Bull. Aust. Math. Soc. **108** (2023), 254–263
doi:10.1017/S0004972722001393

23-REGULAR PARTITIONS AND MODULAR FORMS WITH COMPLEX MULTIPLICATION

DAVID PENNISTON

(Received 9 September 2022; accepted 29 October 2022; first published online 23 December 2022)

Abstract

A partition of a positive integer n is called ℓ-regular if none of its parts is divisible by ℓ. Denote by $b_\ell(n)$ the number of ℓ-regular partitions of n. We give a complete characterisation of the arithmetic of $b_{23}(n)$ modulo 11 for all n not divisible by 11 in terms of binary quadratic forms. Our result is obtained by establishing a relation between the generating function for these values of $b_{23}(n)$ and certain modular forms having complex multiplication by $\mathbb{Q}(\sqrt{-69})$.

2020 *Mathematics subject classification*: primary 11P83.

Keywords and phrases: partitions, congruences, modular forms.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. For $\ell > 1$, a partition is called ℓ-regular if none of its parts is divisible by ℓ. Denoting by $b_\ell(n)$ the number of ℓ-regular partitions of n and adopting the convention $b_\ell(0) := 1$, we have the identity

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \prod_{n=1}^{\infty}\Big(\frac{1-q^{\ell n}}{1-q^n}\Big).$$

A substantial body of results on the arithmetic of $b_\ell(n)$ modulo m has been established. Perhaps the most elegant of these concerns the parity of $b_2(n)$, where Euler's pentagonal number theorem implies that $b_2(n)$ is odd precisely when $24n+1$ is a perfect square. This result has been extended in a number of directions. For example, Ono and the author [4] gave a description of the behaviour of $b_2(n)$ modulo 8 in terms of the arithmetic of the ring $\mathbb{Z}[\sqrt{-6}]$, while Gordon and Ono [3] proved that if p is a prime divisor of ℓ such that $p^a \mid \ell$ and $p^a \geq \sqrt{\ell}$, then for every positive integer j, the congruence $b_\ell(n) \equiv 0 \pmod{p^j}$ holds for almost all $n \geq 0$, that is, on a set of nonnegative integers of density one.

Addressing the case where ℓ and m are coprime, Ahlgren and Lovejoy [1] showed that if $p \geq 5$ is prime, then $b_2(n) \equiv 0 \pmod{p^j}$ for at least $(p+1)/2p$ of the values of n. In [5], the author extended this work to odd primes $\ell \leq 23$, and also proved that $b_{23}(n) \equiv 0 \pmod{11}$ for at least $10/11$ of the values of n by showing that this congruence holds for almost all n not divisible by 11 (see, for example, [2, 9, 10] for further results). In this paper, we use the theory of modular forms to refine the latter result, giving a complete characterisation of the arithmetic of $b_{23}(n)$ modulo 11 for n with $11 \nmid n$.

Our description can be phrased in terms of binary quadratic forms. To give a flavour here (see Section 3 for complete results), for $a, b, c \in \mathbb{Z}$, let

$$F_{a,b,c}(X,Y) = aX^2 + bXY + cY^2.$$

Given an odd prime p with $(\frac{-69}{p}) = 1$, the classical theory implies that there exist integers x and y with

$$p = F_{a,b,c}(x,y) \tag{1.1}$$

for a unique triple (a, b, c) in the set

$$\{(1,0,69), (3,0,23), (2,2,35), (6,6,13), (5,2,14), (7,2,10)\}.$$

THEOREM 1.1. *Suppose n is a positive integer such that $12n + 11 = p$ is prime.*

(i) *If $(\frac{-69}{p}) = -1$, then $b_{23}(n) \equiv 0 \pmod{11}$.*

(ii) *If $(\frac{-69}{p}) = 1$, let x and y be integers satisfying (1.1) and let $\delta_p \in \{-1, 1\}$ be defined as in Section 3 (see (3.1)). Then*

$$b_{23}(n) \equiv \left(\frac{p}{11}\right) \cdot \delta_p \cdot 9yp^4 \pmod{11}.$$

In particular, $b_{23}(n) \equiv 0 \pmod{11}$ if and only if $11 \mid y$.

