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RESEARCH ARTICLE

Ordering and aging properties of systems with dependent components governed by the Archimedean copula

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Abstract
Copula is one of the widely used techniques to describe the dependency structure between components of a system. Among all existing copulas, the family of Archimedean copulas is the popular one due to its wide range of capturing the dependency structures. In this paper, we consider the systems that are formed by dependent and identically distributed components, where the dependency structures are described by Archimedean copulas. We study some stochastic comparisons results for series, parallel, and general r-out-of-n systems. Furthermore, we investigate whether a system of used components performs better than a used system with respect to different stochastic orders. Furthermore, some aging properties of these systems have been studied. Finally, some numerical examples are given to illustrate the proposed results.

1. Introduction

Modern industries use different kinds of systems which are not only costly but also very complex in nature. The failure of such a system may cause catastrophic damage to the concerned industry. If there are more than one systems of similar types available (which is often the case), then the key question is: how to choose the best one among them, keeping in mind that the lifetime of a system is a random quantity? Stochastic orders are often used as an effective solution to this problem. Another important problem related to the system reliability is: how to analyze the lifetime behavior of a system which has already been operated over a period of time? The notion of stochastic agings can be used to address this problem.

Most of the real-life systems are structurally the same as the coherent systems defined in reliability theory (see [5] for the definition). An r-out-of-n system is a special type of coherent system. A system of n components is said to be the r-out-of-n system if it functions as long as at least r of its n components function. Again, two special cases of r-out-of-n systems are 1-out-of-n system (parallel system) and n-out-of-n system (series system). It is worthwhile to mention here that the lifetime of an r-out-of-n system can be represented by the $(n-r+1)$th order statistic (of lifetimes of n components). This means that the study of an r-out-of-n system is the same as the study of the $(n-r+1)$th order statistic of nonnegative random variables.

The study of stochastic comparisons of coherent systems is considered as one of the important problems in reliability theory. Among all, Pledger and Proschan [43], to the best of our knowledge, are the first who studied stochastic comparisons of two r-out-of-n systems with heterogeneous components. Different variations of this problem were further studied by numerous researchers (see [2,6,10,12,22,23,25,45,46,50,51], to name a few). Stochastic comparisons of coherent systems with independent and nonidentically distributed components were considered in [8,9,34]. Furthermore, the same problem for coherent systems with dependent components was studied by Navarro [35], Navarro *et al.* [38,40,41], Navarro and Rubio [37], Amini-Seresht *et al.* [1], Hazra and Finkelstein [18], Navarro and Mulero [36], Kelkinnama and Asadi [24], and Hazra and Misra [19,20]. Recently, Li and Fang [30] studied ordering properties of order statistics where the dependency structure between random variables was described by an Archimedean copula. They have considered the order statistics from a single sample. Later, this problem for two samples under a specific semi-parametric model was considered by numerous researchers (see [7,14,32], and the references therein). In all these studies, it is assumed that the lifetimes of the components of a system either have a specific distribution (namely, exponential, Weibull, Gamma, etc.) or follow a semi-parametric model (namely, proportional hazard rate model, scale model, proportional odds model, etc.). To the best of our knowledge, no study has been carried out for general systems (namely, r-out-of-n system, etc.) with components' lifetimes having arbitrary distributions (i.e., without any specific distribution/semi-parametric model). Thus, in this paper, we study some stochastic comparisons results for series, parallel, and general r-out-of-n systems with arbitrary components' lifetimes.

The systems that are used in real life are mostly composed either by new components or by used components. Consider two systems, namely, a used system and a system of used components (see the definitions in Subsection 3.2). An important research problem in this context is whether a system of used components has larger lifetime than a used system in some stochastic sense. This problem was first considered in [32]. Later, many other researchers (namely, [16,17,21], and the references therein) have shown their interest to study this problem. Although there is a vast literature on this topic, no research has been carried out for the systems with dependent components governed by the Archimedean copula. Thus, another goal of this paper is to study the aforementioned problem for the systems with d.i.d. components governed by the Archimedean copula.

The study of stochastic agings is another important problem in reliability theory. Closure properties of various aging classes (namely, IFR, DFR, IFRA, DRFR, etc.) under the formation of coherent systems with independent components were studied by Esary and Proschan [13], Barlow and Proschan [5], Sengupta and Nanda [48], Franco *et al.* [15], Lai and Xie [29], and others. The same problem for coherent systems with dependent components was considered in [36,39]. However, the study of aging properties of coherent systems (especially, r-out-of-n systems) with dependent components governed by the Archimedean copula has not yet been considered in the literature. In this paper, we focus to study the closure properties of different aging classes under the formation of r-out-of-n systems with dependent and identically distributed components, where the dependency structures are described by Archimedean copulas.

The rest of the paper is organized as follows. In Section 2, we discuss some useful concepts and introduce some notations and definitions. In Section 3, we discuss the main results of this paper. To be more specific, in Subsection 3.1, we give some stochastic comparisons results for series, parallel, and general r-out-of-n systems. In Subsection 3.2, we investigate whether a system of used components performs better than a used system with respect to different stochastic orders. Different aging properties of series, parallel, and general r-out-of-n systems are discussed in Subsection 3.3. Furthermore, in Section 4, we give some numerical examples. Finally, the concluding remarks are given in Section 5.

All proofs of theorems, wherever given, are deferred to the Appendix.

2. Preliminaries

For an absolutely continuous random variable Z, we denote the probability density function (pdf) by $f_Z(\cdot)$, the cumulative distribution function (cdf) by $F_Z(\cdot)$, the reliability/survival function (sf) by $\bar{F}_Z(\cdot)$, the hazard/failure rate function by $r_Z(\cdot)$, and the reverse hazard/failure rate function by $\tilde{r}_Z(\cdot)$; here $\bar{F}_Z(\cdot) \equiv 1 - F_Z(\cdot)$, $r_Z(\cdot) = f_Z(\cdot)/\bar{F}_Z(\cdot)$, and $\tilde{r}_Z(\cdot) = f_Z(\cdot)/F_Z(\cdot)$.

Copula is a very useful notion in describing the dependency structure between components of a random vector. It builds a bridge between a multivariate distribution function and its corresponding one-dimensional marginal distribution functions. The joint cdf of a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ can be written, in terms of a copula, as

$$\begin{aligned} F_{\boldsymbol{X}}(x_1, x_2, \ldots, x_n) &= P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) \\ &= C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)), \end{aligned}$$

where $C(\cdot)$ is a copula. Similarly, the joint reliability function of $\boldsymbol{X}$ can be represented as

$$\begin{aligned} \bar{F}_{\boldsymbol{X}}(x_1, x_2, \ldots, x_n) &= P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \\ &= \bar{C}(\bar{F}_{X_1}(x_1), \bar{F}_{X_2}(x_2), \ldots, \bar{F}_{X_n}(x_n)), \end{aligned}$$

where $\bar{C}(\cdot)$ is a survival copula. In the literature, different types of survival copulas have been introduced to describe different dependency structures between components of a random vector. The commonly used copulas are the Farlie–Gumbel–Morgenstern (FGM) copula, the extreme-value copulas, the family of Archimedean copulas, the Clayton–Oakes (CO) copula, etc. Among all these copulas, the family of Archimedean copulas is the one that has paid more attention from the researchers due to its wide range of capturing the dependency structures. Moreover, these are mathematically tractable, and there is a large number of results available in the literature which can be used on a ready-made basis in different problems. More information on this topic could be found in the monograph written by Nelsen [42]. In what follows, we give the definition of the Archimedean copula (see [33]).

Definition 2.1. *Let $\phi : [0, +\infty] \longrightarrow [0, 1]$ be a decreasing continuous function such that $\phi(0) = 1$ and $\phi(+\infty) = 0$, and let $\psi \equiv \phi^{-1}$ be the pseudo-inverse of ϕ. Then,*

$$C(u_1, u_2, \ldots, u_n) = \phi\left(\psi(u_1) + \psi(u_2) + \cdots + \psi(u_n)\right), \quad \textit{for } (u_1, u_2, \ldots, u_n) \in [0, 1]^n, \tag{2.1}$$

is called the Archimedean copula with generator ϕ if $(-1)^k \phi^{(k)}(x) \ge 0$, for $k = 0, 1, \ldots, n-2$, and $(-1)^{n-2}\phi^{(n-2)}(x)$ is decreasing and convex in $x \ge 0$.

Stochastic orders are frequently used to compare two random variables/vectors. In the last seven decades, it has widely been used in various disciplines of science and engineering including actuarial science, econometrics, finance, risk management, and reliability theory. In the literature, different types of stochastic orders (namely, usual stochastic order, hazard rate order, dispersive order, Lorenz order, etc.) have been developed to study different kinds of problems. This topic is explicitly covered in the book written by Shaked and Shanthikumar [49]. Below we give the definitions of stochastic orders that are used in this paper (see [44,47,49]).

Definition 2.2. *Let X and Y be two absolutely continuous random variables with nonnegative supports. Then, X is said to be smaller than Y in the*

(a) usual stochastic order, denoted by $X \le_{st} Y$, if $\bar{F}_X(x) \le \bar{F}_Y(x)$ for all $x \in [0, \infty)$;
(b) hazard rate order, denoted by $X \le_{hr} Y$, if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in [0, \infty)$;
(c) reversed hazard rate order, denoted by $X \le_{rhr} Y$, if $F_Y(x)/F_X(x)$ is increasing in $x \in [0, \infty)$;
(d) likelihood ratio order, denoted by $X \le_{lr} Y$, if $f_Y(x)/f_X(x)$ is increasing in $x \in (0, \infty)$;
(e) aging faster order in terms of the failure rate, denoted by $X \le_c Y$, if $r_X(x)/r_Y(x)$ is increasing in $x \in [0, \infty)$;
(f) aging faster order in terms of the reversed failure rate, denoted by $X \le_b Y$, if $\tilde{r}_X(x)/\tilde{r}_Y(x)$ is decreasing in $x \in [0, \infty)$.

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Like stochastic orders, the notion of stochastic agings is another important concept in reliability theory. Stochastic agings largely describe how a system behaves as time progresses. There are three types of agings, namely, positive aging, negative aging, and no aging. A system has the positive aging property if its residual lifetime decreases in some stochastic sense as time progresses. On the other hand, negative aging describes the scenario where the residual lifetime of a system increases in some stochastic sense as the system ages. No aging means that the system does not age over time. A variety of positive and negative aging classes (namely, IFR, IFRA, DFR, DLR, etc.) have been introduced in the literature to describe different aging characteristics of a system (see [5,29], and the references therein). For the sake of completeness, we give the following definitions of aging classes that are used in this paper.

Definition 2.3. *Let X be an absolutely continuous random variable with nonnegative support. Then, X is said to have the*

(a) increasing likelihood ratio (ILR) (resp. decreasing likelihood ratio (DLR)) property if $f'_X(x)/f_X(x)$ is decreasing (resp. increasing) in $x \geq 0$;
(b) increasing failure rate (IFR) (resp. decreasing failure rate (DFR)) property if $r_X(x)$ is increasing (resp. decreasing) in $x \geq 0$;
(c) decreasing reversed failure rate (DRFR) property if $\tilde{r}_X(x)$ is decreasing in $x \geq 0$;
(d) increasing failure rate in average (IFRA) (resp. decreasing failure rate in average (DFRA)) property if $-\ln \bar{F}_X(x)/x$ is increasing (resp. decreasing) in $x \geq 0$.

Throughout the paper, we use the words "increasing" and "decreasing" to mean "nondecreasing" and "nonincreasing", respectively. Furthermore, the words "positive" and "negative" mean "nonnegative" and "nonpositive", respectively. By $a \stackrel{\text{def.}}{=} b$, we mean that a is defined as b. For a twice differentiable function $u(\cdot)$, we write $u'(t)$ and $u''(t)$ to mean the first and the second derivatives of $u(t)$ with respect to t. We use the acronyms "i.i.d." and "d.i.d." to mean "independent and identically distributed" and "dependent and identically distributed", respectively. All random variables considered in this paper are assumed to be absolutely continuous with nonnegative supports.

3. Main results

In this section, we first discuss some stochastic comparisons results for series, parallel, and general r-out-of-n systems. Then, we study whether a system of used components performs better than a used system with respect to different stochastic orders. Lastly, the closure properties of different aging classes under formations of different systems have been studied.

