)k\varphi_2(\xi-y-c_0\tau_1)\mathrm{d}y\bigg)\\
&\quad= h'(\vartheta)\int_{-\infty}^{+\infty} J_1(y)\hat{\varphi}^{\theta_1}(\xi-y-c_0\tau_1)\mathrm{d}y \geqslant 0.
\end{aligned}$$

Similarly, we can get

$$g\bigg(\int_{-\infty}^{+\infty} J_2(y)\phi_1(\xi+\theta_1-y-c_0\tau_2)\mathrm{d}y\bigg) - kg\bigg(\int_{-\infty}^{+\infty} J_2(y)\phi_2(\xi-y-c_0\tau_2)\mathrm{d}y\bigg) \geqslant 0.$$

Thus, $\hat{\Phi}^{\theta_1}$ satisfies

$$\begin{cases} c_0(\hat{\phi}^{\theta_1})'(\xi) - d_1(\hat{\phi}^{\theta_1})''(\xi) + a_1\hat{\phi}^{\theta_1}(\xi) = h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_1(\xi+\theta_1-y-c_0\tau_1)\mathrm{d}y \right) \\ \qquad -kh\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_2(\xi-y-c_0\tau_1)\mathrm{d}y \right) \geqslant 0, \ \xi > 0, \\ c_0(\hat{\varphi}^{\theta_1})'(\xi) - d_2(\hat{\varphi}^{\theta_1})''(\xi) + a_2\hat{\varphi}^{\theta_1}(\xi) = g\left(\int_{-\infty}^{+\infty} J_2(y)\phi_1(\xi+\theta_1-y-c_0\tau_2)\mathrm{d}y \right) \\ \qquad -kg\left(\int_{-\infty}^{+\infty} J_2(y)\phi_2(\xi-y-c_0\tau_2)\mathrm{d}y \right) \geqslant 0, \ \xi > 0, \\ (\hat{\phi}^{\theta_1}(+\infty), \hat{\varphi}^{\theta_2}(+\infty)) = (1-k)(u^*, v^*) > \mathbf{0}, \\ (\hat{\phi}^{\theta_1}(\xi), \hat{\varphi}^{\theta_2}(\xi)) \geqslant \mathbf{0}, \ \xi \geqslant 0. \end{cases}$$

By the maximum principle for single equation, we have $(\hat{\phi}^{\theta_1}(\xi), \hat{\varphi}^{\theta_2}(\xi)) > \mathbf{0}$ on $(0, +\infty)$, which contradicts with $\hat{\phi}^{\theta_1}(\xi_1) = 0$. This completes the proof. □

Remark 4.7. Problem (4.5) with fixed $c > 0$ has an increasing solution (ϕ, φ) is equivalent to the evolution system

$$\begin{cases} \partial_t u = d_1\partial_{xx}u - a_1 u + h\left(\int_{-\infty}^{+\infty} J_1(x-y)v(t-\tau_1, y)\mathrm{d}y \right), \ t > 0, \ x \in \mathbb{R}, \\ \partial_t v = d_2\partial_{xx}v - a_2 v + g\left(\int_{-\infty}^{+\infty} J_2(x-y)u(t-\tau_2, y)\mathrm{d}y \right), \ t > 0, \ x \in \mathbb{R} \end{cases} \tag{4.8}$$

admits an increasing travelling wave solution $(u(t, x), v(t, x)) := (\phi(x+ct), \varphi(x+ct))$.

Define

$$c_\tau^* = \inf_{\lambda>0} \frac{\chi_\tau(\lambda)}{\lambda},$$

where $\chi_\tau(\lambda)$ is a real root of

$$\begin{aligned} P(\chi) &:= \chi^2 - [(d_1\lambda^2 - a_1) + (d_2\lambda^2 - a_2)]\chi + (d_1\lambda^2 - a_1)(d_2\lambda^2 - a_2) \\ &\qquad - h'(0)g'(0)e^{-\chi(\tau_1+\tau_2)} \int_{-\infty}^{+\infty} J_1(y)e^{-\lambda y}\mathrm{d}y \int_{-\infty}^{+\infty} J_2(y)e^{-\lambda y}\mathrm{d}y \\ &= 0, \end{aligned} \tag{4.9}$$

and $\chi_\tau(\lambda)$ is greater than the real parts of all other roots.

Note that $\lim_{\chi\to+\infty} P(\chi) = +\infty$ and $P(d_1\lambda^2 - a_1), P(d_2\lambda^2 - a_2) < 0$. Then

$$\chi_\tau(\lambda) > \max\{d_1\lambda^2 - a_1, d_2\lambda^2 - a_2\},$$

which implies that $\lim_{\lambda\to+\infty} \frac{\chi_\tau(\lambda)}{\lambda} = +\infty$. Similarly, since $\mathcal{R}_0 \geqslant \mathcal{R}^* > 1$, we can deduce $\chi_\tau(0) > 0$, and then $\lim_{\lambda\to 0+} \frac{\chi_\tau(\lambda)}{\lambda} = +\infty$. Thus, $\inf_{\lambda>0} \frac{\chi_\tau(\lambda)}{\lambda}$ is attainable

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at some $\lambda^* \in (0, +\infty)$, i.e.,

$$c^*_\tau = \inf_{\lambda>0} \frac{\chi_\tau(\lambda)}{\lambda} = \frac{\chi_\tau(\lambda^*)}{\lambda^*}.$$

Let $c = \frac{\chi(\lambda)}{\lambda}$, we have $\frac{dc}{d\lambda}|_{\lambda=\lambda^*}=0$. Define

$$\begin{aligned}\Delta(\lambda, c) &:= (d_1\lambda^2 - c\lambda - a_1)(d_2\lambda^2 - c\lambda - a_2)\\ &\quad - h'(0)g'(0)e^{-c\lambda(\tau_1+\tau_2)} \int_{-\infty}^{+\infty} J_1(y)e^{-\lambda y}\mathrm{d}y \int_{-\infty}^{+\infty} J_2(y)e^{-\lambda y}\mathrm{d}y.\end{aligned}$$

Then $(c^*_\tau,\ \lambda^*)$ can be determined as the positive solution to the system

$$\Delta(\lambda, c) = 0, \quad \frac{\partial\Delta(\lambda, c)}{\partial\lambda} = 0.$$

THEOREM 4.8. *The semi-wave problem* (1.8) *admits an increasing solution for* $0 < c < c^*_\tau$, *but has no increasing solution for* $c \geqslant c^*_\tau$.

Proof. By the theory of monotone semiflows developed in [**21**], there exists $c_0 > 0$ such that c_0 is the asymptotic spreading speed. Moreover, the asymptotic spreading speed c_0 coincides with the minimal wave speed, that is, (4.8) has an increasing travelling wave solution for $c \geqslant c_0$, but no such a solution for $0 < c < c_0$. If $c_0 = c^*_\tau$, then we can get the desired result by applying remark 4.7 and theorem 4.6.

