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On a Conjecture of Jacquet, Lai, and Rallis: Some Exceptional Cases

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Abstract. We prove two spectral identities. The first one relates the relative trace formula for the spherical variety $\mathrm{GSpin}(4,3)/G_2$ with a weighted trace formula for GL_2. The second relates a spherical variety pertaining to F_4 to one of $\mathrm{GSp}(6)$. These identities are in accordance with a conjecture made by Jacquet, Lai, and Rallis, and are obtained without an appeal to a geometric comparison.

1 Introduction

Let G be a reductive group over a number field F and let H be a subgroup of G which is obtained as the fixed points of an involution θ defined over F. One can relate the double coset space $H\backslash G/H$ to conjugacy classes of a third group G' [KR71]. In a paper whose importance, we believe, is underestimated, Jacquet, Lai, and Rallis [JLR93] conjectured that there is a functorial relation between automorphic representations π' of $G'(\mathbb{A})$ and automorphic representations π which are H-distinguished, *i.e.,* for which the functional $\ell_H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\,dh$ (suitably regularized if π is non-cuspidal) is non-zero on V_π. (The group G' may be non-algebraic in this setup.) Moreover, roughly speaking, for test functions f and f' on $G(\mathbb{A})$ and $G'(\mathbb{A})$ satisfying certain compatibility conditions, there should exist trace identities

$$\int_{(H(F)\backslash H(\mathbb{A}))^2} K_f(h_1,h_2)\,dh_1dh_2 = \int_{G'\backslash G'(\mathbb{A})} K_{f'}(x,x)\Theta'(x)\,dx, \tag{1}$$

$$\int_{(H(F)\backslash H(\mathbb{A}))^2} K_f(h_1,h_2)\Theta(h_2)\,dh_1dh_2 = \int_{G'\backslash G'(\mathbb{A})} K_{f'}(x,x)\,dx. \tag{2}$$

for appropriate automorphic weight functions Θ and Θ' on $H(\mathbb{A})$ and $G'(\mathbb{A})$, respectively. (It may be necessary to vary over inner forms of either H or G'.) The test case considered in [JLR93] was $G = GL_2/E$, H ranges over unitary similitude groups, $G' = GL_2/F$ and E/F is a quadratic extension. Identities like (1) and (2) lead, with some effort, to spectral identities of the form

$$\mathcal{B}^{\pi}_{\ell_H,\ell_H}(f) = \mathcal{T}_{\Theta'}\pi'(f'), \tag{3}$$

$$\mathcal{B}^{\pi}_{\ell_H,\ell_{\Theta,H}}(f) = \operatorname{tr}\pi'(f'). \tag{4}$$

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Here $\mathcal{B}^{\pi}_{\ell_H,\ell_H}(f)$ is the Bessel distribution with respect to ℓ_H and ℓ_H which is defined by

$$\mathcal{B}^{\pi}_{\ell_H,\ell_H}(f) = \sum_{\varphi} \ell_H(\pi(f)\varphi)\overline{\ell_H(\varphi)},$$

the sum ranging over an orthonormal basis of V_π; similarly for $\mathcal{B}^{\pi}_{\ell_H,\ell_{\Theta,H}}(f)$, where

$$\ell_{\Theta,H}(\varphi) = \int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\Theta(h)\,dh.$$

Finally, $\mathcal{T}_{\Theta'}\pi'(f')$ is the "weighted trace"

$$\sum_{\varphi'} \int_{G'(F)\backslash G'(\mathbb{A})} [\pi'(f')\varphi'](x)\overline{\varphi'(x)}\Theta'(x)\,dx.$$

Our goal here is to give some more examples where relations like (3), (4) and similar ones hold. The cases we consider are special cases of the following setup. Let P be a maximal subgroup of G with Levi part M. Let ϖ be the fundamental weight corresponding to P, considered as a (fractional power of a) rational character of M. We will assume that G (or, equivalently, M) is quasi-split, and consider Eisenstein series $E(\cdot, \varphi, s\varpi)$ induced from cuspidal representations of $M(\mathbb{A})$ which admit a non-zero, non-degenerate Fourier coefficient. A residue $E_{-1}(\cdot, \varphi)$ of such an Eisenstein series (necessarily belonging to the discrete spectrum of $G(F)\backslash G(\mathbb{A})$) is often distinguished with respect to a spherical subgroup H. (In all known cases, and presumably in general, there is at most one residue point $s_0\varpi$ for $\mathrm{Re}(s) > 0$, and s_0 is either $\frac{1}{2}$ or 1.) Moreover, one can relate the period integral of this residue to a period integral over $M_H = M \cap H$ of the section inducing it. On the other hand, no cuspidal representation of $G(\mathbb{A})$ (generic or not) is distinguished with respect to H.

The first identity (3) was previously considered in [LR04] and [JLR04]. It relies on an additional formula of the period integral of $E_{-1}\varphi$ in terms of a "co-period" of the constant term. In the first part of the current paper we consider the group $G = \mathrm{GSpin}(4, 3)$ of semi-simple rank 3 containing $M = GL_2 \times GL_2$ as a Levi subgroup. The period subgroup H is the exceptional rank two group G_2. The new feature in this case is that we are able to prove the second identity (4) by obtaining a formula for the co-period of $E_{-1}\varphi$ in terms of its constant term. There seem to be difficult analytic, and perhaps conceptual, problems in deriving such formulas (and therefore, identity 4) in the cases considered in [JLR04, LR04]. In the second part of the paper we consider the exceptional group F_4 and its Levi subgroup GSp_6. The period is obtained by integrating a certain Fourier coefficient over its reductive stabilizer. This is the first time a non-reductive period subgroup is considered for spectral identities analogous to the ones above. This gives rise to additional complications in the derivation of the spectral identities from the period and co-periods formulas. This case can be viewed as "unipotent thickening" of the previous case, in the sense that the stabilizer of the Fourier coefficient is G_2 as before. This procedure is common in integral representations of L-functions, and go back to the exterior square L-function of Jacquet–Shalika [JS90]. Finally, we mention that the spectral identities and the period formulas in this paper are very much inspired by the work of Jiang and Ginzberg [Jia98, GJ00].

1.1 Notation and Preliminaries

We will consider split groups G over a number field F and choose a Borel subgroup $B = T_0U_0$, as well as a "good" maximal compact $\mathbf{K}$ of $G(\mathbb{A})$. If X is defined over F we will often write X for the F-points of X as well. For any rational character λ of T_0 we will extend the character $|\lambda|(t) = \prod_v |\lambda(t_v)|_v$ on $T_0(\mathbb{A})$ to a left-$U_0(\mathbb{A})$ and right-$\mathbf{K}$-invariant function on $G(\mathbb{A})$. We will still denote this function by $|\lambda|$.

Let $P = MU$ be a maximal parabolic of G and ϖ the corresponding fundamental weight. The intersection T_M of the center of M with the derived group of G is a one-dimensional torus. We will imbed $\mathbb{R}_+$ in $T_M(\mathbb{A})$ diagonally in the archimedean places. For any cuspidal automorphic function φ on $M(F)U(\mathbb{A})\backslash G(\mathbb{A})$ satisfying

$$\varphi(ag) = \delta_P(a)^{\frac{1}{2}}\varphi(g) \quad \forall a \in \mathbb{R}_+$$

we consider the Eisenstein series defined for $\mathrm{Re}(s) \gg 0$ by

$$E(g, \varphi, s) = \sum_{\gamma \in P\backslash G} \varphi(\gamma g)|\varpi|^s(\gamma g).$$

It admits a meromorphic continuation to the complex plane and a functional equation $s \mapsto -s$, $\varphi \mapsto M(s)\varphi$, where $M(s)$ is the usual intertwining operator.