Let $\tau^{\phi_1}_{r|n}(\boldsymbol{X})$ and $\tau^{\phi_2}_{r|n}(\boldsymbol{Y})$ be the lifetimes of two r-out-of-n systems formed by two different sets of n d.i.d. components with the lifetime vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$, respectively, where the distributions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are described by the Archimedean copulas with generators $\phi_1(\cdot)$ and $\phi_2(\cdot)$, respectively. Furthermore, let $X_i \stackrel{d}{=} X$ and $Y_i \stackrel{d}{=} Y$, $i = 1, 2, \ldots, n$, for some nonnegative random variables X and Y; here $\stackrel{d}{=}$ stands for equality in distribution. In what follows, we introduce some notation:

$$H_i(u) = \frac{u\phi'_i(u)}{1 - \phi_i(u)}, \quad R_i(u) = \frac{u\phi'_i(u)}{\phi_i(u)}, \quad \text{and} \quad G_i(u) = \frac{u\phi''_i(u)}{\phi'_i(u)}, \quad u > 0,\ i = 1, 2.$$

Since $\phi_i(\cdot)$, $i = 1, 2$, are decreasing convex functions, it follows that $H_i(\cdot)$, $R_i(\cdot)$, and $G_i(\cdot)$ are all negative-valued functions. Furthermore, we write $\tau^{\phi_1}_{r|n} = \tau^{\phi_2}_{r|n} = \tau^{\phi}_{r|n}$ (say), $H_1(\cdot) = H_2(\cdot) = H(\cdot)$ (say), $R_1(\cdot) = R_2(\cdot) = R(\cdot)$ (say), and $G_1(\cdot) = G_2(\cdot) = G(\cdot)$ (say) whenever the distributions of both $\boldsymbol{X}$ and $\boldsymbol{Y}$ are described by the same Archimedean copula with generator $\phi_1(\cdot) = \phi_2(\cdot) = \phi(\cdot)$ (say). Furthermore, we introduce a few more notations as follows:

$$C(u) = \left(\frac{u\phi''(u)}{\phi'(u)} + \frac{u\phi'(u)}{1-\phi(u)} + 1\right) = \frac{uH'(u)}{H(u)}, \quad u > 0,$$

$$C^j_{r,n} = (-1)^{j-r}\binom{j-1}{r-1}\binom{n}{j}, \quad 1 \le r \le j \le n,$$

$$P^j_{r,n}(u) = \frac{C^j_{r,n}\phi(ju)}{\sum_{j=r}^n C^j_{r,n}\phi(ju)}, \quad Q^j_{r,n}(u) = \frac{C^j_{r,n}(1-\phi(ju))}{1-\sum_{j=r}^n C^j_{r,n}\phi(ju)}, \quad u > 0,\ 1 \le r \le j \le n,$$

$$K_j(u) = \frac{R(ju)}{R(u)}, \quad L_j(u) = \frac{H(ju)}{H(u)}, \quad u > 0,\ 1 \le j \le n.$$

3.1. Stochastic comparisons of two systems

In this subsection, we study some comparisons results for series, parallel, and general r-out-of-n systems using different stochastic orders.

In the following theorem, we compare two r-out-of-n systems with respect to the usual stochastic order, the hazard rate order, and the reversed hazard rate order. Here, we assume that both systems have the same dependency structure described by the Archimedean copula with generator ϕ. The proof of the first part of this theorem is straightforward, whereas the proof of the third part is similar to that of the second part and hence omitted.

Theorem 3.1. *The following results hold true.*

(a) Assume that $\sum_{j=r}^n C^j_{r,n}\phi(ju)$ *(or* $\sum_{j=n-r+1}^n C^j_{n-r+1,n}\phi(ju)$*) is decreasing in* $u > 0$. *If* $X \le_{st} Y$, *then* $\tau^\phi_{r|n}(\boldsymbol{X}) \le_{st} \tau^\phi_{r|n}(\boldsymbol{Y})$;

(b) Assume that $\sum_{j=r}^n P^j_{r,n}(u)K_j(u)$ *is increasing in* $u > 0$, *or* $\sum_{j=n-r+1}^n Q^j_{n-r+1,n}(u)L_j(u)$ *is decreasing in* $u > 0$. *If* $X \le_{hr} Y$, *then* $\tau^\phi_{r|n}(\boldsymbol{X}) \le_{hr} \tau^\phi_{r|n}(\boldsymbol{Y})$;

(c) Assume that $\sum_{j=n-r+1}^n P^j_{n-r+1,n}(u)K_j(u)$ *is increasing in* $u > 0$, *or* $\sum_{j=r}^n Q^j_{r,n}(u)L_j(u)$ *is decreasing in* $u > 0$. *If* $X \le_{rh} Y$, *then* $\tau^\phi_{r|n}(\boldsymbol{X}) \le_{rh} \tau^\phi_{r|n}(\boldsymbol{Y})$.

Stochastic comparisons of two series/parallel systems (in the sense of the usual stochastic order, the hazard rate order, and the reversed hazard rate order) are given in the following corollary which is obtained from Theorem 3.1.

Corollary 3.1. *The following results hold true.*

(a) If $X \le_{st} Y$, *then* $\tau^\phi_{1|n}(\boldsymbol{X}) \le_{st} \tau^\phi_{1|n}(\boldsymbol{Y})$ *and* $\tau^\phi_{n|n}(\boldsymbol{X}) \le_{st} \tau^\phi_{n|n}(\boldsymbol{Y})$;

(b) Assume that $uH'(u)/H(u)$ *is decreasing in* $u > 0$. *If* $X \le_{hr} Y$, *then* $\tau^\phi_{1|n}(\boldsymbol{X}) \le_{hr} \tau^\phi_{1|n}(\boldsymbol{Y})$;

(c) Assume that $uR'(u)/R(u)$ *is increasing in* $u > 0$. *If* $X \le_{hr} Y$, *then* $\tau^\phi_{n|n}(\boldsymbol{X}) \le_{hr} \tau^\phi_{n|n}(\boldsymbol{Y})$;

(d) Assume that $uR'(u)/R(u)$ *is increasing in* $u > 0$. *If* $X \le_{rh} Y$, *then* $\tau^\phi_{1|n}(\boldsymbol{X}) \le_{rh} \tau^\phi_{1|n}(\boldsymbol{Y})$;

(e) Assume that $uH'(u)/H(u)$ *is decreasing in* $u > 0$. *If* $X \le_{rh} Y$, *then* $\tau^\phi_{n|n}(\boldsymbol{X}) \le_{rh} \tau^\phi_{n|n}(\boldsymbol{Y})$.

In the next theorem, we compare two series/parallel systems with respect to the likelihood ratio order and the aging faster orders. The proof of the third part of this theorem can be done in the same line as in the second part and hence omitted.

Theorem 3.2. *The following results hold true.*

(a) Assume that $(G(nu) - G(u))/R(u)$ *is positive and increasing in* $u > 0$. *If* $X \le_{lr} Y$, *then* $\tau^\phi_{1|n}(\boldsymbol{X}) \le_{lr} \tau^\phi_{1|n}(\boldsymbol{Y})$ *and* $\tau^\phi_{n|n}(\boldsymbol{X}) \le_{lr} \tau^\phi_{n|n}(\boldsymbol{Y})$;

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(b) Assume that $(C(nu) - C(u))/R(u)$ *is positive and increasing in* $u > 0$. *If* $Y \leq_{rh} X$ *and* $X \leq_c Y$, *then* $\tau^{\phi}_{1|n}(\boldsymbol{X}) \leq_c \tau^{\phi}_{1|n}(\boldsymbol{Y})$;

(c) Assume that $(C(nu) - C(u))/R(u)$ *is positive and increasing in* $u > 0$. *If* $X \leq_{hr} Y$ *and* $X \leq_b Y$, *then* $\tau^{\phi}_{n|n}(\boldsymbol{X}) \leq_b \tau^{\phi}_{n|n}(\boldsymbol{Y})$.

The following remark provides sufficient conditions for the assumptions given in Theorem 3.2.

Remark 3.1. *The following observations can be made.*

(a) If $uG'(u)/G(u)$ *is positive and increasing (resp. decreasing) in* $u > 0$, *and* $G(u)/R(u)$ *is increasing (resp. decreasing) in* $u > 0$, *then* $(G(nu) - G(u))/R(u)$ *is positive and increasing (resp. decreasing) in* $u > 0$.

(b) If both $uC'(u)/C(u)$ *and* $C(u)/R(u)$ *are positive and increasing in* $u > 0$, *then* $(C(nu) - C(u))/R(u)$ *is positive and increasing in* $u > 0$.

In the following theorem, we compare two parallel systems with respect to the usual stochastic order, the hazard rate order, and the reversed hazard rate order. Here, we assume that the dependency structure of one system differs from that of the other system. The proof of the third part of this theorem can be done in the same line, as in the second part, and hence omitted.

Theorem 3.3. *The following results hold true.*

(a) Assume that $\phi_2^{-1}(\phi_1(u))$ *is sub-additive in* $u > 0$. *If* $X \leq_{st} Y$, *then* $\tau^{\phi_1}_{1|n}(\boldsymbol{X}) \leq_{st} \tau^{\phi_2}_{1|n}(\boldsymbol{Y})$;

(b) Assume that $\psi_1(w) \leq \psi_2(v)$, *for all* $0 \leq v \leq w \leq 1$, *and* $H_1(u)/H_2(u)$ *is increasing in* $u > 0$. *Furthermore, assume that either* $uH_1'(u)/H_1(u)$ *or* $uH_2'(u)/H_2(u)$ *is decreasing in* $u > 0$. *If* $X \leq_{hr} Y$, *then* $\tau^{\phi_1}_{1|n}(\boldsymbol{X}) \leq_{hr} \tau^{\phi_2}_{1|n}(\boldsymbol{Y})$;

(c) Assume that $\psi_1(w) \leq \psi_2(v)$, *for all* $0 \leq v \leq w \leq 1$, *and* $R_1(u)/R_2(u)$ *is decreasing in* $u > 0$. *Furthermore, assume that either* $uR_1'(u)/R_1(u)$ *or* $uR_2'(u)/R_2(u)$ *is increasing in* $u > 0$. *If* $X \leq_{rh} Y$, *then* $\tau^{\phi_1}_{1|n}(\boldsymbol{X}) \leq_{rh} \tau^{\phi_2}_{1|n}(\boldsymbol{Y})$.

In the next theorem, we study the same set of results, as in Theorem 3.3, for the series system. The proofs are similar to those of Theorem 3.3 and hence omitted.

Theorem 3.4. *The following results hold true.*

(a) Assume that $\phi_1^{-1}(\phi_2(u))$ *is sub-additive in* $u > 0$. *If* $X \leq_{st} Y$, *then* $\tau^{\phi_1}_{n|n}(\boldsymbol{X}) \leq_{st} \tau^{\phi_2}_{n|n}(\boldsymbol{Y})$;

(b) Assume that $\psi_1(w) \geq \psi_2(v)$, *for all* $0 \leq w \leq v \leq 1$, *and* $R_1(u)/R_2(u)$ *is increasing in* $u > 0$. *Furthermore, assume that either* $uR_1'(u)/R_1(u)$ *or* $uR_2'(u)/R_2(u)$ *is increasing in* $u > 0$. *If* $X \leq_{hr} Y$, *then* $\tau^{\phi_1}_{n|n}(\boldsymbol{X}) \leq_{hr} \tau^{\phi_2}_{n|n}(\boldsymbol{Y})$;

(c) Assume that $\psi_1(w) \geq \psi_2(v)$, *for all* $0 \leq w \leq v \leq 1$, *and* $H_1(u)/H_2(u)$ *is decreasing in* $u > 0$. *Furthermore, assume that* $uH_1'(u)/H_1(u)$ *or* $uH_2'(u)/H_2(u)$ *is decreasing in* $u > 0$. *If* $X \leq_{rh} Y$, *then* $\tau^{\phi_1}_{n|n}(\boldsymbol{X}) \leq_{rh} \tau^{\phi}_{n|n}(\boldsymbol{Y})$.

Remark 3.2. *The following observations can be made.*

(a) If $\phi_2^{-1}(w)/\phi_1^{-1}(w)$ *is increasing in* $w \in (0, 1]$, *then* $\phi_2^{-1}(\phi_1(u))$ *is sub-additive in* $u > 0$;

(b) If $\phi_1(u) \leq \phi_2(u)$, *for all* $u > 0$, *then* $\psi_1(w) \leq \psi_2(v)$, *for all* $0 \leq v \leq w \leq 1$.

3.2. Stochastic comparisons between a used system and a system of used components

Let X be a random variable representing the lifetime of a component/system. Then, its residual lifetime at a time instant t (≥ 0) is given by $X_t = (X - t|X > t)$. We call X_t as a used component/system. Further, let

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$\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random vector representing the lifetimes of n d.i.d. components governed by the Archimedean copula with generator ϕ, where $X_i \stackrel{d}{=} X$, $i = 1, 2, \ldots, n$, for some nonnegative random variable X. We denote the vector of n used components by $\boldsymbol{X}_t = (X_1)_t, (X_2)_t, \ldots, (X_n)_t)$, for some fixed $t > 0$. Consequently, we denote the lifetimes of the parallel and the series systems made by a set of used components with the lifetime vector $\boldsymbol{X}_t$ by $\tau^{\phi}_{1|n}(\boldsymbol{X}_t)$ and $\tau^{\phi}_{n|n}(\boldsymbol{X}_t)$, respectively. Further, we denote the lifetimes of the used parallel system and the used series system formed by a set of components with the lifetime vector $\boldsymbol{X}$ by $(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t$ and $(\tau^{\phi}_{n|n}(\boldsymbol{X}))_t$, respectively, where $(\tau^{\phi}_{i|n}(\boldsymbol{X}))_t = \tau^{\phi}_{i|n}(\boldsymbol{X}) - t | \tau^{\phi}_{i|n}(\boldsymbol{X}) > t$, for $t \geq 0$ and $i = 1, n$.

In the following theorem, we compare a used system and a system made by used components with respect to different stochastic orders. Here, we particularly consider the series and the parallel systems formed by d.i.d. components governed by the Archimedean copula with generator ϕ. The proofs of the fourth, the sixth, and the seventh parts of this theorem are given in the Appendix. Furthermore, the proof of the first part is straightforward, whereas the proofs of the second, the third, and the fifth parts are similar to that of the fourth part and hence omitted.