Theorem 1.1 addresses the case where $12n + 11$ is prime. For general n with $11 \nmid n$, our description requires consideration of the prime factorisation of $12n + 11$.

The remainder of the paper is organised as follows. In Section 2, we provide the necessary background on modular forms and construct the forms we require. In Section 3, we state and prove our main results and illustrate their use.

2. Background and two CM-forms

Let k and N be positive integers and χ a Dirichlet character modulo N. We denote by $M_k(\Gamma_0(N), \chi)$ the space of holomorphic modular forms of weight k on $\Gamma_0(N)$ with Nebentypus χ, and by $S_k(\Gamma_0(N), \chi)$ the subspace of cusp forms (we suppress χ when it is the trivial character). Denote by χ_{23} the character defined by $\chi_{23}(\bullet) = (\frac{23}{\bullet})$ and write $q := e^{2\pi i z}$ for complex z with $\mathrm{Im}(z) > 0$.

We begin by stating a slight refinement of Theorem 2.1 in [5] which implies that the values of $b_{23}(n)$ with $11 \nmid n$ can be realised modulo 11 as the Fourier coefficients of a modular form of integer weight. (Since this result can be established in the same way as Proposition 2.1 in [6], we do not include the proof.)

https://doi.org/10.1017/S0004972722001393 Published online by Cambridge University Press

PROPOSITION 2.1. *There exists a cusp form*

$$H(z) \in S_{1330}(\Gamma_0(144 \cdot 11^3 \cdot 23), \chi_{23}) \cap \mathbb{Z}[[q]]$$

such that

$$H(z) \equiv \sum_{n=0}^{\infty} \left(\frac{n}{11}\right) b_{23}(n) q^{12n+11} \pmod{11}.$$

Following closely the exposition in [7], we now construct two normalised Hecke eigenforms with complex multiplication that turn out to be related to $H(z)$ modulo 11. Let $K = \mathbb{Q}(\sqrt{-69})$, a number field of discriminant $d = -276$, and define the ideal $\mathcal{F}$ of the ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-69}]$ by

$$\mathcal{F} = (6, 2\sqrt{-69}).$$

We note that $N(\mathcal{F}) = 12$, and from now on, we write $\theta := \sqrt{-69}$. One can verify that

$$\{1, 5, 2 + \theta, 4 + \theta\}$$

is a set of representatives of $(\mathcal{O}_K/\mathcal{F})^*$, and since $(\mathcal{O}_K/\mathcal{F})^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, it follows that the function $\rho : (\mathcal{O}_K/\mathcal{F})^* \to \mathbb{C}^*$ defined by

$$\rho(1) = 1, \quad \rho(5) = -1, \quad \rho(2 + \theta) = 1, \quad \rho(4 + \theta) = -1$$

is a group homomorphism.

Next, we define Hecke characters c_1 and c_2 on the fractional ideals of K coprime to $\mathcal{F}$. We begin by defining c_1 on principal ideals of $\mathcal{O}_K$ via

$$c_1((a + b\theta)) = \rho(a + b\theta)(a + b\theta)^{49} \tag{2.1}$$

for $a, b \in \mathbb{Z}$. Since $\mathcal{O}_K$ has class number eight, we must extend c_1. One can check that the class group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ and is generated by the nonprincipal ideals $(5, 1 + \theta)$ and $(23, \theta)$ which satisfy $(5, 1 + \theta)^4 = (2 - 3\theta)$ and $(23, \theta)^2 = (23)$. Letting

$$c_1((5, 1 + \theta)) = \left(\frac{1}{2}\left(\sqrt{10 + 3\sqrt{6}} - i\sqrt{10 - 3\sqrt{6}}\right)\right)^{49} \tag{2.2}$$

and

$$c_1((23, \theta)) = (\sqrt{-23})^{49} = 23^{24}\sqrt{-23} \tag{2.3}$$

allows us to complete c_1 to a Hecke character of K with exponent 49 and conductor $\mathcal{F}$, which in turn yields a Dirichlet character ω_1 defined by

$$\omega_1(n) = c_1(n\mathcal{O}_K)/n^{49}$$

for $n \in \mathbb{Z}$ coprime to 6. We define companion characters c_2 and ω_2 similarly, with c_2 extending to nonprincipal ideals via