2 The Pair $(\mathrm{GSpin}(4,3), G_2)$

We first consider the group $G = \mathrm{GSpin}(4,3)$. Write the simple roots of G as γ_1, γ_2 and γ_3 where γ_1 and γ_2 are contained in GL_3 and the root groups X_{γ_1} and X_{γ_3} commute. We take the maximal parabolic subgroup P defined by γ_2. Its Levi part is $M = GL(2) \times \mathrm{GSpin}(3) = GL(2) \times GL(2)$. Under this identification, $\varpi(g_1, g_2) = \det(g_1)$. The modulus function of $P(\mathbb{A})$ is $|\varpi|^4$. We set $\rho_P = 2$ so that the modulus function is $|\varpi|^{2\rho_P}$. Starting from a cuspidal automorphic representation of π of GL_2 we construct the Eisenstein series on G induced from $\pi \otimes \pi$. We have

$$M(s)\varphi = \frac{L(s, \pi \otimes \tilde{\pi})L(2s, \pi, \mathrm{Ad})}{L(s+1, \pi \otimes \tilde{\pi})L(2s+1, \pi, \mathrm{Ad})} M_S(s)\varphi_S$$

(*cf.* [Asg02]). We consider the residue $E_{-1}(g, \varphi)$ of $E(g, \varphi, s)$ at $s_0 = 1$.

We now consider the period subgroup $H = G_2$ imbedded in G in the usual way (see [Jia98]). In particular, we denote by α and β the simple roots of H so that X_α is contained in $X_{\gamma_1} \cdot X_{\gamma_3}$ and $X_\beta = X_{\gamma_2}$. The other positive roots are $\alpha + \beta$, $2\alpha + \beta$, $3\alpha + \beta$ and $3\alpha + 2\beta$. In this case $M_H = M \cap H = GL(2)$ imbedded diagonally in M. It is the Levi part of the "Heisenberg" parabolic $P_H = P \cap H$ of H defined by β. The unipotent radical is $U_H = U \cap H$. The restriction ϖ_H of ϖ to M_H is the determinant, which is also the fundamental weight of P_H. We will simply identify ϖ_H with ϖ. The modulus function of $P_H(\mathbb{A})$ is $|\varpi|^3$. Set $\rho_{P_H} = 3/2$.

Jiang [Jia98] considered the H-period of $E_{-1}(\,\cdot\,, \varphi)$ and obtained the formula

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h, \varphi)\, dh = \int_{\mathbf{K}_H} \int_{M_H\backslash M_H(\mathbb{A})^1} \varphi(mk)\, dm dk. \tag{5}$$

In particular, this H-period is non-zero.[1]

Consider the Eisenstein series

$$\mathcal{E}(h,s) = \sum_{\gamma \in P_H \backslash H} |\varpi|^{s+\rho_{P_H}}(\gamma h)$$

(unitarily) induced from $|\varpi|^s$ on M_H. This Eisenstein series was considered in [GJ00]. Its singularities for $\mathrm{Re}(s) > 0$ are a simple pole at $s = \rho_{P_H} = \frac{3}{2}$, where the residue is the constant function $\lambda_{-1}/\zeta_F(6)$ (with $\lambda_{-1} = \mathrm{res}_{s=1}\zeta_F(s) = \mathrm{vol}(F^* \backslash \mathbb{I}_F^1)$), and a double pole at $s = \frac{1}{2}$.[2] We write the Laurent expansion near $s = \frac{1}{2}$ as

$$\mathcal{E}(\,\cdot\,,s) = \frac{\mathcal{E}_{-2}(\,\cdot\,)}{(s-\frac{1}{2})^2} + \frac{\mathcal{E}_{-1}(\,\cdot\,)}{s-\frac{1}{2}} + O(1).$$

We also have the right-$K \cap M_H(\mathbb{A})$-invariant Eisenstein series $\mathbb{E}(\,\cdot\,,s)$ on $M_H(\mathbb{A})$ (unitarily) induced from $\mathrm{Ind}|t_1/t_2|^{s/2}$.

The following theorem gives an alternative formula for the H-period, as well as formulas for certain co-periods.

Theorem 2.1 *We have*

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\,dh = \frac{\zeta_F(2)}{\zeta_F(3)}\frac{\zeta_F(6)}{\lambda_{-1}} \int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^1} M_{-1}\varphi(mk)\mathbb{E}(m,5)\,dm dk. \tag{6}$$

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}(h,s)\,dh = 0 \text{ for } |\mathrm{Re}(s)| < \frac{1}{2}. \tag{7}$$

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}_{-2}(h)\,dh = 0. \tag{8}$$

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}_{-1}(h)\,dh = \int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^1} M_{-1}\varphi(mk)\,dm dk. \tag{9}$$

In contrast, for cusp forms of $G(\mathbb{A})$, $\int_{H\backslash H(\mathbb{A})} \phi(h)\mathcal{E}(h,s)\,dh$ gives the L-function corresponding to the second fundamental weight [GR94].

To prove Theorem 2.1 we define, for a holomorphic function $\sigma \in \mathcal{P}(\mathbb{C})$ of the Paley–Wiener type, a function θ_σ on $H\backslash H(\mathbb{A})$ by

$$\theta_\sigma(h) = \int_{\mathrm{Re}(s)\gg 0} \sigma(s)\mathcal{E}(h,s)\,ds = \sum_{\gamma\in P_H\backslash H} \mathcal{F}_\sigma(\gamma h), \tag{10}$$

where

$$\mathcal{F}_\sigma(h) = \int_{\mathrm{Re}(s)=0} \sigma(s)|\varpi|^{s+\rho_{P_H}}(h)\,ds.$$

[1] Strictly speaking [Jia98,GJ00,GR94] deal with $G = \mathrm{SO}(4,3)$, but the extension to $\mathrm{GSpin}(7)$ is straightforward.

[2] Note that our normalization differs from that of [GJ00].

(The function θ_σ is *not* rapidly decreasing.) Set

$$\mathfrak{P}(\sigma,\varphi) = \int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\theta_\sigma(h)\,dh.$$

The main assertion is the following.

Theorem 2.2 *We have*

$$(11)\quad \mathfrak{P}(\sigma,\varphi) = 2\pi i\sigma\Big(\frac{1}{2}\Big)\int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^1} M_{-1}\varphi(mk)\,dmdk \\ + 2\pi i\sigma\Big(\frac{3}{2}\Big)\frac{\zeta_F(2)}{\zeta_F(3)}\int_{\mathbf{K}_H}\int_{M_H\backslash M_H(\mathbb{A})^1} M_{-1}\varphi(mk)\mathbb{E}(m,5)\,dmdk.$$

We will prove Theorem 2.2 in the next section. To see that Theorem 2.2 implies Theorem 2.1, we write

$$\mathfrak{P}(\sigma,\varphi) = \int_{H\backslash H(\mathbb{A})}\int_{\operatorname{Re} s\gg 0} E_{-1}(h,\varphi)\sigma(s)\mathcal{E}(h,s)\,dsdh.$$

Shifting the contour of integration to the imaginary axis we obtain

$$2\pi i\sigma\Big(\frac{3}{2}\Big)\frac{\lambda_{-1}}{\zeta_F(6)}\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\,dh + 2\pi i\sigma\Big(\frac{1}{2}\Big)\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}_{-1}(h)\,dh \\ + 2\pi i\sigma'\Big(\frac{1}{2}\Big)\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}_{-2}(h)\,dh \\ + \int_{\operatorname{Re} s=0}\sigma(s)\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}(h,s)\,dhds.$$

Compare this with (11). We obtain Theorem 2.1 as in [JLR04, Lemma 3]. For this we need to know the following lemma.