Theorem 3.5. *The following results hold true.*

(a) $(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t \leq_{st} \tau^{\phi}_{1|n}(\boldsymbol{X}_t)$ *and* $(\tau^{\phi}_{n|n}(\boldsymbol{X}))_t \leq_{st} \tau^{\phi}_{n|n}(\boldsymbol{X}_t)$, *for any fixed* $t \geq 0$;

(b) *If* $uH'(u)/H(u)$ *is decreasing in* $u > 0$, *then* $(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t \leq_{hr} \tau^{\phi}_{1|n}(\boldsymbol{X}_t)$, *for any fixed* $t \geq 0$;

(c) *If* $uR'(u)/R(u)$ *is increasing in* $u > 0$, *then* $(\tau^{\phi}_{n|n}(\boldsymbol{X}))_t \leq_{hr} \tau^{\phi}_{n|n}(\boldsymbol{X}_t)$, *for any fixed* $t \geq 0$;

(d) *If* $uR'(u)/R(u)$ *is positive and increasing in* $u > 0$, *then* $(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t \leq_{rh} \tau^{\phi}_{1|n}(\boldsymbol{X}_t)$, *for any fixed* $t \geq 0$;

(e) *If* $uH'(u)/H(u)$ *is negative and decreasing in* $u > 0$, *then* $(\tau^{\phi}_{n|n}(\boldsymbol{X}))_t \leq_{rh} \tau^{\phi}_{n|n}(\boldsymbol{X}_t)$, *for any fixed* $t \geq 0$;

(f) *If* $(G(nu) - G(u))/R(u)$ *is positive and increasing in* $u > 0$, *then* $(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t \leq_{lr} \tau^{\phi}_{1|n}(\boldsymbol{X}_t)$ *and* $(\tau^{\phi}_{n|n}(\boldsymbol{X}))_t \leq_{lr} \tau^{\phi}_{n|n}(\boldsymbol{X}_t)$, *for any fixed* $t \geq 0$;

(g) *If* $(C(nu) - C(u))/R(u)$ *is positive and increasing in* $u > 0$, *then* $\tau^{\phi}_{1|n}(\boldsymbol{X}_t) \leq_{c} (\tau^{\phi}_{1|n}(\boldsymbol{X}))_t$, *for any fixed* $t \geq 0$.

3.3. *Preservation of aging classes under the formation of a system*

In this subsection, we discuss the closure properties of different aging classes under the formation of r-out-of-n systems.

In the following theorem, we provide some sufficient conditions to show that the IFR, the DFR, and the DRFR classes are preserved under the formation of an r-out-of-n system. The proof of the second part of this theorem is similar to that of the first part and hence omitted.

Theorem 3.6. *The following results hold true.*

(a) *Assume that* $\sum_{j=r}^{n} P^{j}_{r,n}(u) K_j(u)$ *is increasing (resp. decreasing) in* $u > 0$, *or* $\sum_{j=n-r+1}^{n} Q^{j}_{n-r+1,n}(u) L_j(u)$ *is decreasing (resp. increasing) in* $u > 0$. *If* X *is IFR (resp. DFR), then* $\tau^{\phi}_{r|n}(\boldsymbol{X})$ *is IFR (resp. DFR);*

(b) *Assume that* $\sum_{j=n-r+1}^{n} P^{j}_{n-r+1,n}(u) K_j(u)$ *is increasing in* $u > 0$, *or* $\sum_{j=r}^{n} Q^{j}_{r,n}(u) L_j(u)$ *is decreasing in* $u > 0$. *If* X *is DRFR, then* $\tau^{\phi}_{r|n}(\boldsymbol{X})$ *is DRFR.*

The following corollary immediately follows from Theorem 3.6.

Corollary 3.2. *The following results hold true.*

(a) *Assume that* $uR'(u)/R(u)$ *is increasing (resp. decreasing) in* $u > 0$. *If* X *is IFR (resp. DFR), then* $\tau^{\phi}_{n|n}(\boldsymbol{X})$ *is IFR (resp. DFR);*

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(b) *Assume that $uH'(u)/H(u)$ is decreasing (resp. increasing) in $u > 0$. If X is IFR (resp. DFR), then $\tau^{\phi}_{1|n}(\boldsymbol{X})$ is IFR (resp. DFR);*
(c) *Assume that $uH'(u)/H(u)$ is decreasing in $u > 0$. If X is DRFR, then $\tau^{\phi}_{n|n}(\boldsymbol{X})$ is DRFR;*
(d) *Assume that $uR'(u)/R(u)$ is increasing in $u > 0$. If X is DRFR, then $\tau^{\phi}_{1|n}(\boldsymbol{X})$ is DRFR.*

In the following theorem, we show that the ILR, the DLR, the IFRA, and the DFRA classes are preserved under the formation of the parallel and the series systems.

Theorem 3.7. *The following results hold true.*

(a) *Assume that $(G(nu) - G(u))/R(u)$ is positive and increasing (resp. decreasing) in $u > 0$. If X is ILR (resp. DLR), then both $\tau^{\phi}_{n|n}(\boldsymbol{X})$ and $\tau^{\phi}_{1|n}(\boldsymbol{X})$ are ILR (resp. DLR);*
(b) *Assume that $uR'(u)/R(u)$ is increasing (resp. decreasing) in $u > 0$. If X is IFRA (resp. DFRA), then $\tau^{\phi}_{n|n}(\boldsymbol{X})$ is IFRA (resp. DFRA);*
(c) *Assume that $H(u)/\ln(1-\phi(u))$ is increasing (resp. decreasing) in $u > 0$. If X is IFRA (resp. DFRA), then $\tau^{\phi}_{1|n}(\boldsymbol{X})$ is IFRA (resp. DFRA).*

4. Examples

In this section, we give some examples to illustrate the sufficient conditions used in the previous section. Here, we specifically consider the copulas that are frequently used in practice, namely, the Clayton copula $C(\boldsymbol{u}) = (\prod_{i=1}^{n} u_i^{-\theta} - n + 1)^{-1/\theta}$ with the generator $\phi(t) = (\theta t + 1)^{-1/\theta}$, for $\theta \geq 0$, the Ali-Mikhail-Haq (AMH) copula $C(\boldsymbol{u}) = ((1-\theta)\prod_{i=1}^{n} u_i)/(\prod_{i=1}^{n}(1-\theta+\theta u_i) - \theta \prod_{i=1}^{n} u_i)$ with the generator $\phi(t) = (1-\theta)/(e^t - \theta)$, for $\theta \in [0, 1)$, etc.

We begin with the following example that demonstrates the conditions given in Theorem 3.1(b) and (c), and Theorem 3.6.

Example 4.1. *Consider the Archimedean copula with generator*

$$\phi(u) = \frac{1-\alpha_1}{e^u - \alpha_1}, \quad \alpha_1 \in [-1, 1),\ u > 0.$$

Then,

$$k_1(u) \stackrel{def.}{=} \sum_{j=2}^{3} P_{2,3}^{j}(u) K_j(u) = \frac{(e^u - \alpha_1)\left(\frac{6e^u}{(e^{2u}-\alpha_1)^2} - \frac{6e^{2u}}{\left(e^{3u}-\alpha_1\right)^2}\right)}{\left(\frac{3}{e^{2u}-\alpha_1} - \frac{2}{e^{3u}-\alpha_1}\right)}, \quad u > 0,$$

and

$$k_2(u) \stackrel{def.}{=} \sum_{j=2}^{3} Q_{2,3}^{j}(u) L_j(u) = \frac{(e^u - \alpha_1)(e^u - 1)\left(\frac{6e^u}{(e^{2u}-\alpha_1)^2} - \frac{6e^{2u}}{(e^{3u}-\alpha_1)^2}\right)}{1 - (1-\alpha_1)\left(\frac{3}{e^{2u}-\alpha_1} - \frac{2}{e^{3u}-\alpha_1}\right)}, \quad u > 0.$$

In Figure 1, we plot $k_1(-\ln(v))$ against $v \in (0, 1]$, for fixed $\alpha_1 = 0.3, 0.4, 0.5,$ and 0.6. This shows that $k_1(-\ln(v))$ is decreasing in $v \in (0, 1]$ and hence, $k_1(u)$ is increasing in $u > 0$. Furthermore, we plot $k_2(-\ln(v))$ against $v \in (0, 1]$, for fixed $\alpha_1 = 0.8, 0.85,$ and 0.9. From Figure 2, we see that $k_2(-\ln(v))$ is increasing in $v \in (0, 1]$ and hence, $k_2(u)$ is decreasing in $u > 0$. Thus, the required conditions are satisfied.

The following example demonstrates the condition given in Corollary 3.1(c) and (d), Theorem 3.5(c) and (d), Corollary 3.2(a) and (d), and Theorem 3.7(b).

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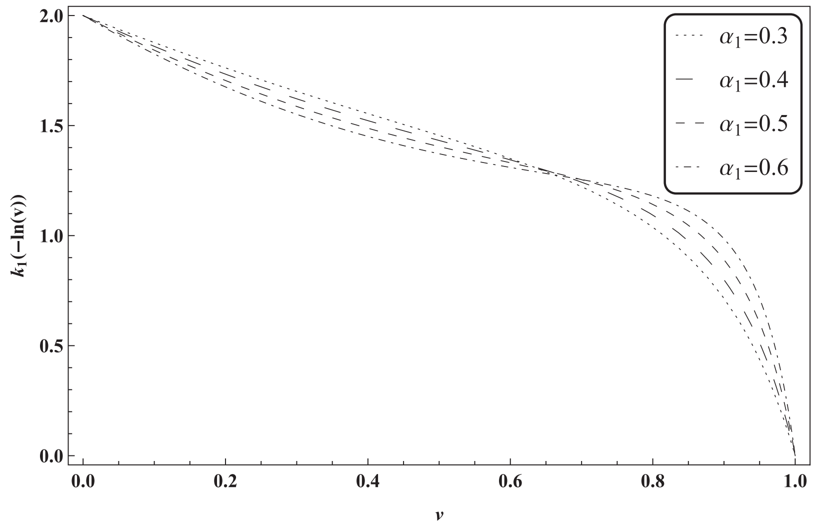

Figure 1. *Plot of $k_1(-\ln(v))$ against $v \in (0, 1]$.*

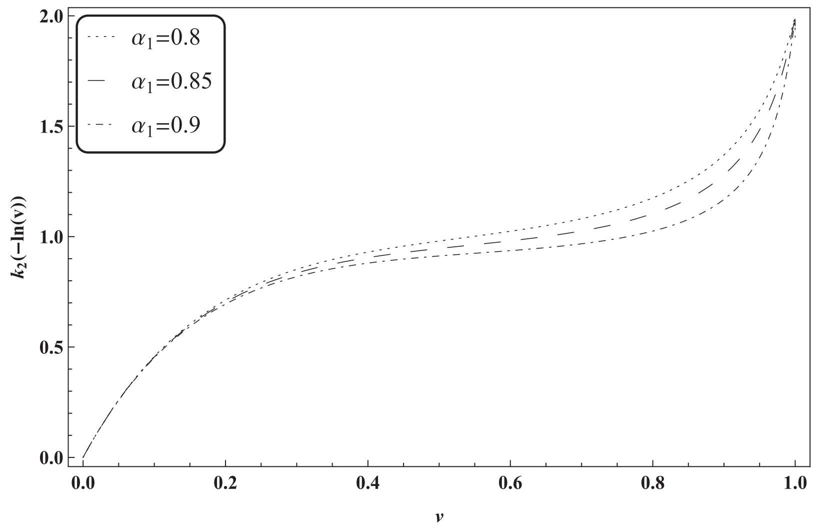

Figure 2. *Plot of $k_2(-\ln(v))$ against $v \in (0, 1]$.*

Example 4.2. *Consider the Archimedean copula with generator*

$$\phi(u) = e^{1-(1+u)^{1/\beta_1}}, \quad \beta_1 \in (0, \infty), \ u > 0.$$

From this, we have

$$R(u) = -\frac{1}{\beta_1} u(1+u)^{1/\beta_1 - 1}, \quad \textit{for all } u > 0,$$

and

$$l_1(u) \overset{def.}{=} \frac{uR'(u)}{R(u)} = 1 + \left(\frac{1}{\beta_1} - 1\right)\frac{u}{u+1}, \quad \textit{for all } u > 0.$$

It can be easily shown that $l_1(u)$ is positive and increasing in $u > 0$, for all $\beta_1 \in (0, 1)$. Thus, the required condition holds.

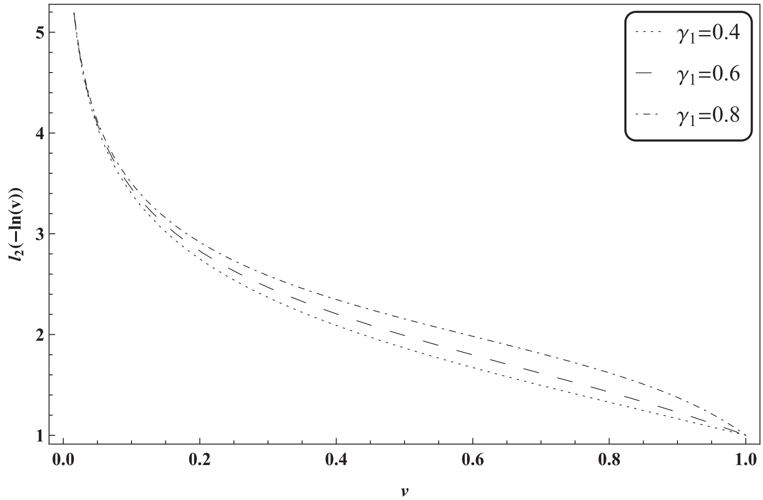

Figure 3. *Plot of* $l_2(-\ln(v))$ *against* $v \in (0,1]$.