$$c_2((5, 1 + \theta)) = ic_1((5, 1 + \theta))$$

https://doi.org/10.1017/S0004972722001393 Published online by Cambridge University Press

and

$$c_2((23,\theta)) = -c_1((23,\theta)).$$

For $v \in \{1,2\}$, let

$$g_v(z) = \sum c_v(\mathcal{I})q^{N(\mathcal{I})} := \sum_{n=1}^{\infty} s_v(n)q^n,$$

where the first sum is over all ideals $\mathcal{I}$ of $\mathcal{O}_K$ coprime to $\mathcal{F}$. Then g_v is a normalised Hecke eigenform of weight 50, level $|d| \cdot N(\mathcal{F}) = 144 \cdot 23$ and character $\epsilon_K\omega_v$, where $\epsilon_K(p) = (\frac{-69}{p})$ for any prime $p \notin \{2,3,23\}$. Note that for such a prime, we have $\omega_v(p) = \rho(p) = (\frac{p}{3})$. Then since $(\frac{-69}{p})(\frac{p}{3}) = (\frac{23}{p})$ by quadratic reciprocity, it follows that

$$g_v(z) \in S_{50}(\Gamma_0(144\cdot 23), \chi_{23}).$$

The fact that $g_v(z)$ is an eigenform implies that

$$s_v(mn) = s_v(m)s_v(n) \quad \text{when } (m,n) = 1 \tag{2.4}$$

and

$$s_v(p^{j+1}) = s_v(p)s_v(p^j) - \chi_{23}(p)p^{49}s_v(p^{j-1}) \tag{2.5}$$

for all primes $p \geq 5$ and $j \geq 1$.

3. Proofs of the main results

We begin by finding explicit expressions for the Fourier coefficients $s_v(p)$ of the CM-forms $g_v(z)$ which can be leveraged to evaluate the coefficients of two auxiliary forms (see $h_v(z)$ below) modulo 11. An odd prime p satisfying $(\frac{-69}{p}) = -1$ is inert in K, which implies that $s_v(p) = 0$. Since 23 ramifies in K, we have $s_v(23) = c_v((23,\theta))$. Now suppose p is an odd prime with $(\frac{-69}{p}) = 1$ and that $x, y \in \mathbb{Z}$ are as in (1.1). Define δ_p by

$$\delta_p = \begin{cases} \rho(x+y\theta) & \text{if } (a,b,c) = (1,0,69),\\ \rho((9x+16y)-(x-y)\theta) & \text{if } (a,b,c) = (6,6,13),\\ \rho((5x+y)-y\theta) & \text{if } (a,b,c) = (5,2,14),\\ \rho((7x+y)+y\theta) & \text{if } (a,b,c) = (7,2,10),\\ \rho(23y+x\theta) & \text{if } (a,b,c) = (3,0,23),\\ \rho((5x-32y)+(x+3y)\theta) & \text{if } (a,b,c) = (2,2,35), \end{cases} \tag{3.1}$$

https://doi.org/10.1017/S0004972722001393 Published online by Cambridge University Press

and for $n \geq 1$, let

$$t_1(n) = \begin{cases} s_1(n) & \text{if } n \equiv 1 \pmod{12}, \\ s_1(n)/i\sqrt{10-3\sqrt{6}} & \text{if } n \equiv 5 \pmod{12}, \\ s_1(n)/\sqrt{14-5\sqrt{6}} & \text{if } n \equiv 7 \pmod{12}, \\ s_1(n)/\sqrt{-23} & \text{if } n \equiv 11 \pmod{12}. \end{cases}$$

We define $t_2(n)$ in the same way as $t_1(n)$ except that we divide $s_2(n)$ by $i\sqrt{10+3\sqrt{6}}$ when $n \equiv 5 \pmod{12}$, and by $\sqrt{14+5\sqrt{6}}$ when $n \equiv 7 \pmod{12}$. Our next result (in the proof of which we will see that $t_1(n) \in \mathbb{Z}[\sqrt{6}]$) gives expressions for $t_1(p)$ modulo 11 in terms of the six reduced binary quadratic forms of discriminant -276 mentioned in the introduction. Recall that $\theta = \sqrt{-69}$.