Lemma 2.3 *The integrals*

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\,dh, \quad \int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}_{-1}(h)\,dh,$$

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}_{-2}(h)\,dh, \quad \int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}(h,s)\,dh \quad \textit{for } |\operatorname{Re} s| < \frac{1}{2},$$

are absolutely convergent. Moreover, for any $0 < s_1 < \frac{1}{2}$,

$$\sup_{\operatorname{Re} s=s_1}\int_{H\backslash H(\mathbb{A})} |E_{-1}(h,\varphi)\mathcal{E}(h,s)|\,dh < \infty.$$

Proof The only exponent of $E_{-1}(h,\varphi)$ is along P and it is $-s_0\varpi$. Since the Siegel set of H can be chosen to be contained in that of G, the condition on an automorphic form f on H for $\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)f(h)\,dh$ to converge is that $-s_0+\rho_P+\operatorname{Re}\mu+\rho_{P_H}-2\rho_{P_H}<0$ for any exponent of $\mu\varpi$ of f along P_H. Substituting, this becomes $\operatorname{Re}\mu<1/2$. The exponents of $\mathcal{E}(\,\cdot\,,s)$ along the Borel are

$$\Bigl(s-\frac{1}{2},s+\frac{1}{2}\Bigr),\ \Bigl(s-\frac{1}{2},-1\Bigr),\ \Bigl(-1,s-\frac{1}{2}\Bigr),$$
$$\Bigl(-1,-\frac{1}{2}-s\Bigr),\ \Bigl(-\frac{1}{2}-s,-1\Bigr),\ \Bigl(-\frac{1}{2}-s,\frac{1}{2}-s\Bigr),$$

where (x,y) corresponds to the character $t_1,t_2\mapsto |t_1|^x|t_2|^y$ of the standard torus of $M_H=GL_2$. Hence, the exponents of $\mathcal{E}(\,\cdot\,,s)$ along P_H are

$$s\varpi,\quad \frac{1}{2}\Bigl(s-\frac{3}{2}\Bigr)\varpi,\quad \frac{1}{2}\Bigl(-s-\frac{3}{2}\Bigr)\varpi,\quad -s\varpi.$$

For $\mathcal{E}_{-1}$ and $\mathcal{E}_{-2}$ the exponent $\frac{1}{2}\varpi$ does not appear [GJ00]. Therefore the exponents of $f=1,\mathcal{E}_{-1}(h),\mathcal{E}_{-2}(h)$ and $\mathcal{E}(h,s)$ with $|\operatorname{Re}(s)|<\frac{1}{2}$ all satisfy $\operatorname{Re}\mu<\frac{1}{2}$ as required. The last part of the lemma is proved exactly as in [LR04]. ■

As a conclusion from Theorem 2.1, we obtain the following.

Theorem 2.4 *Let Π be the automorphic representation of $G(\mathbb{A})$ on $E_{-1}(\,\cdot\,,\varphi)$. Then*

$$\mathcal{B}^{\Pi}_{\ell_H,\ell_H}(f)=\frac{\zeta_F(2)}{\zeta_F(3)}\frac{\zeta_F(6)}{\lambda_{-1}}\cdot\mathcal{B}^{\pi}_{\ell_{M_H},\ell_{M_H,\mathcal{E}(\cdot,5)}}(f'_{\mathbf{K}_H}),$$

where $f'_{\mathbf{K}_H}$ is the function on $M(\mathbb{A})$ defined by

$$f'_{\mathbf{K}_H}(m)=e^{\langle\varpi+\rho_P,H_M(m)\rangle}\cdot\int_{\mathbf{K}_H}\int_{\mathbf{K}_H}\int_{U(\mathbb{A})}f(k'muk)\,du\,dk'\,dk.$$

Similarly,

$$\mathcal{B}^{\Pi}_{\ell_H,\ell_{H,\mathcal{E}_{-1}}}(f)=\mathcal{B}^{\pi}_{\ell_{M_H},\ell_{M_H}}(f'_{\mathbf{K}_H}).$$

This is proved exactly as in [JLR04].

Before proving Theorem 2.2 in the next section, we recall a useful computational trick.

2.1 Exchange of Roots

This procedure was first employed by Jacquet and Shalika [JS90]. It is often used in the study of integral representations [GR94, Gin95a, Gin95b, BG00, GR00], but it is very useful in other instances as well [GRS99, GRS02]. For the convenience of the reader, we record it in much greater generality than needed for the purpose of this paper. It is a global version of [GRS99, Lemma 2.2].

Whenever U is a unipotent group and ψ_U is a character on $U\backslash U(\mathbb{A})$, we write $f^{\psi_U}(g)$ for the Fourier coefficient $\int f(ug)\psi_U(u)\,du$.

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Lemma 2.5 *Let X, Y, X', Y' be unipotent subgroups of G and let ψ be a character of $Y'\backslash Y'(\mathbb{A})$. Suppose that the following conditions are satisfied.*

(i) *X (resp. X') is a normal subgroup of Y (resp. Y') and Y/X (resp Y'/X') is Abelian.*
(ii) *$Y \cap Y' = 1$.*
(iii) *Y normalizes X' and Y' normalizes $W = Y \ltimes X'$.*
(iv) *The character $\psi|_{X'\backslash X'(\mathbb{A})}$ is normalized by $Y(\mathbb{A})$; we denote its extension to $W\backslash W(\mathbb{A})$ (trivial on $Y(\mathbb{A})$) by ψ_W.*
(v) *(Y', ψ) is normalized by X; we denote by $\psi_{W'}$ the extension of ψ to $W'\backslash W'(\mathbb{A})$ (trivial on $X(\mathbb{A})$) where $W' = X \ltimes Y'$.*
(vi) *The set $\{\psi_W(\gamma \cdot \gamma^{-1}) : \gamma \in Z = Y'/X'\}$ ranges over the characters of $X(\mathbb{A})Y\backslash Y(\mathbb{A})$.*

Then for any smooth function on $G\backslash G(\mathbb{A})$ we have

$$\phi^{\psi_{W'}}(g) = \int_{Z(\mathbb{A})} \phi^{\psi_W}(zg)\psi(z)\, dz. \tag{12}$$