Now it is sufficient to prove $c_0 = c^*_\tau$. Set $\mathcal{C} = C([-\tau_2,\ 0] \times \mathbb{R}, \mathbb{R}) \times C([-\tau_1,\ 0] \times \mathbb{R}, \mathbb{R})$, $\bar{\mathcal{C}} = C([-\tau_2,\ 0], \mathbb{R}) \times C([-\tau_1,\ 0], \mathbb{R})$. Let $M_t = (M^u_t, M^v_t) : \mathcal{C} \to \mathcal{C}$ be the solution map at time t of the following linear equations

$$\begin{cases}\partial_t u = d_1\partial_{xx}u - a_1 u + h'(0)J_1 * v_t,\\ \partial_t v = d_2\partial_{xx}v - a_2 v + g'(0)J_2 * u_t.\end{cases}$$

For $\lambda \geqslant 0$, we define the linear map $B_t = (B^u_t,\ B^v_t) : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ by

$$\begin{aligned}B^u_t[(\varphi_1, \varphi_2)](\theta) &= M^u_t[(\varphi_1, \varphi_2)e^{-\lambda x}](\theta, 0),\ \forall\theta \in [-\tau_2, 0],\\ B^v_t[(\varphi_1, \varphi_2)](\theta) &= M^v_t[(\varphi_1, \varphi_2)e^{-\lambda x}](\theta, 0),\ \forall\theta \in [-\tau_1, 0].\end{aligned}$$

Then $B_t = (B^u_t,\ B^v_t) : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ is the solution map of the following equations

$$\begin{cases}u'(t) = d_1\lambda^2 u(t) - a_1 u(t) + h'(0)\left(\int_{-\infty}^{+\infty} J_1(y)e^{-\lambda y}\mathrm{d}y\right)v_t,\\[2ex] v'(t) = d_2\lambda^2 v(t) - a_2 v(t) + g'(0)\left(\int_{-\infty}^{+\infty} J_2(y)e^{-\lambda y}\mathrm{d}y\right)u_t.\end{cases} \tag{4.10}$$

Let

$$A(\chi) = \begin{pmatrix} d_1\lambda^2 - a_1 & h'(0)e^{-\chi\tau_1}\int_{-\infty}^{+\infty} J_1(y)e^{-\lambda y}\mathrm{d}y\\ g'(0)e^{-\chi\tau_2}\int_{-\infty}^{+\infty} J_2(y)e^{-\lambda y}\mathrm{d}y & d_2\lambda^2 - a_2\end{pmatrix}.$$

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Since (4.10) is a cooperative and irreducible delay equations, it follows that

$$\det(\chi I - A(\chi)) = 0,$$

i.e.,

$$\begin{aligned} P(\chi) = \chi^2 - [(d_1\lambda^2 - a_1) + (d_2\lambda^2 - a_2)]\chi + (d_1\lambda^2 - a_1)(d_2\lambda^2 - a_2) \\ -h'(0)g'(0)e^{-\chi(\tau_1+\tau_2)} \int_{-\infty}^{+\infty} J_1(y)e^{-\lambda y}\mathrm{d}y \int_{-\infty}^{+\infty} J_2(y)e^{-\lambda y}\mathrm{d}y = 0, \end{aligned}$$

admits a real root $\chi_\tau(\lambda)$ which is greater than the real parts of all other ones (see Theorem 5.5.1 in [**23**]).

By Theorem 3.10 in [**21**], we know that the spreading speed $c_0 = \inf_{\lambda>0} \frac{\chi_\tau(\lambda)}{\lambda}$. Thus, $c_0 = c_\tau^*$, which completes the proof. □

THEOREM 4.9. *For any $c \in (0,\, c_\tau^*)$, the solution of* (1.8) *obtained in theorem* 4.8 *is unique and strictly increasing on* $\mathbb{R}_+$.

Proof. (*i*) (Strict monotonicity) For any $\theta > 0$, we have

$$\begin{aligned} &\phi(\xi+\theta) \\ &= \frac{1}{d_1(\beta_{12}-\beta_{11})}\Bigg[\int_0^{\xi+\theta}(e^{\beta_{11}(\xi+\theta-s)} - e^{\beta_{11}(\xi+\theta)-\beta_{12}s}) \\ &\quad \times h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(s-y-c\tau_1)\mathrm{d}y\right)\mathrm{d}s \\ &\quad + \int_{\xi+\theta}^{+\infty}(e^{\beta_{12}(\xi+\theta-s)} - e^{\beta_{11}(\xi+\theta)-\beta_{12}s})h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(s-y-c\tau_1)\mathrm{d}y\right)\mathrm{d}s\Bigg] \\ &= \frac{1}{d_1(\beta_{12}-\beta_{11})}\Bigg[\int_{-\theta}^{\xi}(e^{\beta_{11}(\xi-\tilde{s})} - e^{\beta_{11}(\xi+\theta)-\beta_{12}(\tilde{s}+\theta)})\times \\ &\quad h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\tilde{s}+\theta-y-c\tau_1)\mathrm{d}y\right)\mathrm{d}\tilde{s} + \int_{\xi}^{+\infty}(e^{\beta_{12}(\xi-\tilde{s})} - e^{\beta_{11}(\xi+\theta)-\beta_{12}(\tilde{s}+\theta)}) \\ &\quad \times h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\tilde{s}+\theta-y-c\tau_1)\mathrm{d}y\right)\mathrm{d}\tilde{s}\Bigg] \\ &> \frac{1}{d_1(\beta_{12}-\beta_{11})}\Bigg[\int_0^{\xi}(e^{\beta_{11}(\xi-\tilde{s})} - e^{\beta_{11}\xi-\beta_{12}\tilde{s}})h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\tilde{s}-y-c\tau_1)\mathrm{d}y\right)\mathrm{d}\tilde{s} \\ &\quad + \int_{\xi}^{+\infty}(e^{\beta_{12}(\xi-\tilde{s})} - e^{\beta_{11}\xi-\beta_{12}\tilde{s}})h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\tilde{s}-y-c\tau_1)\mathrm{d}y\right)\mathrm{d}\tilde{s}\Bigg] = \phi(\xi). \end{aligned}$$

In a similar way, we can obtain $\varphi(\xi+\theta) > \varphi(\xi)$ for $\xi \in \mathbb{R}_+$.

(*ii*) (uniqueness) Fix $c \in (0,\, c_\tau^*)$, suppose that $(\phi_1,\, \varphi_1)$ and $(\phi_2,\, \varphi_2)$ are two strictly increasing solutions of (1.8). Then, for $i = 1,\, 2$, $(0,\, 0) < (\phi_i(\xi),\, \varphi_i(\xi)) <$

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(u^*, v^*) for $\xi > 0$ and $(\phi_i(+\infty), \varphi_i(+\infty)) = (u^*, v^*)$. Moreover, by the Hopf's boundary lemma, we have $(\phi_i)'_+(0) > 0$, $(\varphi_i)'_+(0) > 0$ for $i = 1, 2$.

We define

$$\rho_1 := \inf\{\rho \geqslant 1 : \rho\phi_1(\xi) > \phi_2(\xi), \forall \xi > 0\},$$
$$\rho_2 := \inf\{\rho \geqslant 1 : \rho\varphi_1(\xi) > \varphi_2(\xi), \forall \xi > 0\},$$

and $\rho^* := \max\{\rho_1, \rho_2\}$.