THEOREM 3.1. *Suppose p is an odd prime with $(\frac{-69}{p}) = 1$. Let $x, y \in \mathbb{Z}$ be defined by (1.1), replacing x and y by $-x$ and $-y$ if necessary so that $-x - 2y + \sqrt{6}y > 0$ if $(a, b, c) = (5, 2, 14)$ and $x - 2y - \sqrt{6}y > 0$ if $(a, b, c) = (7, 2, 10)$. Then, modulo the ideal of $\mathbb{Z}[\sqrt{6}]$ generated by 11,*

$$\delta_p t_1(p) \equiv \begin{cases} 2xp^4 & \text{if } (a,b,c) = (1,0,69), \\ (2x+y)p^4 \cdot \sqrt{6} & \text{if } (a,b,c) = (6,6,13), \\ (x+2y+\sqrt{6}y)p^4 & \text{if } (a,b,c) = (5,2,14), \\ (x-2y-\sqrt{6}y)p^4 & \text{if } (a,b,c) = (7,2,10), \\ 9yp^4 & \text{if } (a,b,c) = (3,0,23), \\ yp^4 \cdot \sqrt{6} & \text{if } (a,b,c) = (2,2,35). \end{cases}$$

PROOF. For the sake of brevity, we provide details for only two of the cases, as the others can be handled similarly.

For $(a, b, c) = (2, 2, 35)$, we begin with the ideal calculation

$$(5, 1+\theta)^2(23, \theta)(47, 5+\theta) = (46 - 19\theta).$$

Combining this with (2.1), (2.2), (2.3) and the multiplicativity of c_1 yields

$$c_1((47, 5+\theta)) = \left(\frac{5+\theta}{\sqrt{2}}\right)^{49}.$$

Now suppose that $p > 47$ is a prime with $p = 2x^2 + 2xy + 35y^2$ for some $x, y \in \mathbb{Z}$, and denote by $\mathfrak{P}$ and $\mathfrak{P}'$ the prime ideals of O_K lying above p. It is easy to verify that if $a + b\theta \in (47, 5+\theta)$ for some $a, b \in \mathbb{Z}$, then $a \equiv 5b \pmod{47}$. Using this and the equalities

$$47p = (5x - 32y)^2 + 69(x+3y)^2 = (5x+37y)^2 + 69(x-2y)^2,$$

we find that the elements

https://doi.org/10.1017/S0004972722001393 Published online by Cambridge University Press

$$\mu = (5x - 32y) + (x + 3y)\theta \quad \text{and} \quad \nu = (5x + 37y) + (x - 2y)\theta$$

generate the principal ideals $(47, 5+\theta)\mathfrak{P}$ and $(47, 5+\theta)\mathfrak{P}'$. Since

$$\mu\nu = -44x^2 - 44xy - 770y^2 + (10x^2 + 10xy + 175y^2)\theta,$$

it follows that

$$\rho(\mu)\rho(\nu) = \rho(\mu\nu) = \rho(4+\theta) = -1,$$

and hence $s_1(p) = c_1(\mathfrak{P}) + c_1(\mathfrak{P}')$ is equal to

$$\rho(\mu)\bigg[\bigg(\sqrt{2}x + \frac{\sqrt{2}}{2}y + \frac{y}{2}\sqrt{-138}\bigg)^{49} - \bigg(\sqrt{2}x + \frac{\sqrt{2}}{2}y - \frac{y}{2}\sqrt{-138}\bigg)^{49}\bigg].$$

A calculation now shows that $s_1(p)$ has the form $\rho(\mu)yC(x,y)\sqrt{-138}$, where $C(x,y) \in \mathbb{Z}[x,y]$ is a homogeneous polynomial of degree 48. Using Fermat's Little Theorem we find that $yC(x,y)$ is congruent modulo 11 to $y(2x^2 + 2xy + 35y^2)^4$, and since $p \equiv 11 \pmod{12}$, our result follows.