Proof It is enough to check this for $g = e$. Using (iv) we write the right-hand side as

$$\begin{aligned}
&\int_{Z\backslash Z(\mathbb{A})} \sum_{\gamma \in Z} \int_{W\backslash W(\mathbb{A})} \phi(\gamma^{-1} w \gamma z)\psi_W(w)\psi(z)\, dwdz \\
&\qquad = \int_{Z\backslash Z(\mathbb{A})} \sum_{\gamma \in Z} \int_{W\backslash W(\mathbb{A})} \phi(wz)\psi_W(\gamma w \gamma^{-1})\psi(z)\, dwdz \\
&\qquad = \int_{Z\backslash Z(\mathbb{A})} \sum_{\gamma \in Z} \int_{Y\backslash Y(\mathbb{A})} \int_{X'\backslash X'(\mathbb{A})} \phi(yx'z)\psi(y)\psi_W(\gamma x' \gamma^{-1})\psi(z)\, dydx'dz \\
&\qquad = \int_{Z\backslash Z(\mathbb{A})} \sum_{\gamma \in Z} \int_{X(\mathbb{A})Y\backslash Y(\mathbb{A})} \int_{X\backslash X(\mathbb{A})} \int_{X'\backslash X'(\mathbb{A})} \phi(yxx'z)\psi(y) \\
&\qquad\qquad \times \psi_W(\gamma x' \gamma^{-1})\psi(z)\, dydxdx'dz.
\end{aligned}$$

By assumption (vi), the sum over γ is the Fourier expansion along Y/X of the function $\int_{X\backslash X(\mathbb{A})} \int_{X'\backslash X'(\mathbb{A})} \phi(yx \cdot z)\psi(y)\, dydx$. By (v), we obtain

$$\begin{aligned}
&\int_{Z\backslash Z(\mathbb{A})} \int_{X\backslash X(\mathbb{A})} \int_{X'\backslash X'(\mathbb{A})} \phi(yxz)\psi(y)\psi(z)\, dydxdz \\
&\qquad\qquad = \int_{X'(\mathbb{A})XY'\backslash XY'(\mathbb{A})} \int_{X'\backslash X'(\mathbb{A})} \phi(yz)\psi(y)\psi_{W'}(z)\, dydz.
\end{aligned}$$

This is equal to the left-hand side of (12). ■

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3 Proof of Theorem 2.2

To begin the proof of Theorem 2.2 we unfold $\theta_\sigma(h)$. Thus,

$$\mathfrak{P}(\sigma,\varphi) = \int_{U_H(\mathbb{A})M_H\backslash H(\mathbb{A})} \int_{U_H\backslash U_H(\mathbb{A})} E_{-1}(uh,\varphi)\mathfrak{F}_\sigma(h)\,dudh. \tag{13}$$

Using Fourier analysis, we expand $\int_{U_H\backslash U_H(\mathbb{A})} E_{-1}(uh,\varphi)$ along the quotient group $U_H\backslash U$ with points in $F\backslash\mathbb{A}$. The space $U_H\backslash U$ is two dimensional and M_H acts on it as the standard representation. Thus there are two orbits on the dual space. The trivial orbit gives

$$\int_{U_H(\mathbb{A})M_H\backslash H(\mathbb{A})} E^U_{-1}(h,\varphi)\mathfrak{F}_\sigma(h)\,dh,$$

which can be written as

$$\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_{\mathbb{R}_+} |\varpi|^{-s_0+\rho_P-2\rho_{P_H}}(z)\mathfrak{F}_\sigma(z)\,dzM_{-1}\varphi(mk)\,dmdk.$$

Since $-s_0+\rho_P-2\rho_{P_H} = -2 = -(\frac{1}{2}+\rho_{P_H})$, the inner integral is $2\pi i\sigma(1/2)$ which gives the first contribution to (11).

To compute the contribution from the other orbit, we take as a representative the character $\psi_U(u) = \psi_G(x_{\gamma_1+\gamma_2}(r_1)+x_{\gamma_2+\gamma_3}(r_2)) = \psi(r_1+r_2)$. Its stabilizer in M_H is of the form $T'U'$ where U' is the unipotent subgroup corresponding to α and T' is a one dimensional torus. We denote by $\psi_{UU'}$ its extension to $UU'(\mathbb{A})$, which is trivial on $U'(\mathbb{A})$. The contribution is

$$\int_{U_H(\mathbb{A})T'U'\backslash H(\mathbb{A})} E^{\psi_U}_{-1}(h,\varphi)\mathfrak{F}_\sigma(h)\,dh = \int_{N_H(\mathbb{A})T'\backslash H(\mathbb{A})} E^{\psi_{UU'}}_{-1}(h,\varphi)\mathfrak{F}_\sigma(h)\,dh,$$

where $N_H = U_0\cap H$ is the maximal unipotent of H.

Let V be the unipotent radical of the rank-one parabolic Q containing the root γ_2 in its Levi part.

We now use Lemma 2.5 with $X = U'$, $Y = M\cap V$, $X' = U\cap V$, $Y' = U$. We obtain

$$\int_{N_H(\mathbb{A})T'\backslash H(\mathbb{A})}\int_{X_{\gamma_2}(\mathbb{A})} E^{\psi_V}_{-1}(uh,\varphi)\,du\mathfrak{F}_\sigma(h)\,dh,$$

where ψ_V is trivial on $M(\mathbb{A})\cap V(\mathbb{A})$ and coincides with ψ_U on $U(\mathbb{A})\cap V(\mathbb{A})$. We can write this as

$$\int_{V(\mathbb{A})T'\backslash H(\mathbb{A})} E^{\psi_V}_{-1}(h,\varphi)\mathfrak{F}_\sigma(h)\,dh.$$

Note that $E^{\psi_V}_{-1}(h) = E^{\psi'_V}_{-1}(w_\beta h)$ where ψ' is trivial on $U\cap V$ and is non-degenerate on γ_1,γ_3. We have $E^{\psi'_V}_{-1} = (E^U_{-1})^{U_0\cap M,\psi'} = (M_{-1}\varphi)^{U_0\cap M,\psi}$. Indeed, when we expand $E^{\psi'_V}_{-1}$ along X_{γ_2} only the trivial orbit contributes since E_{-1} is not generic. Thus, we obtain

$$\int_{V(\mathbb{A})T'\backslash H(\mathbb{A})} (M_{-1}\varphi)^{\psi_{U_0^M}}(w_\beta h)\mathfrak{F}_\sigma(h)\,dh.$$

Changing the variable and noting that $w_\beta T' w_\beta^{-1}$ is the center Z of M_H, we obtain

$$\int_{V(\mathbb{A})Z\backslash H(\mathbb{A})} (M_{-1}\varphi)^{\psi_{U_0^M}}(h)\mathcal{F}_\sigma(w_\beta h)\,dh = \int_{N_H(\mathbb{A})Z\backslash H(\mathbb{A})} (M_{-1}\varphi)^{\psi_{U_0^M}}(h) M_{w_\beta}\mathcal{F}_\sigma(h)\,dh,$$

where $M_{w_\beta}\mathcal{F}_\sigma(h) = \int_{X_\beta(\mathbb{A})} \mathcal{F}_\sigma(w_\beta u h)\,du$. We have

$$M_{w_\beta}\mathcal{F}_\sigma(h) = \int_{\mathrm{Re}(s)\gg 0} \sigma(s)\frac{\zeta_F(s+\frac{1}{2})}{\zeta_F(s+\frac{3}{2})}\big|(s+\rho_{P_H})w_\beta\varpi+\beta\big|(h).$$

Using the Iwasawa decomposition with resect to P_H, we obtain

$$\int_{K_H}\int_{X_\alpha(\mathbb{A})Z\backslash M_H(\mathbb{A})^1}\int_{\mathbb{R}_+} |\varpi|^{-s_0+\rho_P-2\rho_{P_H}}(z) M_{w_\beta}\mathcal{F}_\sigma(zm)\,dz (M_{-1}\varphi)^{\psi_{U_0^M}}(mk)\,dmdk.$$