We will show that $\rho^* = 1$. Otherwise, $\rho^* > 1$. Denote $\tilde{\phi} = \rho^*\phi_1 - \phi_2$ and $\tilde{\varphi} = \rho^*\varphi_1 - \varphi_2$. Obviously, $\tilde{\phi}(\xi) \geqslant 0$, $\tilde{\varphi}(\xi) \geqslant 0$ for $\xi \geqslant 0$, $\tilde{\phi}(0) = \tilde{\varphi}(0) = 0$, $\tilde{\phi}(+\infty) = (\rho^* - 1)u^*$ and $\tilde{\varphi}(+\infty) = (\rho^* - 1)v^*$. Since h is subhomogeneous, we obtain, for $\xi > 0$,

$$\begin{aligned}
&c\tilde{\phi}'(\xi) - d_1\tilde{\phi}''(\xi) + a_1\tilde{\phi}(\xi) \\
&\quad = \rho^* h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_1(\xi - y - c\tau_1)\mathrm{d}y\right) - h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_2(\xi - y - c\tau_1)\mathrm{d}y\right) \\
&\quad = \rho^* h\left(\frac{1}{\rho^*}\int_{-\infty}^{+\infty} J_1(y)\rho^*\varphi_1(\xi - y - c\tau_1)\mathrm{d}y\right) \\
&\qquad - h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_2(\xi - y - c\tau_1)\mathrm{d}y\right) \\
&\quad \geqslant h\left(\int_{-\infty}^{+\infty} J_1(y)\rho^*\varphi_1(\xi - y - c\tau_1)\mathrm{d}y\right) - h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_2(\xi - y - c\tau_1)\mathrm{d}y\right) \\
&\quad \geqslant 0.
\end{aligned}$$

Similarly, we can deduce that $c\tilde{\varphi}'(\xi) - d_2\tilde{\varphi}''(\xi) + a_2\tilde{\varphi}(\xi) \geqslant 0$ for $\xi > 0$. By the Hopf's boundary lemma, we have $\tilde{\phi}'_+(0) > 0$ and $\tilde{\varphi}'_+(0) > 0$.

In view of the L'Hôpital's rule, $\lim_{\xi\to 0+} \frac{\tilde{\phi}(\xi)}{\phi_2(\xi)} = \frac{\tilde{\phi}'_+(0)}{(\phi_2)'_+(0)} > 0$ and $\lim_{\xi\to 0+} \frac{\tilde{\varphi}(\xi)}{\varphi_2(\xi)} = \frac{\tilde{\varphi}'_+(0)}{(\varphi_2)'_+(0)} > 0$. Note that $\lim_{\xi\to +\infty} \frac{\tilde{\phi}(\xi)}{\phi_2(\xi)} = \frac{(\rho^*-1)u^*}{u^*} > 0$, $\lim_{\xi\to +\infty} \frac{\tilde{\varphi}(\xi)}{\varphi_2(\xi)} = \frac{(\rho^*-1)v^*}{v^*} > 0$. Thus, there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that $\frac{\tilde{\phi}}{\phi_2} > \varepsilon_1$ and $\frac{\tilde{\varphi}}{\varphi_2} > \varepsilon_2$ for $\xi > 0$. It follows that

$$\frac{\rho^*}{1+\varepsilon_1}\phi_1(\xi) \geqslant \phi_2(\xi), \quad \frac{\rho^*}{1+\varepsilon_2}\varphi_1(\xi) \geqslant \varphi_2(\xi) \quad \text{for } \xi > 0,$$

which contradicts the definition of ρ^*. Thus, $\rho^* = 1$, which implies that $(\phi_1(\xi), \varphi_1(\xi)) \geqslant (\phi_2(\xi), \varphi_2(\xi))$ for $\xi \geqslant 0$. Clearly, the same method can be used to prove $(\phi_1(\xi), \varphi_1(\xi)) \leqslant (\phi_2(\xi), \varphi_2(\xi))$ for $\xi \geqslant 0$. Hence, we get the uniqueness of solution. □

In what follows, we exhibit some properties of the strictly increasing solution of (1.8).

LEMMA 4.10. *For any fixed* $c \in (0, c_\tau^*)$, *let* $(\phi_\tau^c, \varphi_\tau^c)$ *be the unique strictly increasing solution of* (1.8).

(i) *For* $0 < c_1 < c_2 < c_\tau^*$, *then* $((\phi_\tau^{c_1})'_+(0), (\varphi_\tau^{c_1})'_+(0)) > ((\phi_\tau^{c_2})'_+(0), (\varphi_\tau^{c_2})'_+(0))$, *and* $(\phi_\tau^{c_1}(\xi), \varphi_\tau^{c_1}(\xi)) > (\phi_\tau^{c_2}(\xi), \varphi_\tau^{c_2}(\xi))$ *for* $\xi > 0$.

(ii) *For any fixed* $\mu, \rho > 0$, *there exists a unique* $c_\tau = c_\tau^{\mu,\rho} \in (0, c_\tau^*)$ *such that*

$$\mu[(\phi_\tau^{c_\tau})'_+(0) + \rho(\varphi_\tau^{c_\tau})'_+(0)] = c_\tau.$$

(iii) *If* $(\tau_1, \tau_2) \leqslant (\tilde{\tau}_1, \tilde{\tau}_2)$, *then* $c_{\tilde{\tau}}^* \leqslant c_\tau^*$, $c_{\tilde{\tau}} \leqslant c_\tau$ *with* $\tilde{\tau} = \tilde{\tau}_1 + \tilde{\tau}_2$ *and* $\tau = \tau_1 + \tau_2$.

Proof. Similarly as the proof of Theorem 4.6 in [**30**], we can prove (i) and (ii), here we omit the details.

Next we prove (iii). Recall that in (4.9), $\chi_\tau(\lambda)$ can be seen as an intersection of two curves:

$$f_1(\chi) = \chi^2 - [(d_1\lambda^2 - a_1) + (d_2\lambda^2 - a_2)]\chi + (d_1\lambda^2 - a_1)(d_2\lambda^2 - a_2),$$
$$f_2(\chi) = h'(0)g'(0)e^{-\chi\tau} \int_{-\infty}^{+\infty} J_1(y)e^{-\lambda y}\mathrm{d}y \int_{-\infty}^{+\infty} J_2(y)e^{-\lambda y}\mathrm{d}y.$$

The function f_1 is independent of τ, and f_2 is decreasing in τ. If $\tau \leqslant \tilde{\tau}$, then the two intersections satisfy $\chi_\tau(\lambda) \geqslant \chi_{\tilde{\tau}}(\lambda)$, which implies

$$c_\tau^* = \inf_{\lambda>0} \frac{\chi_\tau(\lambda)}{\lambda} \geqslant \inf_{\lambda>0} \frac{\chi_{\tilde{\tau}}(\lambda)}{\lambda} = c_{\tilde{\tau}}^*.$$

Now we prove $c_\tau \geqslant c_{\tilde{\tau}}$. Note that $c_{\tilde{\tau}} \in (0, c_{\tilde{\tau}}^*)$ and $c_\tau \in (0, c_\tau^*)$. If $c_\tau \geqslant c_{\tilde{\tau}}^*$, then we have $c_\tau \geqslant c_{\tilde{\tau}}^* > c_{\tilde{\tau}}$, which completes the proof.