The next example illustrates the condition given in Theorems 3.2(a), 3.5(f), and 3.7(a).

Example 4.3. *Consider the Archimedean copula with generator*

$$\phi(u) = e^{(1/\gamma_1)(1-e^u)}, \quad \gamma_1 \in (0,1],\ u > 0.$$

Then

$$l_2(u) \overset{def.}{=} \frac{uG'(u)}{G(u)} = \frac{\gamma_1 - e^u - ue^u}{\gamma_1 - e^u} \quad and \quad l_3(u) \overset{def.}{=} \frac{G(u)}{R(u)} = 1 - \frac{\gamma_1}{e^u}, \quad u > 0.$$

Let us fix $\gamma_1 = 0.4$, 0.6, *and* 0.8. *In Figure 3, we plot* $l_2(-\ln(v))$ *against* $v \in (0,1]$. *This shows that* $l_2(-\ln(v))$ *is positive and decreasing in* $v \in (0,1]$ *and hence,* $l_2(u)$ *is positive and increasing in* $u > 0$. *Furthermore, it can easily be checked that* $l_3(u)$ *is increasing in* $u > 0$. *Thus, the required condition holds from Remark 3.1.*

Below we cite an example that illustrates the condition given in Theorem 3.2(b) and (c), and Theorem 3.5(g).

Example 4.4. *Consider the Archimedean copula with generator*

$$\phi(u) = e^{-u^{1/\alpha_2}}, \quad \alpha_2 \in [1,\infty),\ u > 0,$$

which gives

$$R(u) = -\frac{1}{\alpha_2}u^{1/\alpha_2} \quad and \quad C(u) = \frac{1}{\alpha_2} - \frac{u^{1/\alpha_2}e^{-u^{1/\alpha_2}}}{\alpha_2\left(1 - e^{-u^{1/\alpha_2}}\right)} - \frac{1}{\alpha_2}u^{1/\alpha_2}, \quad u > 0.$$

Let us fix $\alpha_2 = 7$ *and* 8. *Furthermore, let* $k_3(u) = uC'(u)/C(u)$ *and* $k_4(u) = C(u)/R(u)$, *for* $u > 0$. *From Figures 4 and 5, we see that both* $k_3(-\ln(v))$ *and* $k_4(-\ln(v))$ *are decreasing and positive for all* $v \in (0,1]$. *Hence, both* $k_3(u)$ *and* $k_4(u)$ *are increasing and positive for all* $u > 0$. *Thus, the required conditions hold from Remark 3.1.*

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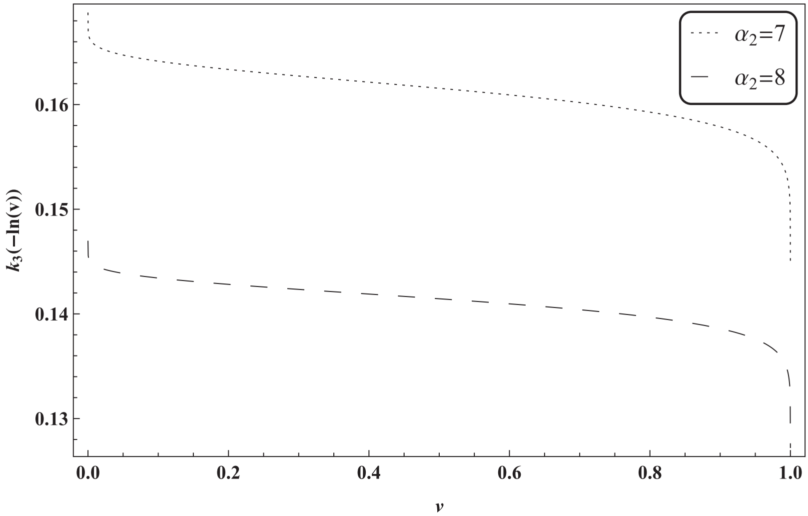

Figure 4. *Plot of* $k_3(-\ln(v))$ *against* $v \in (0, 1]$.

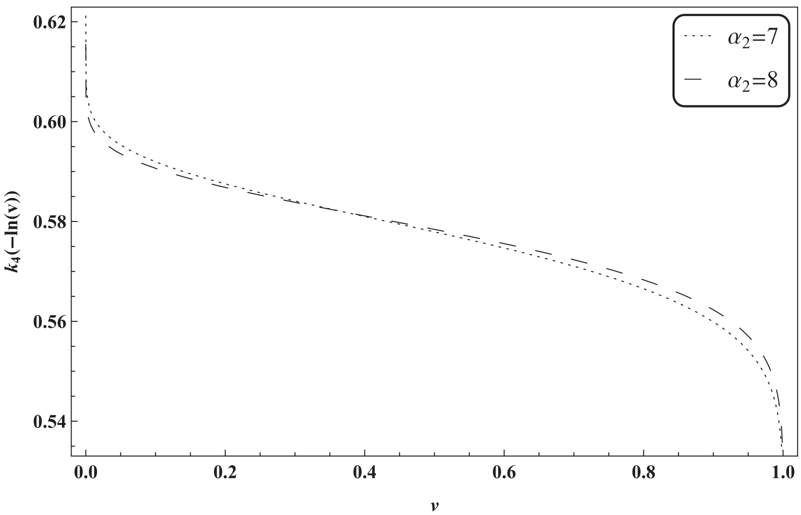

Figure 5. *Plot of* $k_4(-\ln(v))$ *against* $v \in (0, 1]$.

The next example demonstrates the condition given in Theorem 3.3(b). Furthermore, this may also be used to illustrate the condition given in Corollary 3.1(b) and (e), Theorem 3.5(b) and (e), and Corollary 3.2(b) and (c).

Example 4.5. *Consider two parallel systems* $\tau_{1|n}^{\phi_1}(\boldsymbol{X})$ *and* $\tau_{1|n}^{\phi_2}(\boldsymbol{Y})$*, where* $X \leq_{hr} Y$*. Consider the Archimedean copulas with generators*

$$\phi_1(u) = 1 - (1 - e^{-u})^{1/\beta_2}, \quad \beta_2 \in [1, \infty),\ t > 0,$$

and

$$\phi_2(u) = \frac{1 - \gamma_2}{e^t - \gamma_2}, \quad \gamma_2 \in [-1, 1),\ t > 0,$$

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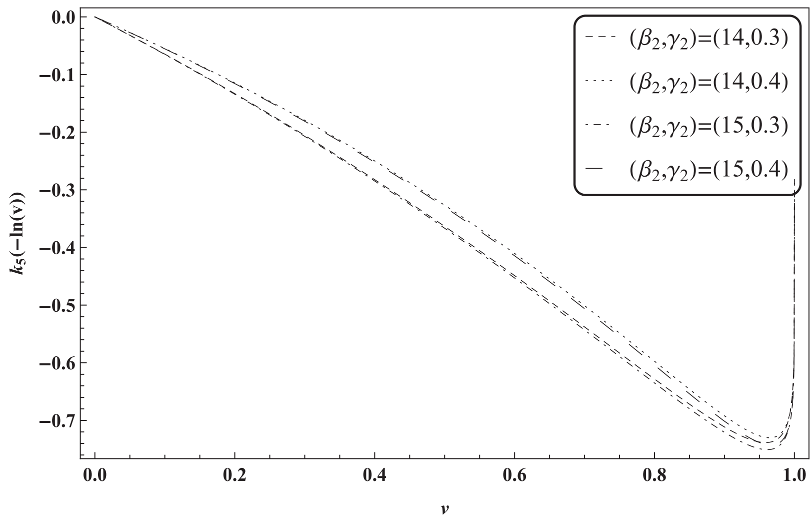

Figure 6. *Plot of* $k_5\,(-\ln(v))$ *against* $v \in (0,1]$.

respectively. Consequently,

$$k_5(u) \overset{def.}{=} \phi_1(u) - \phi_2(u) = (1-(1-e^{-u})^{1/\beta_2}) - \left(\frac{1-\gamma_2}{e^u-\gamma_2}\right), \quad u > 0.$$

Let us fix $(\beta_2, \gamma_2) = (14, 0.3), (14, 0.4), (15, 0.3)$, *and* $(15, 0.4)$. *Figure 6 shows that* $k_5\,(-\ln(v))$ *is negative for all* $v \in (0,1]$ *and hence,* $k_5(u)$ *is negative for all* $u > 0$. *Thus,* $\phi_1(u) \le \phi_2(u)$, *for all* $u > 0$, *which, by Remark 3.2, gives* $\psi_1(w) \le \psi_2(v)$, *for all* $0 \le v \le w \le 1$. *Again,*

$$k_6(u) \overset{def.}{=} \frac{H_1(u)}{H_2(u)} = \frac{e^u - \gamma_2}{\beta_2\,(1-\gamma_2)\,e^u}$$

$$\text{and} \quad k_7(u) \overset{def.}{=} \frac{uH_2'(u)}{H_2(u)} = -\frac{ue^u}{e^u-\gamma_2} - \frac{ue^u}{e^u-1} + 1 + u, \quad u > 0.$$

From Figure 7, we see that $k_6(-\ln(v))$ *is decreasing in* $v \in (0,1]$, *whereas Figure 8 shows that* $k_7(-\ln(v))$ *is increasing and negative for all* $v \in (0,1]$. *Hence,* $k_6(u)$ *is increasing in* $u > 0$, *and* $k_7(u)$ *is decreasing and negative for all* $u > 0$. *Thus, the condition of Theorem 3.3(b) holds.*

Below we give an example that illustrates the condition given in Theorem 3.4(a).

Example 4.6. *Consider the Archimedean copulas with generators*

$$\phi_1(u) = -\frac{1}{\alpha_3}\ln(e^{-u}(e^{-\alpha_3}-1)), \quad \alpha_3 \in (-\infty,\infty)\setminus\{0\},\ u > 0,$$

and

$$\phi_2(u) = (1+\beta_3 u)^{-1/\beta_3}, \quad \beta_3 \in [0,\infty),\ u > 0,$$

respectively. By writing $l_5(w) = \phi_1^{-1}(w)/\phi_2^{-1}(w)$, *we have*

$$l_5(w) = \beta_3\frac{\ln(e^{-\alpha_3}-1) - \ln(e^{-\beta_3 w}+1)}{(w^{-\beta_3}-1)}, \quad w \in (0,1].$$

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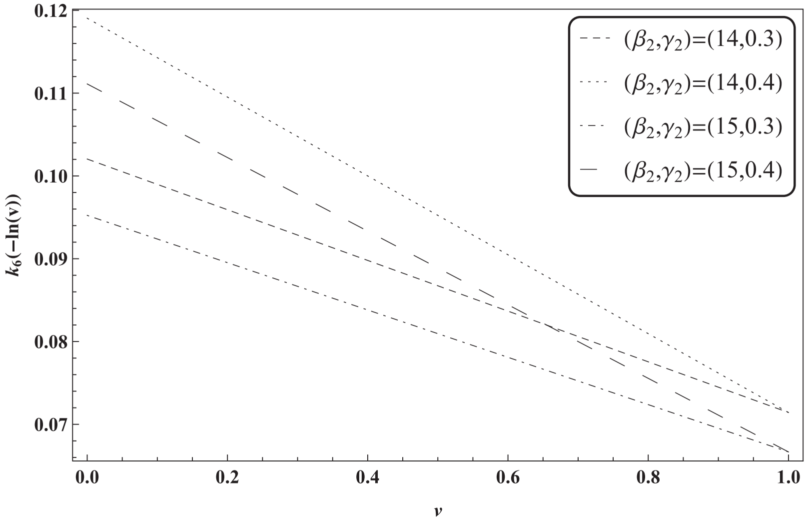

Figure 7. *Plot of* $k_6(-\ln(v))$ *against* $v \in (0, 1]$.

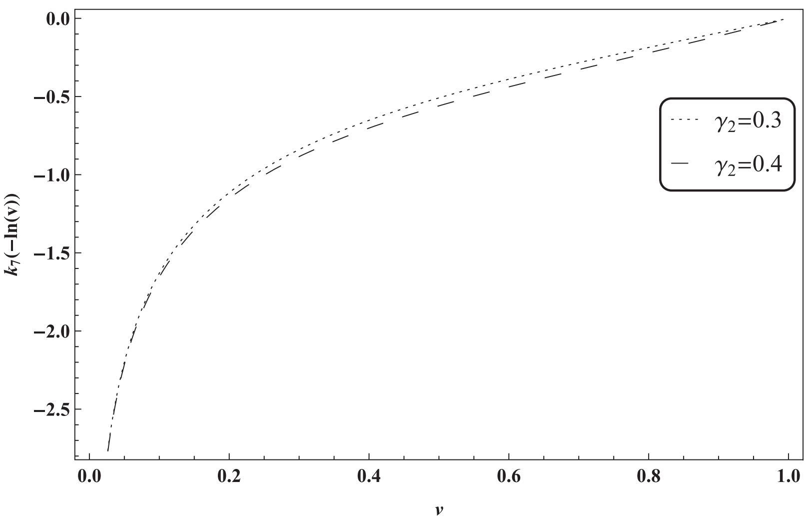

Figure 8. *Plot of* $k_7(-\ln(v))$ *against* $v \in (0, 1]$.