The inner integral is $2\pi i\sigma(s_1)|(s_1+\rho_{P_H})w_\beta\varpi+\beta|(m)$ for the s_1 such that

$$(-s_0+\rho_P-2\rho_{P_H})\varpi+(s_1+\rho_{P_H})w_\beta\varpi+\beta=\kappa\alpha$$

for some κ. We obtain $s_1=3/2$ and $\kappa=3$, and the result is

$$2\pi i\sigma(3/2)\frac{\zeta_F(2)}{\zeta_F(3)}\int_{K_H}\int_{X_\alpha(\mathbb{A})Z\backslash M_H(\mathbb{A})^1} (M_{-1}\varphi)^{\psi_{U_0^M}}(mk)|\kappa\alpha|(m)\,dmdk.$$

Using the usual unfolding of the Rankin–Selberg integral, we obtain

$$2\pi i\sigma(3/2)\frac{\zeta_F(2)}{\zeta_F(3)}\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1} (M_{-1}\varphi)(mk)\mathbb{E}(m,2\kappa-1)\,dmdk$$

as required.

4 The F_4 Case

We let $G=F_4$ and enumerate its simple roots as $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ with

$$\langle\alpha_1,\alpha_2^\vee\rangle=\langle\alpha_2,\alpha_1^\vee\rangle=\langle\alpha_3,\alpha_4^\vee\rangle=\langle\alpha_4,\alpha_3^\vee\rangle=\frac{1}{2}\langle\alpha_2,\alpha_3^\vee\rangle=-1.$$

For integers n_1,n_2,n_3,n_4, we denote by $(n_1n_2n_3n_4)$ the weight $n_1\alpha_1+n_2\alpha_2+n_3\alpha_3+n_4\alpha_4$ and by $(n_1n_2n_3n_4)^\vee$ the co-weight $n_1\alpha_1^\vee+n_2\alpha_2^\vee+n_3\alpha_3^\vee+n_4\alpha_4^\vee$. (Note that if $(n_1n_2n_3n_4)$ is a root, then $(n_1n_2n_3n_4)^\vee$ is *not* necessarily a co-root.) Also, for a sequence $x_i\in\{1,2,3,4\}$ we define $w[x_1x_2\cdots x_m]$ to be the element $w[x_1]\cdots w[x_n]$ in the Weyl group of G where $w[j]$ is the simple reflection corresponding to α_j.

We let P be the maximal parabolic of G corresponding to α_1. Its Levi part is $M = GSp_6$. The fundamental weight $\varpi = (2342)$ is the similitude factor. Here $\rho_P = 4$. Let τ be a cuspidal representation on $M(\mathbb{A})$.

Let $E_\tau(g, s)$ denote the Eisenstein series on $G(\mathbb{A})$ induced from $\tau|\varpi|^s$. The poles of this Eisenstein series are determined by $L(s, \pi, \mathrm{Spin})L(2s, \pi, St)$ where the first L-function is the eight dimensional Spin L-function and the second one is the seven dimensional Standard L-function.

Let $Q' = L'V'$ be the maximal parabolic subgroup of G corresponding to α_4 and let Q denote the (non-standard) parabolic $wQ'w^{-1}$ where $w = w[231234]$. Its Levi subgroup $L = wL'w^{-1}$ is isomorphic to GSpin_7 and its unipotent radical $V = wV'w^{-1}$ is a two-step nilpotent group of dimension 15. Set $\gamma_1 = w(1000) = (0120)$, $\gamma_2 = w(0100) = (1000)$, $\gamma_3 = w(0010) = (0111)$. We define the character ψ_V to be ψ on the roots (0010), (0001). Its stabilizer H in L is isomorphic to G_2. In this embedding, $\alpha^\vee$ corresponds to $(0110)^\vee$ and $(0211)^\vee$ while $\beta^\vee = (1000)^\vee$. We write the simple roots of H as (0120)(0111) and (1000). The subgroup $P_H = P \cap H$ is the parabolic subgroup of H considered in Section 2, and the center Z of M_H coincides under the embedding above with the center of M. It corresponds to the co-root $(2321)^\vee$. The restriction ψ_{V_M} of ψ_V to $V_M = M \cap V$ is the character considered in [BG92, §2] (see [Vo97]). Let $V_1 = \bar{U} \cap V$ where $\bar{U}$ is the unipotent radical of the parabolic subgroup $\bar{P}$ opposite to P containing M. Explicitly, V_1 (which is commutative) is generated by the roots $-(1221)$, $-(1110)$, $-(1220)$, $-(1100)$.

Theorem 4.1 *We have*

$$\int_{H\backslash H(\mathbb{A})} E_{-1}^{\psi_V}(h,\varphi)\,dh = \int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_{V_1(\mathbb{A})} \varphi^{\psi_{V_M}}(mvk)\,dvdmdk$$
$$= \frac{\zeta_F(2)}{\zeta_F(3)}\frac{\zeta_F(6)}{\lambda_{-1}}\int_{V_1(\mathbb{A})}\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1} (M_{-1}\varphi)^{\psi_{V_M}}(mvk)\mathbb{E}(m,5)\,dmdkdv,$$

$$\int_{H\backslash H(\mathbb{A})} E_{-1}^{\psi_V}(h,\varphi)\mathcal{E}_{-2}(h)\,dh = 0,$$
$$\int_{H\backslash H(\mathbb{A})} E_{-1}^{\psi_V}(h,\varphi)\mathcal{E}_{-1}(h)\,dh = \int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_{V_1(\mathbb{A})} (M_{-1}\varphi)^{\psi_{V_M}}(mvk)\,dvdmdk.$$

Remark 1 For a cusp form φ on $M\backslash M(\mathbb{A})$, the integral

$$\int_{M_H\backslash M_H(\mathbb{A})^1} \varphi^{\psi_{V_M}}(m)\mathbb{E}(m, 2s-1)\,dm$$

is related to the spin L-function at s [BG92]. The residue at $s = 1$ is essentially $\int_{M_H\backslash M_H(\mathbb{A})^1} \varphi^{\psi_{V_M}}(m)\,dm$. The integrals over V_1 appearing in the theorem have no bearing on the unramified computation (*cf.* [GJ00]). We also recall from [GJ00] that for a cusp form φ on $G\backslash G(\mathbb{A})$, the integral $\int_{H\backslash H(\mathbb{A})} \phi^{\psi_V}(h,\varphi)\mathcal{E}(h,s)\,dh$ represents the L-function corresponding to the 26-dimensional representation of F_4.

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Proof We first consider the integral $I = \int_{H\backslash H(\mathbb{A})} \Theta_\phi^{\psi_V}(h,\varphi)\,dh$ for a pseudo-Eisenstein series $\Theta_\phi(g) = \sum_{P\backslash G} f(\gamma g)$.