Next, we assume $c_\tau < c_{\tilde{\tau}}^*$. In such a case, $c_\tau, c_{\tilde{\tau}} \in (0, c_{\tilde{\tau}}^*)$. In view of (ii), to get the desired result, we only need to prove that $((\phi_\tau^c)'_+(0), (\varphi_\tau^c)'_+(0)) \geqslant ((\phi_{\tilde{\tau}}^c)'_+(0), (\varphi_{\tilde{\tau}}^c)'_+(0))$ for any $c \in (0, c_{\tilde{\tau}}^*)$.

Since $(\phi_{\tilde{\tau}}^c(\xi), \varphi_{\tilde{\tau}}^c(\xi))$ is increasing on $\mathbb{R}$, we have

$$\begin{cases} c(\phi_{\tilde{\tau}}^c)'(\xi) - d_1(\phi_{\tilde{\tau}}^c)''(\xi) + a_1\phi_{\tilde{\tau}}^c(\xi) = h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_{\tilde{\tau}}^c(\xi - y - c\tilde{\tau}_1)\mathrm{d}y \right) \\ \leqslant h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_{\tilde{\tau}}^c(\xi - y - c\tau_1)\mathrm{d}y \right), \quad \xi > 0, \\ c(\varphi_{\tilde{\tau}}^c)'(\xi) - d_2(\varphi_{\tilde{\tau}}^c)''(\xi) + a_2\varphi_{\tilde{\tau}}^c(\xi) = g\left(\int_{-\infty}^{+\infty} J_2(y)\phi_{\tilde{\tau}}^c(\xi - y - c\tilde{\tau}_2)\mathrm{d}y \right) \\ \leqslant g\left(\int_{-\infty}^{+\infty} J_2(y)\phi_{\tilde{\tau}}^c(\xi - y - c\tau_2)\mathrm{d}y \right), \quad \xi > 0, \\ \phi_{\tilde{\tau}}^c(\xi) = \varphi_{\tilde{\tau}}^c(\xi) = 0, \ \xi \leqslant 0, \end{cases}$$

which implies that $(\phi^c_{\tilde\tau}(\xi), \varphi^c_{\tilde\tau}(\xi))$ is a lower solution of the following problem

$$\begin{cases} \phi_t = d_1\phi_{\xi\xi} - c\phi_\xi - a_1\phi + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\xi - y - c\tau_1)\mathrm{d}y\right), \ t > 0,\ \xi > 0, \\ \varphi_t = d_1\varphi_{\xi\xi} - c\varphi_\xi - a_1\varphi + h\left(\int_{-\infty}^{+\infty} J_2(y)\phi(\xi - y - c\tau_2)\mathrm{d}y\right), \ t > 0,\ \xi > 0, \\ \phi(t,\xi) = \varphi(t,\xi) = 0,\ t > 0,\ \xi \leqslant 0, \\ (\phi(0,\xi), \varphi(0,\xi)) = (\phi^c_{\tilde\tau}(\xi), \varphi^c_{\tilde\tau}(\xi)). \end{cases} \tag{4.11}$$

By the maximum principle, the solution $(\phi(t,\, \xi), \varphi(t,\, \xi))$ of (4.11) is increasing in $t \geqslant 0$ and satisfies $\lim_{t\to+\infty}(\phi(t,\, \xi), \varphi(t,\, \xi)) = (\phi^*(\xi), \varphi^*(\xi))$, where $(\phi^*(\xi), \varphi^*(\xi))$ is a solution of (1.8). Clearly, the uniqueness of the solution to (1.8) ensures that $(\phi^*(\xi), \varphi^*(\xi)) = (\phi^c_\tau(\xi), \varphi^c_\tau(\xi))$. Thus, for all $\xi > 0$, we have

$$\begin{aligned} (\phi^c_{\tilde\tau}(\xi), \varphi^c_{\tilde\tau}(\xi)) &= (\phi(0,\xi), \varphi(0,\xi)) \leqslant (\phi(t,\xi), \varphi(t,\xi)) \\ &\leqslant (\phi(+\infty,\xi), \varphi(+\infty,\xi)) = (\phi^c_\tau(\xi), \varphi^c_\tau(\xi)). \end{aligned}$$

Let $\hat\phi(\xi) = \phi^c_\tau(\xi) - \phi^c_{\tilde\tau}(\xi)$ and $\hat\varphi(\xi) = \varphi^c_\tau(\xi) - \varphi^c_{\tilde\tau}(\xi)$, then $(\hat\phi,\, \hat\varphi)$ satisfies

$$\begin{cases} c\hat\phi'(\xi) - d_1\hat\phi''(\xi) + a_1\hat\phi(\xi) = h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi^c_\tau(\xi - y - c\tau_1)\mathrm{d}y\right) \\ \qquad\qquad -h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi^c_{\tilde\tau}(\xi - y - c\tilde\tau_1)\mathrm{d}y\right) \geqslant 0,\ \xi > 0, \\ c\hat\varphi'(\xi) - d_2\hat\varphi''(\xi) + a_2\hat\varphi(\xi) = g\left(\int_{-\infty}^{+\infty} J_2(y)\phi^c_\tau(\xi - y - c\tau_2)\mathrm{d}y\right) \\ \qquad\qquad -g\left(\int_{-\infty}^{+\infty} J_2(y)\phi^c_{\tilde\tau}(\xi - y - c\tilde\tau_2)\mathrm{d}y\right) \geqslant 0,\ \xi > 0, \\ \hat\phi(0) = 0,\ \hat\varphi(0) = 0. \end{cases}$$

The Hopf boundary lemma yields $\hat\phi'(0) > 0$ and $\hat\varphi'(0) > 0$, that is, $(\phi^c_\tau)'_+(0) > (\phi^c_{\tilde\tau})'_+(0)$ and $(\varphi^c_\tau)'_+(0) > (\varphi^c_{\tilde\tau})'_+(0)$. This completes the proof. □

5. Asymptotic spreading speed

In this section, by employing the semi-wave solutions, we determine the asymptotic spreading speeds of free boundaries when spreading occurs.

Proof of theorem 1.4. We divide the proof into the following two steps.

Step 1. We prove $\liminf_{t\to+\infty} \frac{s_1(t)}{t} \geqslant -c_\tau$ and $\limsup_{t\to+\infty} \frac{s_2(t)}{t} \leqslant c_\tau$.