Now let $(a,b,c) = (7,2,10)$. We begin by using the equality

$$(5, 1+\theta)(23,\theta)(7, 6+\theta) = (23 - 2\theta)$$

as above to deduce that

$$c_1((7, 6+\theta)) = \bigg(\frac{1}{2}\bigg(\sqrt{14 - 5\sqrt{6}} + i\sqrt{14 + 5\sqrt{6}}\bigg)\bigg)^{49}.$$

Combining this with

$$c_1((7, 6+\theta))c_1((7, 1+\theta)) = c_1((7)) = \rho(7)\cdot 7^{49} = 7^{49}$$

then gives

$$c_1((7, 1+\theta)) = \bigg(\frac{1}{2}\bigg(\sqrt{14 - 5\sqrt{6}} - i\sqrt{14 + 5\sqrt{6}}\bigg)\bigg)^{49}.$$

Next suppose that $p > 7$ is a prime with $p = 7x^2 + 2xy + 10y^2$. Then $7p = (7x+y)^2 + 69y^2$, and by replacing x by $-x$ and y by $-y$ if necessary we may ensure that $x - 2y - \sqrt{6}y > 0$. Let z be an integer such that

$$z(7x + y) \equiv -69y \pmod{p}.$$

One can verify that $z^2 \equiv -69 \pmod{p}$ (which implies that $(p, \pm z + \theta)$ are the prime ideals of $\mathcal{O}_K$ above p) and

$$(7, 1+\theta)(p, z+\theta) = ((7x+y) + y\theta).$$

It follows that

$$\begin{aligned} c_1((p, z+\theta)) &= \rho((7x+y) + y\theta)) \\ &\quad\cdot \bigg(\frac{1}{2}\bigg(\sqrt{2p + (-5x^2 - 8xy + 6y^2)\sqrt{6}} + s\cdot\sqrt{-2p + (-5x^2 - 8xy + 6y^2)\sqrt{6}}\bigg)\bigg)^{49}, \end{aligned}$$

where $s = \operatorname{sign}(x^2 - 4xy - 2y^2)$.

https://doi.org/10.1017/S0004972722001393 Published online by Cambridge University Press

Since $p \equiv 7 \pmod{12}$,

$$c_1((p, z+\theta))c_1((p, -z+\theta)) = c_1((p)) = \rho(p)p^{49} = p^{49}.$$

Hence, $c_1((p, -z+\theta)) = \overline{c_1((p, z+\theta))}$, and a calculation reveals that

$$s_1(p) = \sqrt{14 - 5\sqrt{6}} \cdot \rho((7x+y)+y\theta)(x - 2y - \sqrt{6}y)D(x, y) \tag{3.2}$$

with $D(x, y) \in \mathbb{Z}[\sqrt{6}][x, y]$. Our result now follows as in the previous case. □

REMARK 3.2. For our claim that $t_1(n) \in \mathbb{Z}[\sqrt{6}]$, note that if $p = 5x^2 + 2xy + 14y^2$, then $p \equiv 5 \pmod{12}$, and an argument similar to the one given in our proof of (3.2) for primes represented by $F_{7,2,10}$ shows that

$$s_1(p) = i\sqrt{10 - 3\sqrt{6}} \cdot \rho((5x+y) - y\theta)(x + 2y + \sqrt{6}y)E(x, y) \tag{3.3}$$

for some $E(x, y) \in \mathbb{Z}[\sqrt{6}][x, y]$. Moreover, observe that

$$i\sqrt{10 - 3\sqrt{6}} \cdot \sqrt{14 - 5\sqrt{6}} = (-2 + \sqrt{6})\sqrt{-23}.$$

REMARK 3.3. The parallel result to Theorem 3.1 for $t_2(p)$ states that for each odd prime p with $\left(\frac{-69}{p}\right) = 1$,

$$\delta_p t_2(p) \equiv \begin{cases} 2xp^4 & \text{if } (a, b, c) = (1, 0, 69), \\ -(2x+y)p^4 \cdot \sqrt{6} & \text{if } (a, b, c) = (6, 6, 13), \\ (-x - 2y + \sqrt{6}y)p^4 & \text{if } (a, b, c) = (5, 2, 14), \\ (-x + 2y - \sqrt{6}y)p^4 & \text{if } (a, b, c) = (7, 2, 10), \\ -9yp^4 & \text{if } (a, b, c) = (3, 0, 23), \\ yp^4 \cdot \sqrt{6} & \text{if } (a, b, c) = (2, 2, 35). \end{cases}$$

We now show that the behaviour of $b_{23}(n)$ modulo 11 for $11 \nmid n$ is controlled by the Fourier coefficients of the CM-forms $g_\nu(z)$ defined in Section 2. We do this by establishing a relation between the coefficients of the form $H(z)$ from Proposition 2.1 and those of two auxiliary forms built from the $g_\nu(z)$ which we now define.