We write

$$G = \bigcup_\eta P\eta Q = \bigcup_{\eta\in P\backslash G/Q}\ \bigcup_{\gamma\in(L\cap\eta^{-1}P\eta)\backslash L/H} P\eta\gamma HV.$$

It is easy to see that $\eta = e, w[12], w[1234]$ are representatives for $P\backslash G/Q$. The subgroup $L\cap\eta^{-1}P\eta$ is a maximal parabolic of L which corresponds to γ_2 if $\eta = 1$ and to γ_1 otherwise. Thus, if $\eta \neq 1$, then $L = H\cdot(L\cap\eta^{-1}P\eta)$ [PSRS92, §1]. For $\eta = w[12]$ the contribution of η to I factors through

$$\int_{(V\cap\eta^{-1}U\eta)\backslash(V(\mathbb{A})\cap\eta^{-1}U(\mathbb{A})\eta)} \psi_V(v)\,dv,$$

which is zero since $V\cap\eta^{-1}U\eta$ contains the root (0010). For $\eta = w[1234]$ the integral factors through $\int_{U_1\backslash U_1(\mathbb{A})} f(u\cdot)\,du$, where U_1 is the subgroup generated by the roots

$$(0011),\ (0111),\ (0121),\ (0122),\ (0010)(1122),\ (0110)(1222),\ (0120)(1232).$$

Since f is left-$U(\mathbb{A})$-invariant, this will be a constant term of f along a parabolic of M which is zero by cuspidality. For $\eta = 1$ there are two orbits in $(L\cap P)\backslash L/H$. The open orbit is $\gamma = w[1323]$. Once again, its contribution factors through

$$\int_{(V\cap\gamma^{-1}U\gamma)\backslash(V(\mathbb{A})\cap\gamma^{-1}U(\mathbb{A})\gamma)} \psi_V(v)\,dv,$$

which is zero since $V\cap\gamma^{-1}U\gamma$ contains the root (0001).

We conclude that only $\eta = \gamma = 1$ contributes, and its contribution is

$$\int_{P_H\backslash H(\mathbb{A})}\int_{V\cap P\backslash V(\mathbb{A})} f(vh)\psi_V(v)\,dv = \int_{P_H\backslash H(\mathbb{A})}\int_{V_1(\mathbb{A})} f^{\psi_{V_M}}(vh)\,dv.$$

By Iwasawa decomposition we can write this as

$$\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_{\mathbb{R}_+}\int_{V_1(\mathbb{A})} |\varpi|^{-2\rho_P^H}(z) f^{\psi_{V_M}}(vzmk)\,dzdvdmdk.$$

(Recall that $\mathbb{R}_+$ is identified with a subgroup of H via $\varpi^\vee$.) Conjugating z across V_1 we obtain a factor of $|\varpi|^{-2}(z)$ from the change of variables. We finally get

$$I = \int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_{V_1(\mathbb{A})} \hat{f}(s_0)^{\psi_{V_M}}(vmk)\,dvdmdk, \tag{14}$$

where $-(s_0+\rho_P) = -2\rho_{P_H} - 2$, *i.e.*, $s_0 = 1$. The first part of Theorem 4.1 follows from (14) by a standard argument.

For the other parts we need to analyze

$$J = \int_{H\backslash H(\mathbb{A})} E_{-1}^{\psi_V}(h)\theta_\sigma(h)\,dh.$$

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When we unfold the Eisenstein series on H and proceed as in [GJ00, §2], we obtain that $J = J_1 + J_2$, where

$$J_1 = \int_{U_H(\mathbb{A})P_H\backslash H(\mathbb{A})} \int_{V_1(\mathbb{A})} E_{-1}^{\psi_{UV_M}}(vh)\mathcal{F}_\sigma(h)\,dvdh,$$

$$J_2 = \int_{N_H(\mathbb{A})T'\backslash H(\mathbb{A})} \int_{V_2(\mathbb{A})} E_{-1}^{\psi_N}(vw[1]h)\mathcal{F}_\sigma(h)\,dvdh.$$

Here, V_2 is the unipotent subgroup of G generated by V_1 and the roots

$$-(1000),\ -(0100),\ -(0110);$$

N is the maximal unipotent subgroup of G; ψ_N is trivial on U and is Whittaker inside M; and T' is the torus of H defined in §2.

As before, we can write J_1 as

$$\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_{V_1(\mathbb{A})}\int_{\mathbb{R}_+} |\varpi|^{-s_0+\rho_P-2\rho_{P_H}}(z)|\varpi|^{-2}(z)E_{-1}^{\psi_{UV_M}}(vmk)\mathcal{F}_\sigma(zm)\,dzdvdmdk$$
$$= 2\pi i\sigma(s_1)\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_{V_1(\mathbb{A})} E_{-1}^{\psi_{UV_M}}(vmk)\,dvdmdk$$

for s_1, so that $-(s_1+\rho_{P_H}) = -s_0+\rho_P-2\rho_{P_H}-2$, *i.e.*, $s_1 = \frac{1}{2}$.

As for J_2, we write it as

$$\int_{V(\mathbb{A})T'\backslash H(\mathbb{A})}\int_{V_2'(\mathbb{A})} E_{-1}^{\psi_N}(vw[1]h)\mathcal{F}_\sigma(h)\,dvdh,$$

where V_2' is generated by V_1, and the roots $-(0100)$ and $-(0110)$. As in §2 we rewrite this as

$$\int_{V(\mathbb{A})Z\backslash H(\mathbb{A})}\int_{V_2'(\mathbb{A})} E_{-1}^{\psi_N}(vh)M_{w_\beta}\mathcal{F}_\sigma(h)\,dvdh$$
$$= \int_{K_H}\int_{X_\alpha(\mathbb{A})Z\backslash M_H(\mathbb{A})^1}\int_{V_2'(\mathbb{A})}\int_{\mathbb{R}_+} |\varpi|^{-s_0+\rho_P-2\rho_{P_H}-2}(z)M_{w_\beta}\mathcal{F}_\sigma(zm)\,dz$$
$$E_{-1}^{\psi_N}(vmk)\,dvdmdk.$$

Exactly the same as before, we obtain (with $\kappa = 3$)

$$2\pi i\sigma(3/2)\frac{\zeta_F(2)}{\zeta_F(3)}\int_{K_H}\int_{X_\alpha(\mathbb{A})Z\backslash M_H(\mathbb{A})^1}\int_{V_2'(\mathbb{A})} (M_{-1}\varphi)^{\psi_{U_0^M}}(vmk)|\kappa\alpha|(m)\,dmdkdv.$$

Finally, we may conjugate m and use the integral representation of [BG92] to obtain

$$2\pi i\sigma(3/2)\frac{\zeta_F(2)}{\zeta_F(3)}\int_{V_1(\mathbb{A})}\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1} (M_{-1}\varphi)^{\psi_{V_M}}(mvk)\mathbb{E}(m, 2\kappa-1)\,dmdkdv.$$

We continue as in §2. ■

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4.1 Matching of Functions

Finally we discuss how to obtain an identity of Bessel distributions from Theorem 4.1. The recipe for f' is slightly more complicated than before due to the appearance of the extra unipotent integration over V_1.

4.2 Nilpotent Groups

Let N be a 3-dimensional Heisenberg group over a local field F. We write $N = Z \cdot \tilde{V} \cdot V'$ where Z is the center, and we identify Z, $\tilde{V}$ and V' with F in such a way that the commutator $[v, v']$ becomes multiplication. We also fix a non-trivial character ψ of Z.

Let D be a function on N satisfying

$$\int_{V'} D^{\overline{\psi}}(vv')\,dv' = 1 \text{ for all } v \in \tilde{V}. \tag{15}$$

For any function f on N define $\mathcal{T}_N f(n) = \int_N D(x) f(xn)\,dx$.