In Figure 9, we plot $l_5(w)$ *against* $w \in (0, 1]$*, for fixed* $(\alpha_3, \beta_3) = (0.3, 1.1), (0.3, 1.2), (0.4, 1.1)$*, and* $(0.4, 1.2)$*. This shows that* $l_5(w)$ *is increasing in* $w \in (0, 1]$*, and hence, the condition of Theorem 3.4(a) is satisfied.*

The following example demonstrates the conditions given in Theorem 3.4(b).

Example 4.7. *Consider the Archimedean copulas with generators*

$$\phi_1(u) = \frac{1}{1 + u^{1/\alpha_4}}, \quad \alpha_4 \in [1, \infty),\ u > 0,$$

and

$$\phi_2(u) = e^{-u^{1/\beta_4}}, \quad \beta_4 \in [1, \infty),\ u > 0,$$

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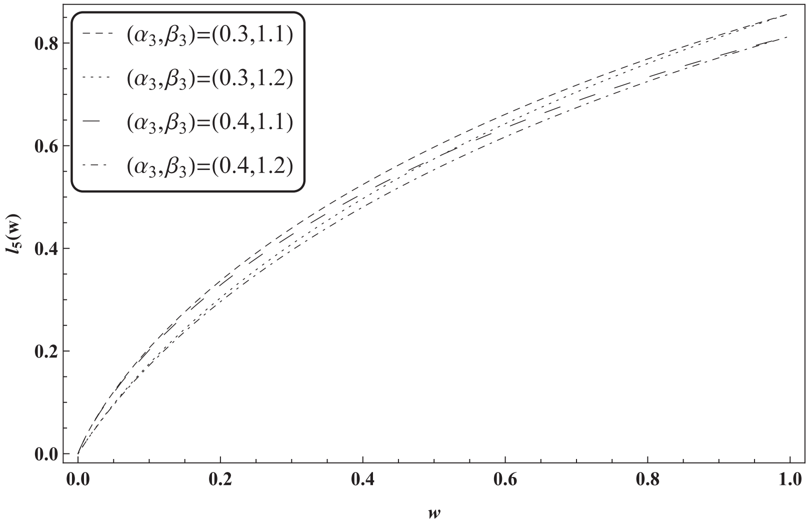

Figure 9. *Plot of $l_5(w)$ against $w \in (0,1]$.*

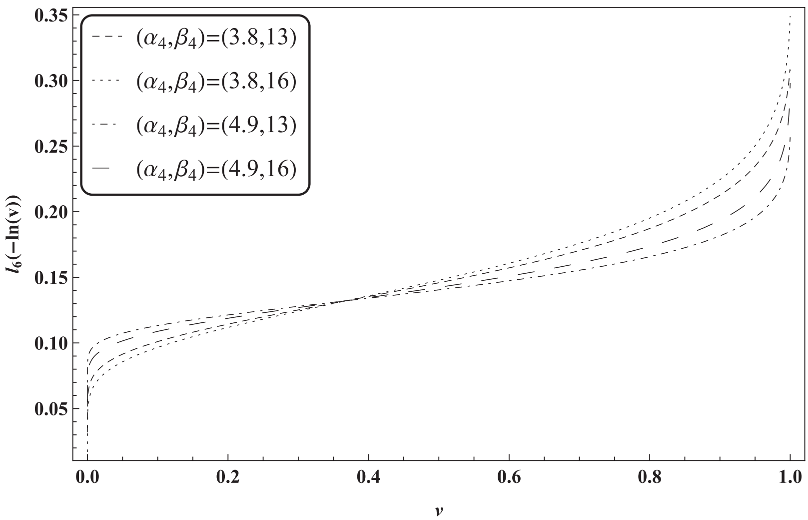

Figure 10. *Plot of $l_6(-\ln(v))$ against $v \in (0,1]$.*

respectively. Consequently,

$$l_6(u) \overset{def.}{=} \phi_1(u) - \phi_2(u) = \frac{1}{1+u^{1/\alpha_4}} - e^{-u^{1/\beta_4}}, \quad u > 0.$$

Let us fix $(\alpha_4, \beta_4) = (3.8, 13), (3.8, 16), (4.9, 13)$, and $(4.9, 16)$. In Figure 10, we plot $l_6(-\ln(v))$ against $v \in (0,1]$. This shows that $l_6(-\ln(v))$ is positive for all $v \in (0,1]$ and hence, $l_6(t)$ is positive for all $u > 0$. Thus, $\phi_1(u) \geq \phi_2(u)$, for all $u > 0$, which, by Remark 3.2, gives $\psi_1(w) \geq \psi_2(v)$, for all $0 \leq w \leq v \leq 1$. Again,

$$l_7(u) \overset{def.}{=} \frac{R_1(u)}{R_2(u)} = \frac{\beta_4 t^{1/\alpha_4}}{\alpha_4(1+u^{1/\alpha_4})u^{1/\beta_4}} \quad \text{and} \quad \frac{uR_2'(u)}{R_2(u)} = \frac{1}{\beta_4}, \quad u > 0.$$

In Figure 11, we plot $l_7(-\ln(v))$ against $v \in (0,1]$. This shows that $l_7(-\ln(v))$ is decreasing in

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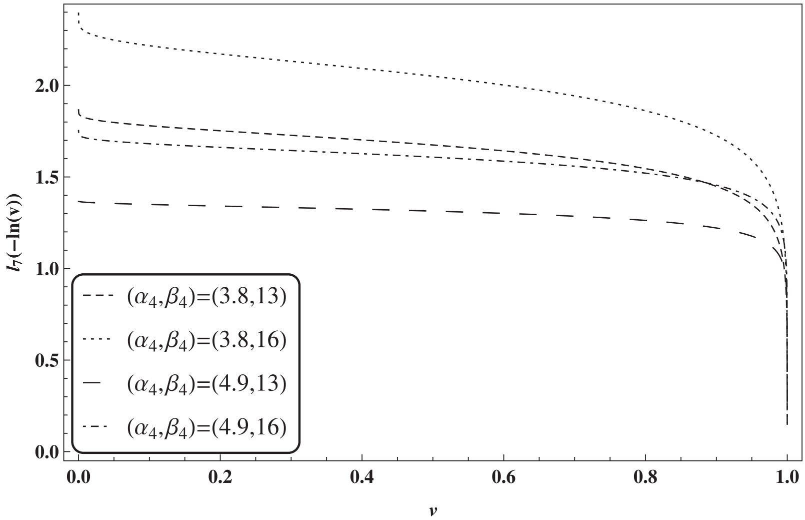

Figure 11. *Plot of* $l_7(-\ln(v))$ *against* $v \in (0, 1]$.

$v \in (0, 1]$ *and hence,* $l_7(u)$ *is increasing in* $u > 0$*. Furthermore, it is trivially true that* $uR_2'(u)/R_2(u)$ *is increasing in* $u > 0$*. Thus, the required conditions are satisfied.*

The following example demonstrates the condition given in Theorem 3.7(c).

Example 4.8. *Consider the Archimedean copula with generator*

$$\phi(u) = \frac{\gamma_3}{\ln(u + e^{\gamma_3})}, \quad \gamma_3 \in (0, \infty),\ u > 0,$$

which gives

$$H(u) = -\frac{\gamma_3 u}{(u + e^{\gamma_3})(\ln(u + e^{\gamma_3}) - \gamma_3)\ln(u + e^{\gamma_3})}, \quad \textit{for all } u > 0.$$

Let us fix $\gamma_3 = 3$, 4, *and* 5*. By writing* $l_8(u) = H(u)/(\ln(1 - \phi(u)))$, $u > 0$, *we plot* $l_8(-\ln(v))$ *against* $v \in (0, 1]$*. From Figure 12, we see that* $l_8(-\ln(v))$ *is decreasing in* $v \in (0, 1]$ *and hence,* $l_8(u)$ *is increasing in* $u > 0$*. Thus, the required condition is satisfied.*

5. Concluding remarks

In this paper, we consider different coherent systems (especially, series, parallel, and general r-out-of-n systems) formed by d.i.d. components, where the dependency structures are described by Archimedean copulas. We provide some sufficient conditions (in terms of the generators of Archimedean copulas) to show that one system performs better than another one with respect to the usual stochastic order, the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the aging faster orders in terms of the failure rate and the reversed failure rate. In the same spirit, we compare a used system and a system made by used components with respect to different stochastic orders. Furthermore, we study the closure properties of different aeing classes (namely, IFR, DFR, DRFR, ILR, DLR, IFRA and DFRA) under the formation of r-out-of-n systems. Moreover, we illustrate the proposed results through various examples.

As we discussed, the main idea of this paper is to consider the dependency structure between components of a system by an Archimedean copula. Most of the systems used in real life have inter-dependency

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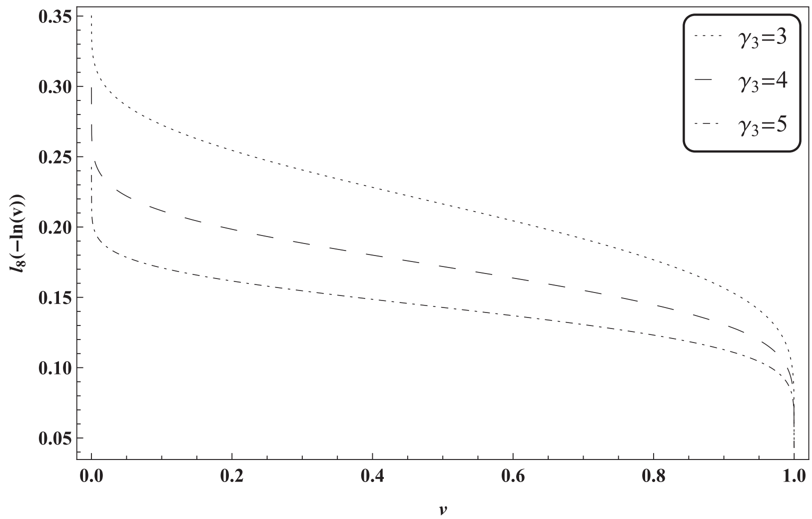

Figure 12. *Plot of* $l_8(-\ln(v))$ *against* $v \in (0, 1]$.

structures between their components, and hence, the assumption of "independent components" sometimes oversimplifies the actual scenario. Thus, we consider the coherent systems that are formed by dependent components governed by the Archimedean copula. As mentioned in the Introduction section, Archimedean copulas are extensively used in the literature due to their wide spectrum of capturing the dependency structures. Furthermore, Archimedean copulas enjoy nice mathematical properties which make them popular. Thus, our study based on the Archimedean copulas may be useful in different practical scenarios where systems with dependent components are considered.

Even though we derived a large number of results in this paper, there remains ample scope to develop further results for systems with dependent components under the Archimedean copula. Here, we only consider the systems with dependent and identically distributed components. The same study for the systems with dependent and nonidentically distributed components may be considered in future.

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Competing interest. The authors declare no conflict of interest.

References

[1] Amini-Seresht, E., Zhang, Y., & Balakrishnan, N. (2018). Stochastic comparisons of coherent systems under different random environments. *Journal of Applied Probability* 55: 459–472.
[2] Balakrishnan, N. & Zhao, P. (2013). Hazard rate comparison of parallel systems with heterogeneous gamma components. *Journal of Multivariate Analysis* 113: 153–160.
[3] Balakrishnan, N. & Zhao, P. (2013). Ordering properties of order statistics from heterogeneous populations: A review with an emphasis on some recent developments. *Probability in the Engineering and Informational Sciences* 27: 403–443.
[4] Balakrishnan, N., Barmalzan, G., & Haidari, A. (2014). Stochastic orderings and ageing properties of residual life lengths of live components in $(n-k+1)$-out-of-n systems. *Journal of Applied Probability* 51: 58–68.
[5] Barlow, R.E. & Proschan, F (1975). *Statistical theory of reliability and life testing*. New York: Holt, Rinehart and Winston.
[6] Barmalzan, G., Haidari, A., & Balakrishnan, N. (2018). Univariate and multivariate stochastic orderings of residual lifetimes of live components in sequential $(n-r+1)$-out-of-n systems. *Journal of Applied Probability* 55: 834–844.
[7] Barmalzan, G., Ayat, S.M., Balakrishnan, N., & Roozegar, R. (2020). Stochastic comparisons of series and parallel systems with dependent heterogeneous extended exponential components under Archimedean copula. *Journal of Computational and Applied Mathematics* 380: 112965.
[8] Belzunce, F., Franco, M., Ruiz, J.M., & Ruiz, M.C. (2001). On partial orderings between coherent systems with different structures. *Probability in the Engineering and Informational Sciences* 15: 273–293.

https://doi.org/10.1017/S0269964821000425 Published online by Cambridge University Press