For $n \geq 1$, write

$$t_1(n) = \alpha_n + \beta_n\sqrt{6}$$

with $\alpha_n, \beta_n \in \mathbb{Z}$. Using Theorem 3.1 and Remark 3.3 along with (2.4) and (2.5), it is straightforward to show that $t_2(n) = -\alpha_n + \beta_n\sqrt{6}$ when $n \equiv 11 \pmod{12}$. For each $\nu \in \{1, 2\}$, we define the modular form

$$h_\nu(z) = \sum_{n \equiv 11 \pmod{12}} t_\nu(n)q^n \in M_{50}(\Gamma_0(144 \cdot 23), \chi_{23}).$$

Next, let

$$F(z) := \frac{h_1(z) - h_2(z)}{2} = \sum_{n \equiv 11 \pmod{12}} \alpha_n q^n$$

and

$$G(z) := \frac{h_1(z) + h_2(z)}{2\sqrt{6}} = \sum_{n \equiv 11 \pmod{12}} \beta_n q^n.$$

THEOREM 3.4. *Suppose n is a positive integer with $11 \nmid n$. Then*

$$b_{23}(n) \equiv \left(\frac{n}{11}\right)(\alpha_{12n+11} + 9\beta_{12n+11}) \pmod{11}.$$

PROOF. Note first that if $H(z)$ is the modular form in Proposition 2.1, then our result is equivalent to the integral power series congruence

$$H(z) \equiv F(z) + 9G(z) \pmod{11}.$$

Recall the weight ten normalised Eisenstein series

$$E_{10}(z) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n \in M_{10}(\Gamma_0(1)),$$

where $\sigma_9(n) = \sum_{d|n} d^9$. Since $F(z) + 9G(z) \in M_{50}(\Gamma_0(144 \cdot 23), \chi_{23})$,

$$(F(z) + 9G(z))E_{10}(z)^{128} \in M_{1330}(\Gamma_0(144 \cdot 23), \chi_{23}),$$

and hence both $(F(z) + 9G(z))E_{10}(z)^{128}$ and $H(z)$ lie in the space

$$M_{1330}(\Gamma_0(144 \cdot 11^3 \cdot 23), \chi_{23}).$$

Since $E_{10}(z) \equiv 1 \pmod{11}$, on checking that the coefficients of the Fourier expansions of $H(z)$ and $F(z) + 9G(z)$ agree modulo 11 as far as the Sturm bound [8], our proof is complete. □

REMARK 3.5. Theorems 3.1 and 3.4, when used in conjunction with (2.4), (2.5), the value of $s_1(23)$ and the fact that $s_1(p) = 0$ for odd primes p with $(\frac{-69}{p}) = -1$, allow one to evaluate $b_{23}(n)$ modulo 11 for any n not divisible by 11.

To illustrate, first let $n = 197$. Since $12n + 11 = 5^3 \cdot 19$, we begin by noting that

$$5 = F_{5,2,14}(-1, 0) \quad \text{and} \quad 19 = F_{7,2,10}(-1, -1).$$

By (3.3) and (3.2),

$$s_1(5) = i\sqrt{10 - 3\sqrt{6}} \cdot \rho(-5)(-1)E(-1, 0) = -i\sqrt{10 - 3\sqrt{6}} \cdot E(-1, 0)$$

and

$$s_1(19) = \sqrt{14 - 5\sqrt{6}} \cdot \rho(-8 - \theta)(1 + \sqrt{6})D(-1, -1) = \sqrt{14 - 5\sqrt{6}} \cdot (-1 - \sqrt{6})D(-1, -1).$$

https://doi.org/10.1017/S0004972722001393 Published online by Cambridge University Press