Lemma 4.2 *Let l be a function on N such that $l(zv'n) = \psi(z)l(n)$ for all $z \in Z$, $v' \in V'$, $n \in N$. Then*

$$\int_{\tilde{V}} \int_N f(n)l(vn)\,dndv = \int_N \mathcal{T}_N f(n)l(n)\,dn. \tag{16}$$

In particular, let $\phi \in C_c^\infty(Z)$ and $\phi' \in C_c^\infty(V')$ be such that

$$\int_Z \phi(z)\,dz = \int_{V'} \phi'(v')\,dv' = 1.$$

Set

$$D(zvv') = \psi(z)\frac{\phi(z/v)}{|v|}\phi'(v'). \tag{17}$$

Then D satisfies (15) *and*

$$f \in \mathcal{S}(N) \implies \mathcal{T}_N f \in \mathcal{S}(N). \tag{18}$$

If F is p-adic, ψ has conductor $\mathcal{O}$, and $\phi = \phi' = 1_{\mathcal{O}}$, then $\mathcal{T}_N f = f$ for any f which is left-K-invariant.

Proof The right-hand side of (16) is equal to

$$\int_Z \int_{\tilde{V}} \int_{V'} \int_Z \int_{\tilde{V}} \int_{V'} D(zvv')f(zvv'wu'u)\psi(w)l(u)\,dv'dvdzdu'dudw.$$

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By a change of variables we get

$$\int_{\tilde V}\int_{V'}\int_{\tilde V}\int_{V'} D^{\overline{\psi}}(vv')f^{\psi}(vv'u'u)l(u)\,dv'dvdu'du,$$

or

$$\int_{\tilde V}\int_{V'}\int_{\tilde V}\int_{V'} D^{\overline{\psi}}(vv')f^{\psi}(vu'u)l(u)\,dv'dvdu'du.$$

Using (15) we get

$$\int_{\tilde V}\int_{V'}\int_{\tilde V} f^{\psi}(vu'u)l(u)\,dvdu'du.$$

Evidently, this is equal to the left-hand side of (16).

Clearly, D defined in (17) satisfies (15). It is also obvious that

$$\mathfrak{T}_N(f_1\star f_2)=\mathfrak{T}_N(f_1)\star f_2.$$

Hence, by the Dixmier–Malliavin theorem, it is enough to check (18) for $f\in C_c^\infty(N)$. In fact, it is enough to show that $\mathfrak{T}_N f$ is rapidly decreasing, since left-invariant differential operators commute with $\mathfrak{T}_N$. Write $F(z,v,v')=f(zvv')$. Thus, $F\in C_c^\infty(Z\times\tilde V\times V')$. Similarly, write $F'(w,u,u')=\mathfrak{T}_N f(wuu')$. Then

$$\begin{aligned}F'(w,u,u')&=\int_Z\int_{\tilde V}\int_{V'}\psi(z)\frac{\phi(z/v)}{|v|}\phi'(v')f(zvv'wuu')\,dv'dvdz\\&=\int_Z\int_{\tilde V}\int_{V'}\psi(z)\frac{\phi(z/v)}{|v|}\phi'(v')F(z+w+[v',u],u+v,u'+v')\,dv'dvdz,\end{aligned}$$

which after a change of variable becomes

$$\begin{aligned}&\int_Z\int_{\tilde V}\int_{V'}\psi(vz)\phi(z)\phi'(v')F(vz+w+v'u,u+v,u'+v')\,dv'dvdz\\&=\int_Z\int_{\tilde V}\int_{V'}\psi(vz)\overline{\psi(uz)}\phi(z)\phi'(v'+z)F(vz+w+v'u,v,v'+z+u')\,dv'dvdz.\end{aligned}$$

Clearly this is zero unless u' is confined to a compact set which is independent of w and u. Similarly, z, v and v' are integrated over a fixed compact set, independently of w and u and u'. Hence, in the support of F' we have $|w|\le C(1+|u|)$ for some constant C. Therefore, in order to show that $\mathfrak{T}_N f\in\mathcal{S}(N)$, it is enough to show that $F'(w,u,u')$ is rapidly decreasing in u, uniformly in w. However, $F'(w,u,u')$ is the Fourier transform at u of the function

$$z\mapsto\psi(vz)\phi(z)\phi'(v'+z)F(vz+w+v'u,v,v'+z+u')$$

and the latter is a normal family in $\mathcal{S}(Z)$ when we vary v, v', u, u', w. The result follows.

To show the last part of the lemma, it is enough to check that $\mathfrak{T}1_K = 1_K$. The previous computation gives that $\mathfrak{T}1_K(wuu')$ equals

$$\int_Z\int_{\tilde V}\int_{V'}\psi(vz)\overline{\psi(uz)}1_{\mathcal{O}}(z)1_{\mathcal{O}}(v'+z-u')1_{\mathcal{O}}(vz+w+v'u)1_{\mathcal{O}}(v)1_{\mathcal{O}}(v'+z)\,dv'dvdz$$
$$= \int_Z\int_{\tilde V}\int_{V'}\overline{\psi(uz)}1_{\mathcal{O}}(z)1_{\mathcal{O}}(u')1_{\mathcal{O}}(w+v'u)1_{\mathcal{O}}(v)1_{\mathcal{O}}(v')\,dv'dvdz,$$

which is equal to $1_{\mathcal{O}}(u)1_{\mathcal{O}}(u')1_{\mathcal{O}}(w)$ since

$$\int_Z\overline{\psi(uz)}1_{\mathcal{O}}(z) = 1_{\mathcal{O}}(u). \qquad \blacksquare$$

Remark 2 The lemma still holds for a group N isomorphic to the maximal unipotent of $O(3,2)$ (with roots α, β, $\alpha+\beta$, $\alpha+2\beta$) with $Z = X_{\alpha+\beta}$, $\tilde V = X_\alpha$, $V' = X_\beta \cdot X_{\alpha+2\beta}$. (Note that Z is *not* the center of N.) The proof is only slightly different. We omit the details.

Let $U_1 = X_{-(1221)}$, $U_2 = X_{-(1110)}$, $U_3 = X_{-(1220)}$, $U_4 = X_{-(1100)}$. Also, set

$$N_1 = U_1 \cdot X_{(1231)} \cdot X_{(0010)},$$
$$N_2 = U_2 \cdot X_{(1111)} \cdot X_{(0001)},$$
$$N_3 = U_3 \cdot X_{(1221)} \cdot X_{(0001)} \cdot X_{(1222)},$$
$$N_4 = U_4 \cdot X_{(1110)} \cdot X_{(0010)} \cdot X_{(1120)}.$$

Thus, N_1, N_2, are Heisenberg groups and N_3, N_4 are of the type considered in Remark 2. Thus we may apply Lemma 4.2 with $N = N_i$, $\tilde V = U_i$, $Z = X_{(0010)}$ for $i = 1, 4$ and $Z = X_{(0001)}$ for $i = 2, 3$, and $\psi = \psi_V|_Z$. On each N_i, fix functions D_i as before.

Note that $Y_1 = [N_1, U_2U_3U_4] = X_{-(1100)} \cdot X_{(0121)} \cdot X_{(0011)}$. Thus,

$$Y_1 \subset \mathrm{Ker}(\psi_{V_M}) \cdot U_4 \text{ and } [Y_1, U_1] \subset U_4. \tag{19}$$

Similarly, $Y_2 = [N_2, U_3U_4] = X_{(0011)}$, and hence, ψ_{V_M} is trivial on Y_2 and Y_2 commutes with U_2. Finally, $Y_3 = [N_3, U_4] = X_{(0121)}$. Again ψ_{V_M} is trivial on Y_3 and Y_3 commutes with U_3.