[9] Boland, P.J., El-Neweihi, E., & Proschan, F. (1994). Applications of the hazard rate ordering in reliability and order statistics. *Journal of Applied Probability* 31: 180–192.
[10] Bon, J.L. & Păltănea, E. (2006). Comparisons of order statistics in a random sequence to the same statistics with i.i.d. variables. *ESAIM: Probability and Statistics* 10: 1–10.
[11] David, H.A. & Nagaraja, H.N. (2003). *Order statistics*. New Jersey: Wiley & Sons.
[12] Dykstra, R., Kochar, S., & Rojo, J. (1997). Stochastic comparisons of parallel systems of heterogeneous exponential components. *Journal of Statistical Planning and Inference* 65: 203–211.
[13] Esary, J.D. & Proschan, F. (1963). Reliability between system failure rate and component failure rates. *Technometrics* 5: 183–189.
[14] Fang, R., Li, C., & Li, X. (2016). Stochastic comparisons on sample extremes of dependent and heterogeneous observations. *Statistics* 50: 930–955.
[15] Franco, M., Ruiz, M.C., & Ruiz, J.M. (2003). A note on closure of the ILR and DLR classes under formation of coherent systems. *Statistical Papers* 44: 279–288.
[16] Gupta, N. (2013). Stochastic comparisons of residual lifetimes and inactivity times of coherent systems. *Journal of Applied Probability* 50: 848–860.
[17] Gupta, N., Misra, N., & Kumar, S. (2015). Stochastic comparisons of residual lifetimes and inactivity times of coherent systems with dependent identically distributed components. *European Journal of Operational Research* 240: 425–430.
[18] Hazra, N.K. & Finkelstein, M. (2019). Comparing lifetimes of coherent systems with dependent components operating in random environments. *Journal of Applied Probability* 56: 937–957.
[19] Hazra, N.K. & Misra, N. (2020). On relative ageing of coherent systems with dependent identically distributed components. *Advances in Applied Probability* 52: 348–376.
[20] Hazra, N.K. & Misra, N. (2021). On relative ageing comparisons of coherent systems with identically distributed components. *Probability in the Engineering and Informational Sciences* 35: 481–495.
[21] Hazra, N.K. & Nanda, A.K. (2016). Stochastic comparisons between used systems and systems made by used components. *IEEE Transactions on Reliability* 65: 751–762.
[22] Hazra, N.K., Kuiti, M.R., Finkelstein, M., & Nanda, A.K. (2017). On stochastic comparisons of maximum order statistics from the location-scale family of distributions. *Journal of Multivariate Analysis* 160: 31–41.
[23] Hazra, N.K., Kuiti, M.R., Finkelstein, M., & Nanda, A.K. (2018). On stochastic comparisons of minimum order statistics from the location-scale family of distributions. *Metrika* 81: 105–123.
[24] Kelkinnama, M. & Asadi, M. (2019). Stochastic and ageing properties of coherent systems with dependent identically distributed components. *Statistical Papers* 60: 805–821.
[25] Khaledi, B.E. & Kochar, S. (2000). Some new results on stochastic comparisons of parallel systems. *Journal of Applied Probability* 37: 1123–1128.
[26] Khaledi, B.E. & Kochar, S. (2006). Weibull distribution: some stochastic comparisons results. *Journal of Statistical Planning and Inference* 136: 3121–3129.
[27] Kochar, S. & Xu, M. (2007). Stochastic comparisons of parallel systems when components have proportional hazard rates. *Probability in the Engineering and Informational Sciences* 21: 597–609.
[28] Kundu, A., Chowdhury, S., Nanda, A.K., & Hazra, N.K. (2016). Some results on majorization and their applications. *Journal of Computational and Applied Mathematics* 301: 161–177.
[29] Lai, C.D. & Xie, M. (2006). *Stochastic ageing and dependence for reliability*. New York: Springer.
[30] Li, X. & Fang, R. (2015). Ordering properties of order statistics from random variables of Archimedean copulas with applications. *Journal of Multivariate Analysis* 133: 304–320.
[31] Li, C. & Li, X. (2019). Hazard rate and reversed hazard rate orders on extremes of heterogeneous and dependent random variables. *Statistics and Probability Letters* 146: 104–111.
[32] Li, X. & Lu, X. (2003). Stochastic comparison on residual life and inactivity time of series and parallel systems. *Probability in the Engineering and Informational Sciences* 17: 267–275.
[33] McNeil, A.J. & Něslehová, J. (2009). Multivariate Archimedeaen copulas, D-monotone functions and ℓ_1-norm symmetric distributions. *The Annals of Statistics* 37: 3059–3097.
[34] Nanda, A.K., Jain, K., & Singh, H. (1998). Preservation of some partial orderings under the formation of coherent systems. *Statistics and Probability Letters* 39: 123–131.
[35] Navarro, J. (2018). Stochastic comparisons of coherent systems. *Metrika* 81: 465–482.
[36] Navarro, J. & Mulero, J. (2020). Comparisons of coherent systems under the time-transformed exponential model. *TEST* 29: 255–281.
[37] Navarro, J. & Rubio, R. (2010). Comparisons of coherent systems using stochastic precedence. *TEST* 19: 469–486.
[38] Navarro, J., Águila, Y.D., Sordo, M.A., & Suárez-Liorens, A. (2013). Stochastic ordering properties for systems with dependent identically distributed components. *Applied Stochastic Models in Business and Industry* 29: 264–278.
[39] Navarro, J., Águila, Y.D., Sordo, M.A., & Suárez-Liorens, A. (2014). Preservation of reliability classes under the formation of coherent systems. *Applied Stochastic Models in Business and Industry* 30: 444–454.
[40] Navarro, J., Pellerey, F., & Di Crescenzo, A. (2015). Orderings of coherent systems with randomized dependent components. *European Journal of Operational Research* 240: 127–139.

[41] Navarro, J., Águila, Y.D., Sordo, M.A., & Suárez-Liorens, A. (2016). Preservation of stochastic orders under the formation of generalized distorted distributions: Applications to coherent systems. *Methodology and Computing in Applied Probability* 18: 529–545.
[42] Nelsen, R.B. (2006). *An introduction to copulas*. New York: Springer.
[43] Pledger, G. & Proschan, F. (1971). Comparisons of order statistics and of spacings from heterogeneous distributions. In *Optimizing methods in statistics*, Rustagi, J. S. (ed.). Ohio: Academic Press, pp. 89–113.
[44] Razaei, M., Gholizadeh, B., & Izadkhah, S. (2015). On relative reversed hazard rate order. *Communications in Statistics – Theory and Methods* 44: 300–308.
[45] Samaniego, F.J. (2007). *System signatures and their applications in engineering reliability*. New York: Springer.
[46] Samaniego, F.J. & Navarro, J. (2016). On comparing coherent systems with heterogeneous components. *Advances in Applied Probability* 48: 88–111.
[47] Sengupta, D. & Deshpande, J.V. (1994). Some results on the relative ageing of two life distributions. *Journal of Applied Probability* 31: 991–1003.
[48] Sengupta, D. & Nanda, A.K. (1999). Log-concave and concave distributions in reliability. *Naval Research Logistics* 46: 419–433.
[49] Shaked, M. & Shanthikumar, J. (2007). *Stochastic orders*. New York: Springer.
[50] Zhao, P., Li, X., & Balakrishnan, N. (2009). Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables. *Journal of Multivariate Analysis* 100: 952–962.
[51] Zhao, P., Li, X., & Da, G. (2011). Right spread order of the second order statistic from heterogeneous exponential random variables. *Communications in Statistics – Theory and Methods* 40: 3070–3081.

Appendix

Proof of Theorem 3.1(b). We only prove the result under the assumption that $\sum_{j=r}^{n} P_{r,n}^{j}(u)K_j(u)$ is increasing in $u > 0$. The proof for the other case follows in the same line. From (3.4.3) of David and Nagaraja [11], we have

$$\bar{F}_{\tau_{r|n}^{\phi}(\boldsymbol{X})}(x) = \sum_{j=r}^{n} C_{r,n}^{j}\phi(j\psi(\bar{F}_X(x))), \quad x > 0,$$

which gives

$$\begin{aligned} r_{\tau_{r|n}^{\phi}(\boldsymbol{X})}(x) &= r_X(x)\frac{\sum_{j=r}^{n} C_{r,n}^{j}\phi(j\psi(\bar{F}_X(x)))\frac{R(j\psi(\bar{F}_X(x)))}{R(\psi(\bar{F}_X(x)))}}{\sum_{i=r}^{n} C_{r,n}^{i}\phi(i\psi(\bar{F}_X(x)))} \\ &= r_X(t)\sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_X(x)))K_j(\psi(\bar{F}_X(x))), \quad x > 0, \end{aligned} \tag{A.1}$$

Similarly, we get

$$r_{\tau_{r|n}^{\phi}(\boldsymbol{Y})}(x) = r_Y(x)\sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_Y(x)))K_j(\psi(\bar{F}_Y(x))), \quad x > 0.$$

Since $X \le_{\text{hr}} Y$ and ψ is a decreasing function, we have $r_X(x) \ge r_Y(x)$ and $\psi(\bar{F}_X(x)) \ge \psi(\bar{F}_Y(x))$, for all $x > 0$. Then, by using the assumption "$\sum_{j=r}^{n} P_{r,n}^{j}(u)K_j(u)$ is increasing in $u > 0$", we get

$$\sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_X(x)))K_j(\psi(\bar{F}_X(x))) \ge \sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_Y(x)))K_j(\psi(\bar{F}_Y(x))), \quad \text{for all } x > 0,$$

which is equivalent to mean that $r_{\tau_{r|n}^{\phi}(\boldsymbol{X})}(x) \ge r_{\tau_{r|n}^{\phi}(\boldsymbol{Y})}(x)$, for all $x > 0$. Hence, the result is proved. □

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Proof of Theorem 3.2(a). We only prove the result for the parallel system. The proof for the series system follows in the same line. We have

$$F_{\tau^{\phi}_{1|n}(\boldsymbol{X})}(x) = \phi(n\psi(F_X(x))), \quad x > 0, \tag{A.2}$$

which gives

$$\begin{aligned}\frac{f'_{\tau^{\phi}_{1|n}(\boldsymbol{X})}(x)}{f_{\tau^{\phi}_{1|n}(\boldsymbol{X})}(x)} &= \frac{f'_X(x)}{f_X(x)} + \frac{f_X(x)}{F_X(x)}\left[\frac{F_X(t)\psi''(F_X(x))}{\psi' F_X(x)} + \frac{F_X(x)\psi'(F_X(x))}{\psi(F_X(x))}\frac{n\psi(F_X(x))\phi''(nF_X(x))}{\phi'(nF_X(x))}\right] \\ &= \frac{f'_X(x)}{f_X(x)} + \tilde{r}_X(x)\left[\frac{G(n\psi(F_X(x)))}{R(\psi(F_X(x)))} - \frac{G(\psi(F_X(x)))}{R(\psi(F_X(x)))}\right], \quad x > 0, \end{aligned}\tag{A.3}$$

where the last equality follows from the fact that

$$\frac{p\psi'(p)}{\psi(p)} = \frac{\phi(\psi(p))}{\psi(p)\phi'(\psi(p))} = \frac{1}{R(\psi(p))}, \quad 0 < p < 1.$$

Similarly, we have

$$\frac{f'_{\tau^{\phi}_{1|n}(\boldsymbol{Y})}(x)}{f_{\tau^{\phi}_{1|n}(\boldsymbol{Y})}(x)} = \frac{f'_Y(x)}{f_Y(x)} + \tilde{r}_Y(x)\left[\frac{G(n\psi(F_Y(x)))}{R(\psi(F_Y(x)))} - \frac{G(\psi(F_Y(x)))}{R(\psi(F_Y(x)))}\right], \quad x > 0.$$

Then, the result holds if, and only if,

$$\begin{aligned}&\frac{f'_X(x)}{f_X(x)} + \tilde{r}_X(x)\left[\frac{G(n\psi(F_X(x)))}{R(\psi(F_X(x)))} - \frac{G(\psi(F_X(x)))}{R(\psi(F_X(x)))}\right] \\ &\quad\le \frac{f'_Y(x)}{f_Y(x)} + \tilde{r}_Y(x)\left[\frac{G(n\psi(F_Y(x)))}{R(\psi(F_Y(x)))} - \frac{G(\psi(F_Y(x)))}{R(\psi(F_Y(x)))}\right], \quad x > 0. \end{aligned}\tag{A.4}$$

Since $X \le_{\text{lr}} Y$, we have

$$\tilde{r}_X(x) \le \tilde{r}_Y(x) \quad \text{and} \quad F_X(x) \ge F_Y(x), \quad \text{for all } x > 0, \tag{A.5}$$

and

$$\frac{f'_X(x)}{f_X(x)} \le \frac{f'_Y(x)}{f_Y(x)}, \quad \text{for all } x > 0. \tag{A.6}$$

Since ψ is a decreasing function, we have, from (A.5),

$$\psi(F_X(x)) \le \psi(F_Y(x)), \quad \text{for all } x > 0. \tag{A.7}$$

Again, from the assumption, we have that

$$\frac{G(nu)}{R(u)} - \frac{G(u)}{R(u)} \text{ is positive and increasing in } u > 0,$$

which, by (A.7), gives

$$0 \le \frac{G(n\psi(F_X(x)))}{R(\psi(F_X(x)))} - \frac{G(\psi(F_X(x)))}{R(\psi(F_X(x)))} \le \frac{G(n\psi(F_Y(x)))}{R(\psi(F_Y(x)))} - \frac{G(\psi(F_Y(x)))}{R(\psi(F_Y(x)))}, \tag{A.8}$$

for all $x > 0$. On combining (A.5), (A.6), and (A.8), we get (A.4) and hence, the result follows. □

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Proof of Theorem 3.2(b). From (A.2), we have

$$r_{\tau_{1|n}^{\phi}(\boldsymbol{X})}(x) = r_X(x)\left[\frac{(1-F_X(x))\,\psi'(F_X(x))}{\psi(F_X(x))}\right]\left[\frac{n\psi(F_X(x))\phi'(n\psi(F_X(x)))}{1-\phi(n\psi(F_X(x)))}\right]$$
$$= r_X(x)\frac{H(n\psi(F_X(x)))}{H(\psi(F_X(x)))}, \quad t > 0.$$