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Consider the following auxiliary semi-wave problem

$$\begin{cases} c\phi'(\xi) = d_1\phi''(\xi) - (a_1 - 2\varepsilon)\phi(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\xi - y - c\tau_1)\mathrm{d}y\right),\ \xi > 0, \\ c\varphi'(\xi) = d_2\varphi''(\xi) - (a_2 - 2\varepsilon)\varphi(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(\xi - y - c\tau_2)\mathrm{d}y\right),\ \xi > 0, \\ (\phi(\xi), \varphi(\xi)) = (0, 0),\ \xi \leqslant 0, \\ (\phi(+\infty), \varphi(+\infty)) = (u^*_{2\varepsilon}, v^*_{2\varepsilon}), \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ is a small constant, and $(u^*_{2\varepsilon},\ v^*_{2\varepsilon})$ is the unique positive equilibrium for the first two equations of (5.1). By theorem 4.10 (ii), there exists a unique $c_{\tau,2\varepsilon} > 0$ such that

$$\mu[(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}})'_+(0) + \rho(\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}})'_+(0)] = c_{\tau,2\varepsilon},\quad \lim_{\varepsilon\to 0^+} c_{\tau,2\varepsilon} = c_\tau,$$

where $(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}},\ \varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}})$ is the unique strictly increasing solution of (5.1) with $c = c_{\tau,2\varepsilon}$.

Let $(\bar{u}(t),\ \bar{v}(t))$ be the solution of (3.1). Since $\mathcal{R}_0 \geqslant \mathcal{R}^* > 1$, from Theorem 3.2 in [**42**], we can show that $\lim_{t\to\infty}(\bar{u}(t),\ \bar{v}(t)) = (u^*,\ v^*)$. The comparison principle implies $(u(t,\ x),\ v(t,\ x)) \leqslant (\bar{u}(t),\ \bar{v}(t))$ for $t > 0,\ \ x \in (s_1(t),\ s_2(t))$. Note that $(u^*,\ v^*) < (u^*_\varepsilon,\ v^*_\varepsilon)$. Thus, there exists sufficiently large $T_0 > 0$ such that

$$\begin{aligned} u(t,x) &\leqslant u^*_\varepsilon,\ \forall t \geqslant T_0,\ x \in [s_1(t), s_2(t)], \\ v(t,x) &\leqslant v^*_\varepsilon,\ \forall t \geqslant T_0,\ x \in [s_1(t), s_2(t)]. \end{aligned}$$

Since $(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(+\infty), \varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(+\infty)) = (u^*_{2\varepsilon},\ v^*_{2\varepsilon}) > (u^*_\varepsilon,\ v^*_\varepsilon)$, there exists $\xi_0 > s_2(T_0 + \max\{\tau_1,\ \tau_2\})$ such that

$$\left(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\xi_0 - s_2(T_0 + \max\{\tau_1, \tau_2\})),\quad \varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\xi_0 - s_2(T_0 + \max\{\tau_1, \tau_2\}))\right) > (u^*_\varepsilon, v^*_\varepsilon).$$

Define

$$\begin{aligned} \bar{s}_2(t) &= c_{\tau,2\varepsilon}(t - T_0) + \xi_0,\quad t \geqslant T_0, \\ \bar{u}(t,x) &= \begin{cases} \phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t) + x),\ t \geqslant T_0,\ x \in [-\bar{s}_2(t), 0], \\ \phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t) - x),\ t \geqslant T_0,\ x \in [0, \bar{s}_2(t)], \end{cases} \end{aligned}$$

and

$$\bar{v}(t,x) = \begin{cases} \varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t) + x),\ t \geqslant T_0,\ x \in [-\bar{s}_2(t), 0], \\ \varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t) - x),\ t \geqslant T_0,\ x \in [0, \bar{s}_2(t)]. \end{cases}$$

For $t \geqslant T_0 + \max\{\tau_1, \tau_2\}$ and $x \in [0, \bar{s}_2(t))$, by the symmetry of J_1 and the monotonicity of $\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}$ and h, we have

$$
\begin{aligned}
h\bigg(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\bigg) &= h\bigg(\int_{-\infty}^{+\infty} J_1(y)\bar{v}(t-\tau_1, x+y)\mathrm{d}y\bigg)\\
&= h\bigg(\int_{-\infty}^{-x} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t-\tau_1)+x+y)\mathrm{d}y\\
&\quad + \int_{-x}^{+\infty} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t-\tau_1)-x-y)\mathrm{d}y\bigg)\\
&\leqslant h\bigg(\int_{-\infty}^{-x} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t-\tau_1)-x-y)\mathrm{d}y\\
&\quad + \int_{-x}^{+\infty} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t-\tau_1)-x-y)\mathrm{d}y\bigg)\\
&= h\bigg(\int_{-\infty}^{+\infty} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t-\tau_1)-x-y)\mathrm{d}y\bigg),
\end{aligned}
$$

and then

$$
\begin{aligned}
&\bar{u}_t - d_1\bar{u}_{xx} + a_1\bar{u} - h\bigg(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\bigg)\\
&= c_{\tau,2\varepsilon}(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}})'(\bar{s}_2(t)-x) - d_1(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}})''(\bar{s}_2(t)-x) + a_1\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)-x)\\
&\quad - h\bigg(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\bigg)\\
&= 2\varepsilon\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)-x) + h\bigg(\int_{-\infty}^{+\infty} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)-x-y-c_{\tau,2\varepsilon}\tau_1)\mathrm{d}y\bigg)\\
&\qquad - h\bigg(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\bigg)\\
&= 2\varepsilon\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)-x) + h\bigg(\int_{-\infty}^{+\infty} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t-\tau_1)-x-y)\mathrm{d}y\bigg)\\
&\quad - h\bigg(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\bigg)\\
&\geqslant 0.
\end{aligned}
$$

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For $t \geqslant T_0 + \max\{\tau_1, \tau_2\}$ and $x \in (-\bar{s}_2(t), 0)$, we also have

$$\begin{aligned}
&\bar{u}_t - d_1\bar{u}_{xx} + a_1\bar{u} - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\right) \\
&\quad = c_{\tau,2\varepsilon}(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}})'(\bar{s}_2(t)+x) - d_1(\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}})''(\bar{s}_2(t)+x) + a_1\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)+x) \\
&\qquad - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\bar{v}(t-\tau_1, y)\mathrm{d}y\right) \\
&\quad = 2\varepsilon\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)+x) + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)+x-y-c_{\tau,2\varepsilon}\tau_1)\mathrm{d}y\right) \\
&\qquad - h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{v}(t-\tau_1, x-y)\mathrm{d}y\right) \\
&\quad = 2\varepsilon\phi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t)+x) + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_{2\varepsilon}^{c_{\tau,2\varepsilon}}(\bar{s}_2(t-\tau_1)+x-y)\mathrm{d}y\right) \\
&\qquad - h\left(\int_{-\infty}^{+\infty} J_1(y)\bar{v}(t-\tau_1, x-y)\mathrm{d}y\right) \\
&\quad \geqslant 0.
\end{aligned}$$

The inequality satisfied by $\bar{v}$ can be proved similarly. In terms of the choices of T_0 and ξ_0, we can check that $(\bar{u}(t, x), \bar{v}(t, x); -\bar{s}_2(t), \bar{s}_2(t))$ is an upper solution of (1.1) with $t > 0$ in lemma 2.2 replaced by $t \geqslant T_0 + \max\{\tau_1, \tau_2\}$. Applying the comparison principle, we have $s_1(t) \geqslant -\bar{s}_2(t)$ and $s_2(t) \leqslant \bar{s}_2(t)$ for $t \geqslant T_0 + \max\{\tau_1, \tau_2\}$, and then

$$\liminf_{t\to+\infty} \frac{s_1(t)}{t} \geqslant \liminf_{t\to+\infty} \frac{-\bar{s}_2(t)}{t} \geqslant -c_{\tau,2\varepsilon},$$
$$\limsup_{t\to+\infty} \frac{s_2(t)}{t} \leqslant \limsup_{t\to+\infty} \frac{\bar{s}_2(t)}{t} \leqslant c_{\tau,2\varepsilon}.$$

Taking $\varepsilon \to 0^+$, we can get the desired result.

