Canad. Math. Bull. Vol. **14** (2), 1971

# ON PRIME ONE-SIDED IDEALS

BY
KWANGIL KOH

Let $R$ be a ring and let $L_\gamma(R)$ be the lattice of right ideals. We define that $I \in L_\gamma(R)$ is a *prime right ideal* provided that if $AB \subseteq I$ for some $A$, $B$ in $L_\gamma(R)$ such that $AI \subseteq I$ then either $A \subseteq I$ or $B \subseteq I$. Any prime ideal of a ring $R$ is a prime right ideal and if $R$ is commutative then an ideal is prime if and only if it is a prime right ideal. If $R$ is a ring and $a \in R$, let $aR = \{x \in R \mid x = ar$ for some $r \in R\}$ and $aR_1 = \{x \in R \mid x = na + ar$ for some integer $n$ and $r \in R\}$. The purpose of this note is to prove the following two theorems:

THEOREM 1. *Let $R$ be a ring such that $aR = aR_1$ for each $a \in R$. Then $R$ is a right Noetherian ring if and only if every prime right ideal of $R$ is finitely generated. Every right ideal of $R$ is generated by one element if and only if every prime right ideal of $R$ is generated by one element.*

THEOREM 2. *Let $R$ be a ring. Then every right ideal of $R$ is a prime right ideal if and only if $R$ is a simple ring and $aR = aR_1$ for all $a \in R$.*

**Proof of Theorem 1.** Since $R$ is right Noetherian if and only if every right ideal is finitely generated, if $R$ is right Noetherian then, clearly, every prime right ideal is finitely generated. Assume now that every prime right ideal is finitely generated. If there is $I \in L_\gamma(R)$ such that $I$ is not finitely generated then by Zorn's lemma, one can choose a $I_0 \in L_\gamma(R)$, which is not finitely generated, such that $I_0 \supseteq I$ and if $J \in L_\gamma(R)$ and $J \supseteq I_0$ then either $J = I_0$ or $J$ is finitely generated. We will prove that $I_0$ is a prime right ideal and hence finitely generated and thus the supposition that there is $I \in L_\gamma(R)$ such that $I$ is not finitely generated is impossible.

If $I_0$ is not a prime right ideal then there exists $A$, $B$ in $L_\gamma(R)$ such that $AI_0 \subseteq I_0$, $AB \subseteq I_0$ but $A \nsubseteq I_0$ and $B \nsubseteq I_0$. Let $a \in A$ such that $a \notin I_0$. Then $I_0 + aR$ contains $I_0$ properly. Hence $I_0 + aR = x_1R + x_2R + \cdots + x_nR$ for some $x_1, x_2, \ldots, x_n$ in $R$. Let $J = \{x \in R \mid ax \in I_0\}$. Then $J$ contains $I_0 + B$. Since $B \nsubseteq I_0$, $J$ contains $I_0$ properly and hence $J = y_1R + y_2R + \cdots + y_mR$ for some $y_1, y_2, \ldots, y_m$ in $R$. Now $x_i = b_i + ar_i$ for some $b_i \in I_0$ and $r_i \in R$ for $i = 1, 2, \ldots, n$. Clearly $b_1R + b_2R + \cdots + b_nR + aJ \subseteq I_0$. If $w \in I_0$ then

$$\begin{aligned} w &= x_1c_1 + x_2c_2 + \cdots + x_nc_n = (b_1 + ar_1)c_1 + \cdots + (b_n + ar_n)c_n \\ &= b_1c_1 + \cdots + b_nc_n + a(r_1c_1 + \cdots + ar_nc_n) \end{aligned}$$

for some $c_1, c_2, \ldots, c_n$ in $R$. Since $r_1c_1 + \cdots + r_nc_n \in J$,

$$I_0 \subseteq b_1R + b_2R + \cdots + b_nR + aJ.$$



https://doi.org/10.4153/CMB-1971-047-3 Published online by Cambridge University Press