Similarly,

$$r_{\tau_{1|n}^{\phi}(\boldsymbol{Y})}(x) = r_Y(x)\frac{H(n\psi(F_Y(x)))}{H(\psi(F_Y(x)))}, \quad x > 0.$$

Then,

$$\frac{r_{\tau_{1|n}^{\phi}(\boldsymbol{X})}(x)}{r_{\tau_{1|n}^{\phi}(\boldsymbol{Y})}(x)} = \frac{r_X(x)}{r_Y(x)}\frac{\frac{H(n\psi(F_X(x)))}{H(\psi(F_X(x)))}}{\frac{H(n\psi(F_Y(x)))}{H(\psi(F_Y(x)))}}, \quad x > 0.$$

Since $X \leq_c Y$, we have that $r_X(x)/r_Y(x)$ is increasing in $x > 0$. Thus, to prove the result, it suffices to show that

$$\frac{H(n\psi(F_X(x)))H(\psi(F_Y(x)))}{H(\psi(F_X(x)))H(n\psi(F_Y(x)))} \quad \text{is increasing in } t > 0,$$

or equivalently,

$$\tilde{r}_X(x)\left[\frac{C(n\psi(F_X(x)))}{R(\psi(F_X(x)))} - \frac{C(\psi_1(F_X(x)))}{R(\psi_1(F_X(x)))}\right]$$
$$\geq \tilde{r}_Y(x)\left[\frac{C(n\psi(F_Y(x)))}{R(\psi(F_Y(x)))} - \frac{C(\psi(F_Y(x)))}{R(\psi(F_Y(x)))}\right], \tag{A.9}$$

for all $x > 0$. Since $Y \leq_{\text{rh}} X$, we have

$$\tilde{r}_X(x) \geq \tilde{r}_Y(x) \text{ and } F_X(x) \leq F_Y(x), \quad \text{for all } x > 0. \tag{A.10}$$

Since ψ is a decreasing function, we have, from the above inequality,

$$\psi(F_X(x)) \geq \psi(F_Y(x)), \quad \text{for all } x > 0. \tag{A.11}$$

Again, from the assumption, we have that

$$\frac{C(nu)}{R(u)} - \frac{C(u)}{R(u)} \quad \text{is positive and increasing in } u > 0,$$

which, by (A.11), gives

$$0 \leq \frac{C(n\psi(F_Y(x)))}{R(\psi(F_Y(x)))} - \frac{C(\psi(F_Y(x)))}{R(\psi(F_Y(x)))} \leq \frac{C(n\psi(F_X(x)))}{R(\psi(F_X(x)))} - \frac{C(\psi(F_X(x)))}{R(\psi_1(F_X(x)))}, \tag{A.12}$$

for all $x > 0$. On combining (A.10) and (A.12), we get (A.9) and hence, the result follows. □

Proof of Theorem 3.3(a). We have

$$F_{\tau_{1|n}^{\phi_1}(\boldsymbol{X})}(x) = \phi_1(n\psi_1(F_X(x))) \quad \text{and} \quad F_{\tau_{1|n}^{\phi_2}(\boldsymbol{Y})}(x) = \phi_2(n\psi_2(F_Y(x))), \quad x > 0. \tag{A.13}$$

Since $X \leq_{\text{st}} Y$ and ψ_1 is a decreasing function, we have $\phi_1(n\psi_1(F_X(x))) \geq \phi_1(n\psi_1(F_Y(x)))$ for all $x > 0$. Furthermore, from the assumption "$\phi_2^{-1}(\phi_1(u))$ is sub-additive in $u > 0$", we have

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$\phi_1(n\psi_1(F_Y(x))) \geq \phi_2(n\psi_2(F_Y(x)))$ for all $x > 0$. On combining these two inequalities, we get $\phi_1(n\psi_1(F_X(x))) \geq \phi_2(n\psi_2(F_Y(x)))$ for all $x > 0$ and hence, the result follows $\square$

Proof of Theorem 3.3(b). We only prove the result under the condition that $uH_2'(u)/H_2(u)$ is decreasing in $u > 0$. The proof follows in the same line for the other case. Note that, for all $p \in (0, 1)$,

$$\frac{(1-p)\psi_i'(p)}{\psi_i(p)} = \frac{1-\phi_i(\psi_i(p))}{\psi_i(p)\phi_i'(\psi_i(p))} = \frac{1}{H_i(\psi_i(p))}, \quad \text{for } i = 1, 2. \tag{A.14}$$

Now, from (A.13), we get

$$\begin{aligned} r_{\tau_{1|n}^{\phi_1}(\boldsymbol{X})}(x) &= \frac{f_X(x)\phi_1'(n\psi_1(F_X(x)))n\psi_1'(F_X(x))}{1-\phi_1(n\psi_1(F_X(x)))} \\ &= r_X(x)\left[\frac{(1-F_X(x))\,\psi_1'(F_X(x))}{\psi_1(F_X(x))}\right]\left[\frac{n\psi_1(F_X(x))\phi_1'(n\psi_1(F_X(x)))}{1-\phi_1(n\psi_1(F_X(x)))}\right] \\ &= r_X(x)\frac{H_1(n\psi_1(F_X(x)))}{H_1(\psi_1(F_X(x)))}, \quad x > 0, \end{aligned}$$

where the last equality follows from (A.14). Similarly, from (A.13) and (A.14), we get

$$r_{\tau_{1|n}^{\phi_2}(\boldsymbol{Y})}(x) = r_Y(x)\frac{H_2(n\psi_2(F_Y(x)))}{H_2(\psi_2(F_Y(x)))}, \quad x > 0.$$

Thus, the result holds if, only if,

$$r_X(x)\frac{H_1(n\psi_1(F_X(x)))}{H_1(\psi_1(F_X(x)))} \geq r_Y(x)\frac{H_2(n\psi_2(F_Y(x)))}{H_2(\psi_2(F_Y(x)))}, \quad \text{for all } x > 0.$$

Since $X \leq_{\text{hr}} Y$, we have $r_X(x) \geq r_Y(x)$. Thus, the above inequality holds if

$$\frac{H_1(n\psi_1(F_X(x)))}{H_1(\psi_1(F_X(x)))} \geq \frac{H_2(n\psi_2(F_Y(x)))}{H_2(\psi_2(F_Y(x)))}, \quad \text{for all } x > 0. \tag{A.15}$$

Now, from the assumptions "$\psi_1(w) \leq \psi_2(v)$ for all $0 \leq v \leq w \leq 1$" and "$X \leq_{\text{hr}} Y$", we have

$$\psi_1(F_X(x)) \leq \psi_2(F_Y(x)), \quad \text{for all } x > 0. \tag{A.16}$$

Again, we have that $uH_2'(u)/H_2(u)$ is decreasing in $u > 0$. This implies that

$$\frac{H_2(nu)}{H_2(u)} \quad \text{is decreasing in } u > 0,$$

which further, by (A.16), gives

$$\frac{H_2(n\psi_1(F_X(x)))}{H_2(\psi_1(F_X(x)))} \geq \frac{H_2(n\psi_2(F_Y(x)))}{H_2(\psi_2(F_Y(x)))}, \quad \text{for all } x > 0. \tag{A.17}$$

Again, on using the condition "$H_1(u)/H_2(u)$ is increasing in $u > 0$", we get

$$\frac{H_1(n\psi_1(F_X(x)))}{H_1(\psi_1(F_X(x)))} \geq \frac{H_2(n\psi_1(F_X(x)))}{H_2(\psi_1(F_X(x)))}, \quad \text{for all } x > 0. \tag{A.18}$$

On combining (A.17) and (A.18), we get (A.15). Thus, the result is proved. $\square$

Proof of Theorem 3.5(d). We only prove the result for the parallel system. The proof for the series system can be done in the same line. Now, for any fixed $t \geq 0$, we have

$$F_{(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t}(x) = \frac{\phi(n\psi(F_X(x+t))) - \phi(n\psi(F_X(t)))}{1-\phi(n\psi(F_X(t)))}, \quad x > 0, \tag{A.19}$$

and

$$F_{\tau^{\phi}_{1|n}(\boldsymbol{X}_t)}(x) = \phi\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right), \quad x > 0, \tag{A.20}$$

which give

$$\begin{aligned}
\tilde{r}_{(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t}(x) &= \frac{f_X(x+t)\phi'(n\psi(F_X(x+t))n\psi'(F_X(x+t))}{\phi(n\psi(F_X(x+t))} \\
&\quad \times \frac{\phi(n\psi(F_X(x+t)))}{\phi(n\psi(F_X(x+t))) - \phi(n\psi(F_X(t)))} \\
&= \tilde{r}_X(x+t)\left[\frac{(F_X(x+t))\psi'(F_X(x+t))}{\psi(F_X(x+t))}\right] \\
&\quad \times \left[\frac{n\psi(F_X(x+t))\phi'(n\psi(F_X(x+t))}{\phi(n\psi(F_X(x+t))}\right] \\
&\quad \times \left[\frac{\phi(n\psi(F_X(x+t)))}{\phi(n\psi(F_X(x+t))) - \phi(n\psi(F_X(t)))}\right] \\
&= \tilde{r}_X(x+t)\frac{R(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} \\
&\quad \times \frac{\phi(n\psi(F_X(x+t)))}{\phi(n\psi(F_X(x+t))) - \phi(n\psi(F_X(t)))}, \quad x > 0,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{r}_{\tau^{\phi}_{1|n}(\boldsymbol{X}_t)}(x) &= \frac{f_X(x+t)\phi'\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)n\psi'\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}{\bar{F}_X(t)\phi\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)} \\
&= \tilde{r}_X(x+t)\left[\frac{\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\psi'\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}{\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}\right] \\
&\quad \times \left[\frac{n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\phi'\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{\phi\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}\right] \\
&\quad \times \left[\frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t))) - \phi(\psi(F_X(t)))}\right] \\
&= \tilde{r}_X(x+t)\frac{R\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)} \\
&\quad \times \frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t))) - \phi(\psi(F_X(t)))}, \quad x > 0.
\end{aligned}$$

Thus, to prove the result, it suffices to show that, for any fixed $t \geq 0$,

$$\frac{R(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} \frac{\phi(n\psi(F_X(x+t))}{\phi(n\psi(F_X(x+t)) - \phi\,(n\psi(F_X(t)))} \leq \frac{R\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)} \frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t))) - \phi(\psi(F_X(t)))}, \quad x > 0. \tag{A.21}$$

Since $uR'(u)/R(u) \geq 0$ for all $u \geq 0$, we get that $R(u)$ is decreasing in $u > 0$. This implies that, for any fixed $t \geq 0$,

$$\frac{\phi(n\psi(F_X(x+t))}{\phi(n\psi(F_X(x+t)) - \phi(n\psi(F_X(t)))} \leq \frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t))) - \phi(\psi(F_X(t)))}, \tag{A.22}$$

for all $x > 0$. Note that, for any fixed $t \geq 0$,

$$F_X(x+t) \geq \frac{F_X(x+t) - F_X(t)}{1 - F_X(t)}, \quad x > 0,$$

which, by decreasing property of ψ, gives

$$\psi(F_X(x+t)) \leq \psi\left(\frac{F_X(x+t) - F_X(t)}{1 - F_X(t)}\right), \quad x > 0. \tag{A.23}$$

Again, we have that $uR'(u)/R(u)$ is increasing in $u > 0$. This implies that

$$\frac{R\,(nu)}{R(u)} \quad \text{is increasing in } u > 0,$$

which further, by (A.23), gives

$$\frac{R(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} \leq \frac{R\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}, \tag{A.24}$$

for all $x > 0$ and for fixed $t \geq 0$. On combining (A.22) and (A.24), we get (A.21) and hence, the result is proved. □

Proof of Theorem 3.5(f). From (A.19) and (A.20), we have that, for any fixed $t \geq 0$,

$$\begin{aligned}
\frac{f'_{(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t}(x)}{f_{(\tau^{\phi}_{1|n}(\boldsymbol{X}))_t}(x)} &= \frac{f'_X(x+t)}{f_X(x+t)} + \frac{f_X(x+t)n\psi'(F_X(x+t))\phi''(n\psi(F_X(x+t))}{\phi'(n\psi(F_X(x+t))} \\
&\quad + \frac{f_X(x+t)\psi''(F_X(x+t))}{\psi'(F_X(x+t))} \\
&= \frac{f'_X(x+t)}{f_X(x+t)} + \frac{f_X(x+t)}{F_X(x+t)} \left[\frac{F_X(x+t)\psi''(F_X(x+t))}{\psi'(F_X(x+t))} \right. \\
&\quad \left. + \frac{F_X(x+t)\psi'(F_X(x+t))}{\psi(F_X(x+t))} \frac{n\psi(F_X(x+t))\phi''(n\psi(F_X(x+t))}{\phi'(n\psi(F_X(x+t))} \right] \\
&= \frac{f'_X(x+t)}{f_X(x+t)} + \tilde{r}_X(x+t) \left[\frac{G(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} - \frac{G(\psi(F_X(x+t)))}{R(\psi(F_X(x+t)))} \right], \quad x > 0,
\end{aligned}$$