Step 2. We show $\limsup_{t\to+\infty} \frac{s_1(t)}{t} \leqslant -c_\tau$ and $\liminf_{t\to+\infty} \frac{s_2(t)}{t} \geqslant c_\tau$.

We consider another auxiliary semi-wave problem

$$\begin{cases}
c\phi'(\xi) = d_1\phi''(\xi) - a_1\phi(\xi) + h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi(\xi-y-c\tau_1)\mathrm{d}y - \zeta\right), \ \xi > 0, \\
c\varphi'(\xi) = d_2\varphi''(\xi) - a_2\varphi(\xi) + g\left(\int_{-\infty}^{+\infty} J_2(y)\phi(\xi-y-c\tau_2)\mathrm{d}y - \zeta\right), \ \xi > 0, \\
(\phi(\xi), \varphi(\xi)) = (0, 0), \ \xi \leqslant 0, \\
(\phi(+\infty), \varphi(+\infty)) = (u_\zeta^*, v_\zeta^*),
\end{cases} \tag{5.2}$$

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where $\zeta > 0$ is a small constant, and $(u^*_\zeta,\, v^*_\zeta)$ is the unique positive equilibrium for the first two equations of (5.2). By theorem 4.10 (ii), there exists a unique $c_{\tau,\zeta} > 0$ such that

$$\mu[(\phi_\zeta^{c_{\tau,\zeta}})'_+(0) + \rho(\varphi_\zeta^{c_{\tau,\zeta}})'_+(0)] = c_{\tau,\zeta}, \quad \lim_{\zeta\to 0^+} c_{\tau,\zeta} = c_\tau,$$

where $(\phi_\zeta^{c_{\tau,\zeta}},\, \varphi_\zeta^{c_{\tau,\zeta}})$ is the unique strictly increasing solution of (5.2) with $c = c_{\tau,\zeta}$.

From lemma 3.1, we know that $\lim_{t\to+\infty}(u(t,\,x), v(t,\,x)) = (u^*, v^*)$ locally uniformly for $x \in \mathbb{R}$. Note that $(u^*, v^*) > (u^*_\zeta, v^*_\zeta)$. Then for any $L_0 > 0$, there exists sufficiently large $\mathcal{T}_0 > 0$ such that $s_2(\mathcal{T}_0) > L_0$ and $(u(t,\,x),\, v(t,\,x)) \geqslant (u^*_\zeta,\, v^*_\zeta)$ for any $(t,\,x) \in [\mathcal{T}_0,\, +\infty) \times [-3L_0,\, L_0]$.

We define

$$\begin{aligned}
\underline{s}_2(t) &= c_{\tau,\zeta}(t - \mathcal{T}_0) + L_0,\ t \geqslant \mathcal{T}_0,\\
\underline{u}(t,x) &= \phi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t) - x),\ t \geqslant \mathcal{T}_0,\ x \in [-L_0, \underline{s}_2(t)],\\
\underline{v}(t,x) &= \varphi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t) - x),\ t \geqslant \mathcal{T}_0,\ x \in [-L_0, \underline{s}_2(t)],
\end{aligned}$$

and continuously extend $\underline{u}(t,\,x), \underline{v}(t,\,x)$ to be functions defined on $[\mathcal{T}_0, +\infty) \times (-\infty, \underline{s}_2(t)]$ such that $\underline{u}(t,\,x) \equiv 0$, $\underline{v}(t,\,x) \equiv 0$ on $[\mathcal{T}_0, +\infty) \times (-\infty, -3L_0]$, and

$$(\phi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t) - x), \varphi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t) - x)) \leqslant (\underline{u}(t,x), \underline{v}(t,x)) \leqslant (u^*_\zeta, v^*_\zeta)$$

on $[\mathcal{T}_0, +\infty) \times [-2L_0,\, -L_0]$. The graph of $\underline{v}(t,\,x)$ is plotted in Fig. 1. It follows that

$$\begin{aligned}
\int_{-L_0}^{+\infty} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y &= \int_{-L_0}^{+\infty} J_1(x-y)\varphi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t-\tau_1) - y)\mathrm{d}y,\\
\int_{-2L_0}^{-L_0} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y &\geqslant \int_{-2L_0}^{-L_0} J_1(x-y)\varphi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t-\tau_1) - y)\mathrm{d}y
\end{aligned} \tag{5.3}$$

for any $t \geqslant \mathcal{T}_0 + \max\{\tau_1,\, \tau_2\}$ and $x \in [-L_0,\, \underline{s}_2(t)]$.

Choose L_0 sufficiently large such that

$$v^* \int_{-\infty}^{-L_0} J_1(z)\mathrm{d}z < \zeta, \quad u^* \int_{-\infty}^{-L_0} J_2(z)\mathrm{d}z < \zeta,$$

which imply

$$\begin{aligned}
\int_{-\infty}^{-2L_0} J_1(x-y)\varphi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t-\tau_1) - y)\mathrm{d}y &\leqslant v^* \int_{-\infty}^{-2L_0} J_1(x-y)\mathrm{d}y\\
= v^* \int_{-\infty}^{-2L_0-x} J_1(z)\mathrm{d}z &\leqslant v^* \int_{-\infty}^{-L_0} J_1(z)\mathrm{d}z < \zeta
\end{aligned} \tag{5.4}$$

and

$$\int_{-\infty}^{-2L_0} J_2(x-y)\phi_\zeta^{c_{\tau,\zeta}}(\underline{s}_2(t-\tau_2) - y)\mathrm{d}y \leqslant u^* \int_{-\infty}^{-L_0} J_2(z)\mathrm{d}z < \zeta$$

for $t \geqslant \mathcal{T}_0 + \max\{\tau_1,\, \tau_2\}$ and $x \in [-L_0,\, \underline{s}_2(t)]$.