Since $aJ=ay_1R+ay_2R+\cdots+ay_mR$, $I_0$ is finitely generated. This is impossible. Now to prove the second part of the theorem, assume that every prime right ideal is principal. Suppose there is a right ideal $I$ such that it is not generated by one element. Again, by Zorn's lemma, one can choose, say $I_0 \in L_\gamma(R)$ such that $I_0 \supseteq I$, $I_0$ is not a principal right ideal and if $J \in L_\gamma(R)$ such that $J \supsetneq I_0$ then either $J=I_0$ or $J$ is a principal right ideal. We claim $I_0$ is a prime right ideal. If not, there exists $A$, $B$ in $L_\gamma(R)$, $AI_0 \subseteq I_0$ such that $AB \subseteq I_0$ but $A \nsubseteq I_0$ and $B \nsubseteq I_0$. Let $a \in A$ such that $a \notin I_0$. Then $I_0+aR=bR$ for some $b \in R$ since $I_0+aR$ contains $I_0$ properly. $b=i_0+ar_0$ for some $i_0 \in I_0$ and $r_0 \in R$. Let $b^{-1}I_0=\{x \in R \mid bx \in I_0\}$. Then $b(b^{-1}I_0)=I_0$ since $I_0 \subseteq bR$. We claim that $b^{-1}I_0=I_0$. Since $b=i_0+ar_0$, $I_0 \subseteq b^{-1}I_0$. If $b^{-1}I_0 \neq I_0$ then $b^{-1}I_0=dR$ for some $d \in R$. Therefore, $I_0=b(b^{-1}I_0)=bdR$ which is contrary to the assumption that $I_0$ is not principal. Hence $b^{-1}I_0=I_0$. Now consider $(ar_0)^{-1}I_0$, i.e. the right ideal $\{x \in R \mid ar_0x \in I_0\}$. Then $I_0 \subseteq (ar_0)^{-1}I_0$. Since $B \subseteq (ar_0)^{-1}I_0$, $(ar_0)^{-1}I_0 \neq I_0$. However, since $b=i_0+ar_0$, $(ar_0)^{-1}I_0=b^{-1}I_0=I_0$. This is impossible. Therefore $I_0$ is a prime right ideal and it is principal by hypothesis. Thus the assumption that there is a right ideal which is not a principal right ideal is invalid.

**Proof of Theorem 2.** If $R$ is a simple ring and $aR=aR_1$ for each $a \in R$ then every right ideal $I$ of $R$ is prime. For if $AB \subseteq I$, for some $A$, $B \in L_\gamma(R)$ then $ARB \subseteq I$ and $AR \subseteq I$ if $RB \neq 0$, since $RB$ is a two sided ideal. To prove the converse, we need the following Lemma:

LEMMA. *If every right ideal of $R$ is prime then $aR=aR_1$ and $R$ is semi-simple.*

**Proof.** If $aR \neq aR_1$ for some $a \in R$ then $(aR_1)R \subseteq aR$ implies that $aR$ is not a prime right ideal. Therefore $aR=aR_1$ for each $a \in R$. This means that, first of all, $R$ is not a radical ring. Let $J(R)$ denote the Jacobson radical of $R$. Then $J(R) \neq R$. First, we observe that if $A \in L_\gamma(R)$ and $S$ is an ideal of $R$ then $A(A \cap S) \subseteq A \cap S$. Since $A \cap S$ is a prime right ideal and $AS \subseteq A \cap S$, either $A \subseteq A \cap S$ or $S \subseteq A \cap S$. Furthermore, $A=A^2$ and $S=S^2$ since $A^2$ and $S^2$ are both prime right ideals. Therefore, either $A=A^2 \subseteq A(A \cap S) \subseteq AS \subseteq A$ or $S=S^2 \subseteq (A \cap S)S \subseteq AS \subseteq S$. Thus whether $A \subseteq A \cap S$ or $S \subseteq A \cap S$, $AS=A \cap S$. Now suppose $J(R) \neq 0$ and let $0 \neq x \in J(R)$. Let $I^*(x) \in L_\gamma(R)$ such that $x \notin I^*(x)$ and $I^*(x)$ is maximal with respect to this property. Let $A=xR+I^*(x)$. Then $A/I^*(x)$ is a simple $R$-module. Hence, $AJ(R) \subseteq I^*(x)$. Since $AJ(R)=A \cap J(R)$, $x \in A \cap J(R) \subseteq I^*(x)$. This is impossible. Therefore, $J(R)=0$. Now to conclude the proof of Theorem 2, let $S$ be a two sided ideal of $R$ such that $R \neq S$. Let $I$ be a maximal right ideal of $R$. Then $IS \subseteq I \cap S$ and either $I \subseteq I \cap S$ or $S \subseteq I \cap S$. In any case, $S \subseteq I$. Hence $S \subseteq J(R)=(0)$.

NORTH CAROLINA STATE UNIVERSITY,
RALEIGH, NORTH CAROLINA