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and

$$\begin{aligned}\frac{f'_{\tau^{\phi}_{1|n}(\boldsymbol{X}_t)}(x)}{f_{\tau^{\phi}_{1|n}(\boldsymbol{X}_t)}(x)} &= \frac{f'_X(x+t)}{f_X(x+t)} + \frac{f_X(x+t)n\psi'\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\phi''\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{\bar{F}_X(t)\phi'\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}\\ &\quad + \frac{f_X(x+t)\psi''\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}{\bar{F}_X(t)\psi'\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}\\ &= \frac{f'_X(x+t)}{f_X(x+t)} + \frac{f_X(x+t)}{F_X(x+t)}\left[\frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t)))-\phi(\psi(F_X(t)))}\right]\\ &\quad \times\left[\frac{\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\psi''\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}{\psi'\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)} + \frac{\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\psi'\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}{\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}\right.\\ &\quad \left.\times\frac{n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\phi''\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{\phi'\left(n\psi(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)})\right)}\right]\\ &= \frac{f'_X(x+t)}{f_X(x+t)} + \tilde{r}_X(x+t)\\ &\quad\times\left[\frac{G\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)} - \frac{G\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}\right]\\ &\quad\times\left[\frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t)))-\phi(\psi(F_X(t)))}\right],\quad x>0. \end{aligned}\tag{A.25}$$

Thus, to prove the result, it suffices to show that, for any fixed $t \geq 0$,

$$\begin{aligned}&\frac{G(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} - \frac{G(\psi(F_X(x+t)))}{R(\psi(F_X(x+t)))}\\ &\quad\leq\left[\frac{G\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)} - \frac{G\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}\right]\\ &\quad\times\left[\frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t)))-\phi(\psi(F_X(t)))}\right],\quad x>0. \end{aligned}\tag{A.26}$$

Since ϕ is a decreasing function, we have that, for any fixed $t \geq 0$,

$$\left[\frac{\phi(\psi(F_X(x+t)))}{\phi(\psi(F_X(x+t)))-\phi(\psi(F_X(t)))}\right] \geq 1, \quad \text{for all } x>0. \tag{A.27}$$

Again, from the assumption, we have that

$$\frac{G(nu)}{R(u)} - \frac{G(u)}{R(u)} \quad \text{is positive and increasing in } u>0,$$

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which, by (A.23), gives

$$0 \leq \frac{G(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} - \frac{G(\psi(F_X(x+t)))}{R(\psi(F_X(x+t)))}$$
$$\leq \frac{G\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{F_X(t)}\right)\right)} - \frac{G\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{F_X(t)}\right)\right)}, \tag{A.28}$$

for any fixed $t \geq 0$ and for all $x > 0$. Finally, by combining (A.27) and (A.28), we get (A.26) and hence, the result follows. □

Proof of Theorem 3.5(g). From (A.19) and (A.20), we have that, for any fixed $t \geq 0$,

$$\begin{aligned} r_{(\tau^{\phi}_{1|n}(\mathbf{X}))_t}(x) &= \frac{f_X(x+t)\phi'(n\psi(F_X(x+t))n\psi'(F_X(x+t))}{1-\phi(n\psi(\bar{F}_X(x+t)))} \\ &= r_X(x+t)\left[\frac{(1-F_X(x+t))\psi'(F_X(x+t))}{\psi(F_X(x+t))}\right]\left[\frac{n\psi(F_X(x+t))\phi'(n\psi(F_X(x+t))}{1-\phi(n\psi(F_X(x+t))}\right] \\ &= r_X(x+t)\frac{H(n\psi(F_X(x+t))}{H(\psi(F_X(x+t)))}, \quad x > 0, \end{aligned}$$

and

$$\begin{aligned} r_{\tau^{\phi}_{1|n}(\mathbf{X}_t)}(x) &= \frac{f_X(x+t)\phi'\left(n\psi\left(\frac{F(x+t)-F(t)}{1-F(t)}\right)\right)n\psi'\left(\frac{F(x+t)-F(t)}{1-F(t)}\right)}{\bar{F}(t)\left(1-\phi\left(n\psi\left(\frac{F(x+t)-F(t)}{1-F(t)}\right)\right)\right)} \\ &= r_X(x+t)\left[\frac{\left(1-\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\psi'\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}{\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)}\right] \\ &\quad\times\left[\frac{n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\phi'\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{1-\phi\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}\right] \\ &= r_X(x+t)\frac{H\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{H\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}, \quad x > 0. \end{aligned}$$

Thus, to prove the result, it suffices to show that, for any fixed $t \geq 0$,

$$\frac{r_{\tau^{\phi}_{1|n}(\mathbf{X}_t)}(x)}{r_{(\tau^{\phi}_{1|n}(\mathbf{X}))_t}(x)} = \frac{H\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)H(\psi(F_X(x+t)))}{H\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)H(n\psi(F_X(x+t))} \quad \text{is increasing in } x > 0,$$

or equivalently,

$$\begin{aligned} &\left[\frac{C\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)} - \frac{C\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}\right]\left[\frac{F_X(x+t)}{F_X(x+t)-F_X(t)}\right] \\ &\geq \left[\frac{C(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} - \frac{C(\psi(F_X(x+t)))}{R(\psi(F_X(x+t)))}\right], \quad \text{for all } x > 0. \end{aligned} \tag{A.29}$$

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Since $F(\cdot)$ is a increasing function, we have

$$\frac{F_X(x+t)}{F_X(x+t)-F_X(t)} \geq 1, \quad \text{for all } x>0 \text{ and } t \geq 0. \tag{A.30}$$

Again, from the assumption, we have that

$$\frac{C(nu)}{R(u)} - \frac{C(u)}{R(u)} \quad \text{is positive and increasing in } u>0,$$

which, by (A.23), gives

$$\begin{aligned} 0 &\leq \frac{C(n\psi(F_X(x+t))}{R(\psi(F_X(x+t)))} - \frac{C(\psi(F_X(x+t)))}{R(\psi(F_X(x+t)))} \\ &\leq \frac{C\left(n\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)} - \frac{C\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}{R\left(\psi\left(\frac{F_X(x+t)-F_X(t)}{1-F_X(t)}\right)\right)}, \end{aligned} \tag{A.31}$$

for any fixed $t \geq 0$ and for all $x>0$. On combining (A.30) and (A.31), we get (A.29) and hence, the result follows. □

Proof of Theorem 3.6(a). We only prove the result under the condition that $\sum_{j=r}^{n} P_{r,n}^{j}(u)K_j(u)$ is increasing (resp. decreasing) in $u>0$. The proof follows in the same line for the other case. From (A.1), we have

$$r_{\tau_{r|n}^{\phi}(\boldsymbol{X})}(x) = r_X(x)\sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_X(x)))K_j(\psi(\bar{F}_X(x))), \quad x>0.$$

Since X is IFR (resp. DFR), we have that $r_X(x)$ is increasing (resp. decreasing) in $x>0$. Thus, to prove the result, it suffices to show that

$$\sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_X(x)))K_j(\psi(\bar{F}_X(x))) \quad \text{is increasing (resp. decreasing) in } x>0. \tag{A.32}$$

Now,

$$\frac{d}{dx}\left(\sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_X(x)))K_j(\psi(\bar{F}_X(x)))\right) = \frac{d}{du}\left(\sum_{j=r}^{n} P_{r,n}^{j}(u)K_j(u)\right)\frac{du}{dt},$$

where $u=\psi\left(\bar{F}_X(t)\right)$. Since ψ is a decreasing function, we have $du/dt \geq 0$. On using this together with the condition "$\sum_{j=r}^{n} P_{r,n}^{j}(u)K_j(u)$ is increasing (resp. decreasing) in $u>0$", we get

$$\frac{d}{dx}\left(\sum_{j=r}^{n} P_{r,n}^{j}(\psi(\bar{F}_X(x)))K_j(\psi(\bar{F}_X(x)))\right) \geq (\text{resp. } \leq)\ 0,$$

which implies that (A.32) is true. Hence, the result is proved. □

Proof of Theorem 3.7(a). We only prove the result for the series system. The result for the parallel system can be shown in the same line. We have

$$\bar{F}_{\tau_{n|n}^{\phi}(\boldsymbol{X})}(x) = \phi(n\psi(\bar{F}_X(x))), \quad x>0,$$

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which gives

$$\frac{f'_{\tau^{\phi}_{n|n}(\boldsymbol{X})}(x)}{f_{\tau^{\phi}_{n|n}(\boldsymbol{X})}(x)} = \frac{f'_X(x)}{f_X(x)} - r_X(x)\left[\frac{G(n\psi(\bar{F}_X(x)))}{R(\psi(\bar{F}_X(x)))} - \frac{G(\psi(\bar{F}_X(x)))}{R(\psi(\bar{F}_X(x)))}\right], \quad x > 0.$$

Since X is ILR (resp. DLR), we have that $f'_X(x)/f_X(x)$ is decreasing (resp. increasing) in $x > 0$, and $r_X(x)$ is increasing (resp. decreasing) in $x > 0$. Thus, to prove the result, it suffices to show that

$$\frac{G(n\psi(\bar{F}_X(x)))}{R(\psi(\bar{F}_X(x)))} - \frac{G(\psi(\bar{F}_X(x)))}{R(\psi(\bar{F}_X(x)))} \tag{A.33}$$

is positive and increasing (resp. decreasing) in $x > 0$. Since ψ is a decreasing function, we have that

$$\psi(\bar{F}_X(x)) \quad \text{is increasing in } x > 0. \tag{A.34}$$

Again, from the assumption, we have that

$$\frac{G(nu)}{R(u)} - \frac{G(u)}{R(u)} \quad \text{is positive and increasing (resp. decreasing) in } u > 0. \tag{A.35}$$

On combining (A.34) and (A.35), we get (A.33) and hence, the result is proved. □

Proof of Theorem 3.7(b). To prove the result, we have to show that

$$\bar{F}_{\tau^{\phi}_{n|n}(\boldsymbol{X})}(\alpha x) \geq \text{(resp. } \leq\text{)} \ (\bar{F}_{\tau^{\phi}_{n|n}(\boldsymbol{X})}(x))^{\alpha}, \quad x > 0,$$

or equivalently,

$$\phi(n\psi(\bar{F}_X(\alpha x))) \geq \text{(resp. } \leq\text{)}(\phi(n\psi(\bar{F}_X(x))))^{\alpha}, \quad x > 0 \text{ and } 0 < \alpha < 1. \tag{A.36}$$

Since X is IFRA (resp. DFRA), we have

$$\bar{F}_X(\alpha x) \geq \text{(resp. } \leq\text{)}(\bar{F}_X(x))^{\alpha}, \quad \text{for all } x > 0 \text{ and } 0 < \alpha < 1.$$

On using the decreasing property of ϕ in the above inequality, we get

$$\phi(n\psi(\bar{F}_X(\alpha x))) \geq \text{(resp. } \leq\text{)}\phi(n\psi((\bar{F}_X(x))^{\alpha})), \tag{A.37}$$

for all $x > 0$ and $0 < \alpha < 1$. Again, the assumption "$uR'(u)/R(u)$ is increasing (resp. decreasing) in $u > 0$" implies that

$$-\ln(\phi(n\psi(e^{-u}))) \quad \text{is convex (resp. concave) in } u > 0,$$

which further implies that

$$-\ln(\phi(n\psi(e^{-u}))) \quad \text{is starshaped (resp. anti-starshaped) in } u > 0,$$

or equivalently,

$$-\ln(\phi(n\psi(e^{-\alpha u}))) \leq \text{(resp. } \geq\text{)} - \alpha\ln(\phi(n\psi(e^{-u}))), \quad \text{for all } u > 0 \text{ and } 0 < \alpha < 1.$$

Furthermore, this implies that

$$\phi(n\psi(u^{\alpha})) \geq \text{(resp. } \leq\text{)}(\phi(n\psi(u)))^{\alpha}, \quad \text{for all } 0 < u < 1 \text{ and } 0 < \alpha < 1,$$

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and hence,

$$\phi(n\psi((\bar{F}_X(x))^\alpha)) \geq (\text{resp. } \leq)(\phi(n\psi(\bar{F}_X(x))))^\alpha, \tag{A.38}$$

for all $x > 0$ and $0 < \alpha < 1$. On combining (A.37) and (A.38), we get (A.36). Hence, the result is proved. □

Proof of Theorem 3.7(c). Note that, for all $x > 0$,

$$\frac{-\ln \bar{F}_{\tau_{1|n}(\boldsymbol{X})}(x)}{x} = \left(\frac{-\ln(1-\phi(\psi(F_X(x))))}{x}\right)\left(\frac{-\ln(1-\phi(n\psi(F_X(x))))}{-\ln(1-\phi(\psi(F_X(x))))}\right).$$

Since X is IFRA (resp. DFRA), we have that

$$\frac{-\ln(1-\phi(\psi(F_X(x))))}{x} \quad \text{is increasing (resp. decreasing) in } x > 0. \tag{A.39}$$

Again, from the assumption "$H(u)/\log(1-\phi(u))$ is increasing (resp. decreasing) in $u > 0$", we get that

$$\frac{-\ln(1-\phi(n\psi(F_X(x))))}{-\ln(1-\phi(\psi(F_X(x))))} \quad \text{is increasing (resp. decreasing) in } x > 0. \tag{A.40}$$

On combing (A.39) and (A.40), we get that

$$\frac{-\ln(1-\phi(n\psi(F_X(x))))}{x} \quad \text{is increasing (resp. decreasing) in } x > 0,$$

or equivalently,

$$\frac{-\ln \bar{F}_{\tau_{1|n}(\boldsymbol{X})}(x)}{x} \quad \text{is increasing (resp. decreasing) in } x > 0,$$

and hence, the result follows. □

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