Then (2.5) yields

$$s_1(5^2) = (-10 + 3\sqrt{6})E(-1,0)^2 + 5^{49}$$

and

$$s_1(5^3) = -i\sqrt{10 - 3\sqrt{6}} \cdot [(-10 + 3\sqrt{6})E(-1,0)^3 + 2 \cdot 5^{49}E(-1,0)],$$

and so (2.4) allows us to conclude that $s_1(5^3 \cdot 19)$ is equal to

$$(4 - \sqrt{6})D(-1,-1)[(-10 + 3\sqrt{6})E(-1,0)^3 + 2 \cdot 5^{49}E(-1,0)]\sqrt{-23}.$$

Since $D(-1,-1) \equiv 19^4 \equiv 4 \pmod{11}$, $E(-1,0) \equiv 5^4 \equiv 9 \pmod{11}$ and $5^{49} \equiv 9 \pmod{11}$, it follows that

$$t_1(5^3 \cdot 19) \equiv 4 + \sqrt{6} \pmod{11}.$$

Theorem 3.4 then yields

$$b_{23}(197) \equiv \left(\frac{197}{11}\right)(4 + 9 \cdot 1) \equiv 9 \pmod{11},$$

which one may verify using the exact value $b_{23}(197) = 2626664703430$.

Now let $n = 102$. Since $12n + 11 = 5 \cdot 13 \cdot 19$, we proceed by noting that $13 = F_{6,6,13}(0,1)$, which gives

$$s_1(13) = t_1(13) \equiv \rho(16 + \theta)(1)(13^4)\sqrt{6} \equiv 6\sqrt{6} \pmod{11}.$$

Combining this with our values of $s_1(5)$ and $s_1(19)$, we can calculate

$$t_1(5 \cdot 13 \cdot 19) \equiv 2 + 6\sqrt{6} \pmod{11},$$

and hence Theorem 3.4 implies that

$$b_{23}(102) \equiv \left(\frac{102}{11}\right)(2 + 9 \cdot 6) \equiv 1 \pmod{11},$$

which can be checked against $b_{23}(102) = 226889906$.

References

[1] S. Ahlgren and J. Lovejoy, 'The arithmetic of partitions into distinct parts', *Mathematika* **48**(1–2) (2001), 203–211.

[2] E. Boll and D. Penniston, 'The 7-regular and 13-regular partition functions modulo 3', *Bull. Aust. Math. Soc.* **93**(3) (2016), 410–419.

[3] B. Gordon and K. Ono, 'Divisibility of certain partition functions by powers of primes', *Ramanujan J.* **1**(1) (1997), 25–34.

[4] K. Ono and D. Penniston, 'The 2-adic behavior of the number of partitions into distinct parts', *J. Combin. Theory Ser. A* **92**(2) (2000), 138–157.

[5] D. Penniston, 'Arithmetic of ℓ-regular partition functions', *Int. J. Number Theory* **4**(2) (2008), 295–302.

[6] D. Penniston, '11-regular partitions and a Hecke eigenform', *Int. J. Number Theory* **15**(6) (2019), 1251–1259.

[7] J.-P. Serre, 'Sur la lacunarité des puissances de η', *Glasg. Math. J.* **27** (1985), 203–221.

https://doi.org/10.1017/S0004972722001393 Published online by Cambridge University Press

[8] J. Sturm, 'On the congruence of modular forms', in: *Number Theory*, Lecture Notes in Mathematics, 1240 (eds. D. V. Chudnovsky, G. V. Chudnovsky, H. Cohn and M. B. Nathanson) (Springer, Berlin–Heidelberg, 1984), 275–280.

[9] S. Treneer, 'Congruences for the coefficients of weakly holomorphic modular forms', *Proc. Lond. Math. Soc. (3)* **93**(2) (2006), 304–324.

[10] J. J. Webb, 'Arithmetic of the 13-regular partition function modulo 3', *Ramanujan J.* **25**(1) (2011), 49–56.

DAVID PENNISTON, Department of Mathematics,
University of Wisconsin Oshkosh, Oshkosh, WI 54901-8631, USA
e-mail: pennistd@uwosh.edu