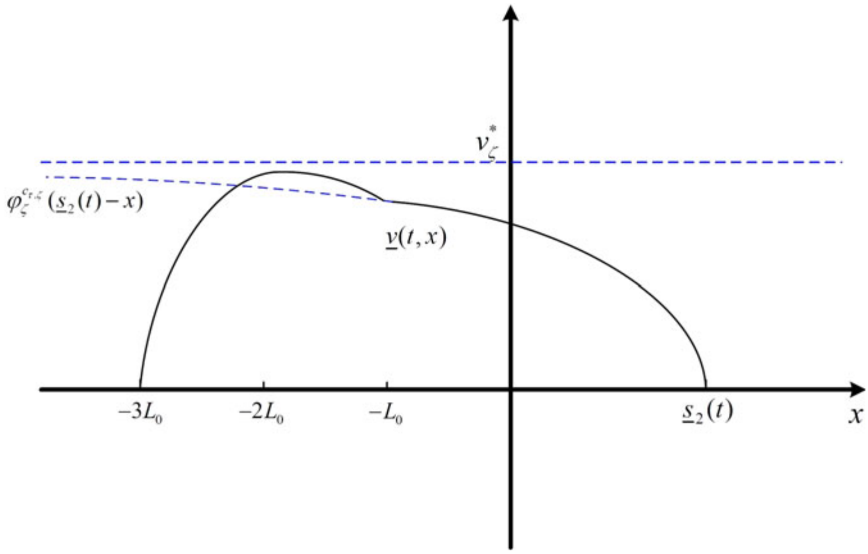

Figure 1. Lower solution $\underline{v}(t,\, x)$.

For $t \geqslant \mathcal{T}_0 + \max\{\tau_1,\, \tau_2\}$ and $x \in [-L_0,\, \underline{s}_2(t)]$, we can deduce

$$\begin{aligned}
&\underline{u}_t - d_1\underline{u}_{xx} + a_1\underline{u} - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y\right)\\
&\quad = c_{\tau,\zeta}(\phi_\zeta^{c_\tau,\zeta})'(\underline{s}_2(t)-x) - d_1(\phi_\zeta^{c_\tau,\zeta})''(\underline{s}_2(t)-x) + a_1\phi_\zeta^{c_\tau,\zeta}(\underline{s}_2(t)-x)\\
&\qquad - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y\right)\\
&\quad = h\left(\int_{-\infty}^{+\infty} J_1(y)\varphi_\zeta^{c_\tau,\zeta}(\underline{s}_2(t)-x-y-c_{\tau,\zeta}\tau_1)\mathrm{d}y - \zeta\right)\\
&\qquad - h\left(\int_{-\infty}^{+\infty} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y\right)\\
&\quad = h'(\eta)\left(\int_{-\infty}^{+\infty} J_1(x-y)\varphi_\zeta^{c_\tau,\zeta}(\underline{s}_2(t-\tau_1)-y)\mathrm{d}y\right.\\
&\qquad \left. - \zeta - \int_{-\infty}^{+\infty} J_1(x-y)\underline{v}(t-\tau_1, y)\mathrm{d}y\right)\\
&\quad \leqslant 0 \quad \text{with some } \eta > 0,
\end{aligned}$$

where the last inequality uses (5.3) and (5.4). The inequality satisfied by $\underline{v}$ can be proved similarly. In terms of the choices of $\mathcal{T}_0$ and L_0, we can check that $(\underline{u}(t,\, x),\, \underline{v}(t,\, x); \underline{s}_2(t))$ is a lower solution of one-side case with $t > 0, 0 < x < \bar{s}_2(t)$

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in lemma 2.3 replaced by $t > \mathcal{T}_0 + \max\{\tau_1, \tau_2\}, -L_0 < x < \underline{s}_2(t)$. Therefore, by the comparison principle we have $s_2(t) \geqslant \underline{s}_2(t)$ for $t \geqslant \mathcal{T}_0 + \max\{\tau_1, \tau_2\}$, which implies

$$\liminf_{t\to+\infty} \frac{s_2(t)}{t} \geqslant \liminf_{t\to+\infty} \frac{\underline{s}_2(t)}{t} \geqslant c_{\tau,\zeta}.$$

By taking $\zeta \to 0^+$, we get the desired result. The limit superior of $\frac{s_1(t)}{t}$ can be proved similarly. This completes the proof. □

6. Partially degenerate diffusion case without delays

In this section, we aim to determine the asymptotic spreading speeds of free boundaries for the partially degenerate diffusion case considered in [**19**]. The upper bounds of spreading speeds were provided in [**19**], but their precise values are still unknown. Here we give a complete answer to the problem. More precisely, we consider the following free boundary model introduced in [**19**]:

$$\begin{cases} u_t(t,x) = d_1 u_{xx} - a_1 u(t,x) + \int_{-\infty}^{+\infty} J_1(x-y)v(t,y)\mathrm{d}y,\ t>0, s_1(t)<x<s_2(t),\\ v_t(t,x) = -a_2 v(t,x) + g(u(t,x)),\ t>0, s_1(t)<x<s_2(t),\\ u(t,s_1(t)) = u(t,s_2(t)) = 0,\ v(t,s_1(t)) = v(t,s_2(t)) = 0,\ t>0,\\ s_1'(t) = -\mu u_x(t,s_1(t)),\ s_2'(t) = -\mu u_x(t,s_2(t)),\ t>0,\\ s_1(0) = -s_0,\ s_2(0) = s_0,\ t>0,\\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ -s_0<x<s_0, \end{cases} \tag{6.1}$$

which is a special case of (1.1).

As in § 4, we consider the corresponding perturbed semi-wave problem

$$\begin{cases} c\phi'(\xi) = d_1\phi''(\xi) - a_1\phi(\xi) + \int_{-\infty}^{+\infty} J_1(y)\varphi(\xi-y)\mathrm{d}y,\ \xi>0,\\ c\varphi'(\xi) = -a_2\varphi(\xi) + g(\phi(\xi)),\ \xi>0,\\ (\phi(\xi),\varphi(\xi)) = (\delta u^*, \delta v^*),\ \xi\leqslant 0,\\ (\phi(+\infty),\varphi(+\infty)) = (u^*,v^*). \end{cases} \tag{6.2}$$

Define $\mathcal{F}_1(\Phi)(\xi)$ similarly as in § 4.1, and

$$\mathcal{F}_2(\Phi)(\xi) = \begin{cases} \delta v^* e^{-\frac{a_2}{c}\xi} + \frac{1}{c}\int_0^\xi e^{\frac{a_2}{c}(s-\xi)} g(\phi(s))\mathrm{d}s,\ \xi>0,\\ \delta v^*,\ \xi\leqslant 0. \end{cases}$$

By applying the monotone iteration method, we can also establish the existence of solutions to the perturbed semi-wave problem (6.2).

Similar as theorem 4.6, there is a dichotomy between increasing semi-wave solution and increasing travelling wave solution $(u(t,x), v(t,x)) = (\phi(x+ct), \varphi(x+ct))$ of

$$\begin{cases} \partial_t u = d_1\partial_{xx}u - a_1 u + \int_{-\infty}^{+\infty} J_1(x-y)v(t,y)\mathrm{d}y, \quad t>0,\ x\in\mathbb{R},\\ \partial_t v = -a_2 v + g(u(t,x)), \quad t>0,\ x\in\mathbb{R}. \end{cases} \tag{6.3}$$

In [**39**], Xu and Zhao proved that there exists $c^* > 0$ such that (6.3) has an increasing travelling wave solution for $c \geqslant c^*$, but no such a solution for $0 < c < c^*$.

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Therefore, we can establish the (non-)existence of semi-wave solution. The critical value of speed c for semi-wave is also c^*.