Set

$$\ell_H^{\psi_V}(\varphi) = \int_{H\backslash H(\mathbb{A})}\varphi^{\psi_{V_M}}(h)\,dh.$$

For $f \in \mathcal{S}(G)$, define $\mathfrak{T}_i f(x) = \int_{N_i} D_i(n) f(nx)\,dn, x \in G$.

Lemma 4.3 *The function $\mathfrak{T}_i f$ belongs to $\mathcal{S}(G)$. We have*

$$\int_{U_i}\int_{N_i(\mathbb{A})} f(nx)\ell_{M_H}^{\psi_{V_M}}\varphi(vnx)\,dndv = \int_{N_i(\mathbb{A})}\mathfrak{T}_i f(nx)\ell_{M_H}^{\psi_{V_M}}\varphi(nx)\,dn.$$

If F is p-adic, then $\mathfrak{T}_i$ acts as the identity on the Hecke algebra.

Proof Let w be a Weyl group element so that $wN_iw^{-1} \subset U_0$. Let C be the variety $T_0 \cdot \prod_\beta X_\beta$, where the product (in a prescribed order) is over the roots β not inside N_i such that $w\beta > 0$. Thus, C is a complement of N_i in $w^{-1}Bw$. Since $\mathcal{T}_i f(nck) = \mathcal{T}_{N_i} f(\cdot\, ck)$, the first and last assertions follow from Lemma 4.2. The second statement follows from (16). ∎

Define $\mathcal{T}f = \mathcal{T}_4(\mathcal{T}_3(\mathcal{T}_2(\mathcal{T}_1 f^{K_H})))$. (We could have taken ^{K_H} at the end.)

Theorem 4.4 *Let Π be the automorphic representation of $G(\mathbb{A})$ on $E_{-1}(\cdot,\varphi)$. Then*

$$\mathcal{B}^{\Pi}_{\ell_H^{\psi_V},\ell_H^{\psi_V}}(f) = \frac{\zeta_F(2)}{\zeta_F(3)}\frac{\zeta_F(6)}{\lambda_{-1}} \cdot \mathcal{B}^{\pi}_{\ell_{M_H}^{\psi_{V_M}},\ell_{M_H,\mathbb{E}(\cdot,5)}^{\psi_{V_M}}}(\mathcal{T}f).$$

Similarly,

$$\mathcal{B}^{\Pi}_{\ell_H^{\psi_V},\ell_{H,\mathcal{E}_{-1}}^{\psi_V}}(f) = \mathcal{B}^{\pi}_{\ell_{M_H}^{\psi_{V_M}},\ell_{M_H}^{\psi_{V_M}}}(\mathcal{T}f).$$

The theorem will follow from the next lemma, as in [JLR04].

Lemma 4.5 *The linear form*

$$\varphi \mapsto \ell_H^{\psi_V}(\Pi(f)E_{-1}(\,\cdot\,,\varphi)) \tag{20}$$

on $I(\pi,\varpi)$ is given by $\Psi \in I(\pi^\vee,-\varpi)$, where

$$\Psi(g) = \ell_{M_H}^{\psi_{V_M}} \circ \pi((R_g\mathcal{T}f)^U), \tag{21}$$

where $R_g f(\,\cdot\,) = f(\,\cdot\, g)$ and

$$f^U(m) = e^{\langle \varpi+\rho_P, H_M(m)\rangle} \cdot \int_{U(\mathbb{A})} f(mu)\,du.$$

Proof First, one easily checks that Ψ lies in $I(\pi^\vee,-\varpi)$.

By Theorem 4.1, $\ell_H^{\psi_V} \circ \Pi(f)$ equals

$$\begin{aligned}
\int_{\mathbf{K}_H}\int_{V_1(\mathbb{A})}\int_{M_H\backslash M_H(\mathbb{A})^1} & I(f,\pi,\varpi)\varphi^{\psi_{V_M}}(lv_1k')\,dldv_1dk' \\
&= \int_{\mathbf{K}_H}\int_{V_1(\mathbb{A})}\int_{G(\mathbb{A})} f(x)\ell_{M_H}^{\psi_{V_M}}\varphi(v_1k'x)\,dxdv_1dk' \\
&= \int_{V_1(\mathbb{A})}\int_{G(\mathbb{A})} f^{K_H}(x)\ell_{M_H}^{\psi_{V_M}}\varphi(v_1x)\,dxdv_1 \\
&= \iiiint_{U_i(\mathbb{A})}\int_{G(\mathbb{A})} f^{K_H}(x)\ell_{M_H}^{\psi_{V_M}}\varphi(u_1u_2u_3u_4x)\,dx \otimes du_i.
\end{aligned}$$

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We now write this as

$$\iiiint_{U_i(\mathbb{A})} \int_{N_1(\mathbb{A})\backslash G(\mathbb{A})} \int_{N_1(\mathbb{A})} f^{K_H}(nx)\ell_{M_H}^{\psi_{V_M}} \varphi(u_1u_2u_3u_4nx)\, dndx \otimes du_i$$
$$= \iiiint f^{K_H}(nx)\ell_{M_H}^{\psi_{V_M}} \varphi(u_1nu_2u_3u_4x)\, dndx \otimes du_i.$$

The last step is justified by (19) using a change of variables in u_4. Using Lemma 4.3, the integrals over n_1 and u_1 become

$$\int_{N_1(\mathbb{A})} \mathcal{T}_1(f^{K_H})(nx)\ell_{M_H}^{\psi_{V_M}} \varphi(nu_2u_3u_4x)\, dn.$$

Commuting n back over $u_2u_3u_4$ and combining the integral over $N_1(\mathbb{A})$ with $N_1(\mathbb{A})\backslash G(\mathbb{A})$, we obtain

$$\int_{U_2(\mathbb{A})} \int_{U_3(\mathbb{A})} \int_{U_4(\mathbb{A})} \int_{G(\mathbb{A})} \mathcal{T}_1(f^{K_H})(x)\ell_{M_H}^{\psi_{V_M}} \varphi(u_2u_3u_4x)\, dxdu_2du_3du_4.$$

Continuing this way with U_2, U_3 and U_4 we finally get

$$\int_{G(\mathbb{A})} \mathcal{T}f(x)\ell_{M_H}^{\psi_{V_M}} \varphi(x)\, dx.$$

Using Iwasawa decomposition we get

$$\int_{\mathbf{K}} \int_{U(\mathbb{A})} \int_{M(\mathbb{A})} \mathcal{T}f(muk)\ell_{M_H}^{\psi_{V_M}} \varphi(mk)\, dmdudk.$$

Viewing φ as an element $I(\pi, \varpi)$ we get

$$\int_{\mathbf{K}} \int_{U(\mathbb{A})} \int_{M(\mathbb{A})} \mathcal{T}f(muk)\ell_{M_H}(\pi(m)\varphi(k))e^{\langle \varpi+\rho_P, H_M(m)\rangle}\, dmdudk$$
$$= \int_{\mathbf{K}} \ell_{M_H}^{\psi_{V_M}} \circ \pi((R_k\mathcal{T}f)^U)(\varphi(k))\, dk = \Psi(\varphi),$$

as required. ∎

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