Similarly as the proof of Lemma 2.13 in [**40**] and Lemma 2.10 in [**10**], we can prove that there exists a unique $c^*_\mu \in (0, c^*)$ such that $\mu(\phi^{c^*_\mu})'_+(0) = c^*_\mu$ for any given $\mu > 0$, where $(\phi^{c^*_\mu}, \varphi^{c^*_\mu})$ is the semi-wave solution with $c = c^*_\mu$. Moreover, $\lim_{\mu\to+\infty} c^*_\mu = c^*$.

As in § 5, by constructing a pair of upper and lower solutions from semi-wave solutions, we can get the asymptotic spreading speeds for (6.1) as follows

$$-\lim_{t\to+\infty} \frac{s_1(t)}{t} = \lim_{t\to+\infty} \frac{s_2(t)}{t} = c^*_\mu.$$

Remark 6.1. We remark that the method in this paper can also be applied to determine the asymptotic speeds for the partially degenerate diffusion case with time delays, i.e., $d_2 = \rho = 0, J_2 = \delta$ (Dirac delta function).

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Appendix A.

Proposition A
For the generalized principal eigenvalue $\mu_1(\Omega)$ defined in (2.6), we have (i) $\mu_1((-l, l)) = \mu^l_0$ for any $l > 0$, where μ^l_0 is the principal eigenvalue of (1.7); (ii) $\mu_1((-l, l)) \to \mu_1(\mathbb{R})$ as $l \to +\infty$.

Proof. (i) From theorem 1.1 (ii), we know $\mu^l_0 \in E^{(-l,l)}$, where $E^{(-l,l)}$ is defined in (2.6). Then $\mu_1((-l, l)) = \sup E^{(-l,l)} \geqslant \mu^l_0$. Now we prove the equality holds.

Assume by contradiction that $\mu_1((-l, l)) > \mu^l_0$, we can choose $\tilde{\mu} \in (\mu^l_0, \mu_1((-l, l)))$ and $(\tilde{\phi}, \tilde{\varphi}) \in C^2((-l, l)) \cap C^1([-l, l])$ such that $(\tilde{\phi}, \tilde{\varphi}) > \mathbf{0}$ in $(-l, l)$ and satisfies

$$-d_1\tilde{\phi}_{xx} + a_1\tilde{\phi} \geqslant \tilde{\mu}h'(0)\int_{-l}^{l} J_1(x-y)\tilde{\varphi}(y)\mathrm{d}y,$$

$$-d_2\tilde{\varphi}_{xx} + a_2\tilde{\varphi} \geqslant \tilde{\mu}g'(0)\int_{-l}^{l} J_2(x-y)\tilde{\phi}(y)\mathrm{d}y$$

for $x \in (-l, l)$.

We claim that there exists $\varepsilon > 0$ such that $(\tilde{\phi}, \tilde{\varphi}) \geqslant \varepsilon(\phi^l_{\mu_0}, \varphi^l_{\mu_0})$ on $(-l, l)$. Indeed, since $(\tilde{\phi}, \tilde{\varphi}), (\phi^l_{\mu_0}, \varphi^l_{\mu_0})$ are positive continuous functions in $(-l, l)$, we only need to prove the inequality near the endpoints $x = \pm l$. If $\tilde{\phi}(-l) > 0$, due to $\phi^l_{\mu_0}(-l) = 0$, we know that $\tilde{\phi} \geqslant \varepsilon\phi^l_{\mu_0}$ on $(-l, -l+\epsilon)$ with some $\varepsilon, \epsilon > 0$. If $\tilde{\phi}(-l) = 0$, by the Hopf boundary lemma, we have $(\tilde{\phi})'_+(-l) > 0$. It follows that $\lim_{x\to -l} \frac{\tilde{\phi}(x)}{\phi^l_{\mu_0}(x)} = \frac{(\tilde{\phi})'_+(-l)}{(\phi^l_{\mu_0})'_+(-l)}$. Let $\varepsilon = \frac{1}{2}\frac{(\tilde{\phi})'_+(-l)}{(\phi^l_{\mu_0})'_+(-l)}$, we can also prove that $\tilde{\phi} \geqslant \varepsilon\phi^l_{\mu_0}$ on $(-l, -l+\epsilon)$ for some $\epsilon > 0$. The other cases can be similarly proved. Thus, the claim holds true.

Let ε^* be the largest $\varepsilon > 0$ such that $(\tilde{\phi}, \tilde{\varphi}) \geqslant \varepsilon(\phi^l_{\mu_0}, \varphi^l_{\mu_0})$ on $(-l, l)$. We define $(\hat{\phi}, \hat{\varphi}) = (\tilde{\phi} - \varepsilon^*\phi^l_{\mu_0}, \tilde{\varphi} - \varepsilon^*\varphi^l_{\mu_0})$. Then $(\hat{\phi}, \hat{\varphi}) \geqslant \mathbf{0}$ on $(-l, l)$ and there exists at least one $x_0 \in (-l, l)$ such that $\hat{\phi}(x_0) = 0$ or $\hat{\varphi}(x_0) = 0$. Since $\tilde{\mu} > \mu^l_0$, we know $\hat{\phi}, \hat{\varphi} \not\equiv 0$ on $(-l, l)$. We may assume that $\hat{\phi}(x_1) > 0$ and $\hat{\varphi}(x_2) > 0$ for some $x_1, x_2 \in (-l, l)$. By the strong maximum principle, we have $\hat{\phi}, \hat{\varphi} > 0$ on $(-l, l)$, which contradicts with $\hat{\phi}(x_0) = 0$ or $\hat{\varphi}(x_0) = 0$. Thus, $\mu_1((-l, l)) = \mu^l_0$.

(ii) By the definition of $E^{\mathbb{R}}$ in (2.6), for any $\hat{\mu} \in E^{\mathbb{R}}$, there exists $(\phi_1, \varphi_1) \in C^2(\mathbb{R})$ such that $(\phi_1, \varphi_1) > \mathbf{0}$ in $\mathbb{R}$ and satisfies

$$-d_1\phi_{1,xx} + a_1\phi_1 \geqslant \hat{\mu}h'(0)\int_{-\infty}^{+\infty} J_1(x-y)\varphi_1(y)\mathrm{d}y,$$

$$-d_2\varphi_{1,xx} + a_2\varphi_1 \geqslant \hat{\mu}g'(0)\int_{-\infty}^{+\infty} J_2(x-y)\phi_1(y)\mathrm{d}y$$

for $x \in \mathbb{R}$. Using (ϕ_1, φ_1) as 'test function' for $\mu_1((-l, l))$, we have $\hat{\mu} \leqslant \mu_1((-l, l)) = \mu^l_0$ for any $l > 0$. Taking $l \to +\infty$, we get $\hat{\mu} \leqslant \mu^*$, and then $\mu_1(\mathbb{R}) = \sup E^{\mathbb{R}} \leqslant \mu^*$. Moreover, from (2.5), we know $\mu^* \in E^{\mathbb{R}}$. Thus, $\mu_1(\mathbb{R}) \geqslant \mu^*$. In summary, $\mu_1(\mathbb{R}) = \mu^*$, which completes the proof. □

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