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SOME RESULTS OF THE $\mathcal{K}_{\mathcal{A}}$-APPROXIMATION PROPERTY FOR BANACH SPACES

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Abstract. Given a Banach operator ideal $\mathcal{A}$, we investigate the approximation property related to the ideal of $\mathcal{A}$-compact operators, $\mathcal{K}_{\mathcal{A}}$-AP. We prove that a Banach space X has the $\mathcal{K}_{\mathcal{A}}$-AP if and only if there exists a $\lambda \geq 1$ such that for every Banach space Y and every $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$,

$$R \in \overline{\{SR : S \in \mathcal{F}(X, X), \|SR\|_{\mathcal{K}_{\mathcal{A}}} \leq \lambda \|R\|_{\mathcal{K}_{\mathcal{A}}}\}}^{\tau_c}.$$

For a surjective, maximal and right-accessible Banach operator ideal $\mathcal{A}$, we prove that a Banach space X has the $\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}$-AP if the dual space of X has the $\mathcal{K}_{\mathcal{A}}$-AP.

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1. Introduction. A Banach space X is said to have the *approximation property* (AP) if

$$\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|}$$

for every Banach space Y, where $\mathcal{K}$ and $\mathcal{F}$ are the ideals of compact and finite rank operators, respectively. Lassalle, Turco and Oja [**16**, **21**] introduced a general notion of the AP. Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal. A Banach space X is said to have the $\mathcal{A}$-AP if $\mathcal{A}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|_{\mathcal{A}}}$ for every Banach space Y.

Carl and Stephani [**1**] introduced a notion of compactness determined by operator ideals. A subset K of a Banach space X is said to be *relatively $\mathcal{A}$-compact* if there exist a Banach space Z, $U \in \mathcal{A}(Z, X)$ and a relatively compact subset C of Z such that $K \subset U(C)$. In fact, this notion is an equivalent statement of the original definition of $\mathcal{A}$-compactness (see [**1**, Definition 1.1 and Theorem 1.2]). Throughout this paper, we use "$\mathcal{A}$-compact" instead of "relatively $\mathcal{A}$-compact" in the notion of $\mathcal{A}$-compactness. A linear map $R : Y \to X$ is said to be *$\mathcal{A}$-compact* if $R(B_Y)$ is an $\mathcal{A}$-compact subset of X (see [**1**]), where B_Y is the unit ball of Y. Let $\mathcal{K}_{\mathcal{A}}(Y, X)$ be the space of all $\mathcal{A}$-compact operators from Y to X.

Lassalle and Turco [**17**] introduced a way to measure the size of $\mathcal{A}$-compact sets. For an $\mathcal{A}$-compact subset K of X, let

$$m_{\mathcal{A}}(K; X) := \inf\{\|U\|_{\mathcal{A}} : U \in \mathcal{A}(Z, X),\ \text{relatively compact } C \subset B_Z, K \subset U(C)\}$$

and let $\|R\|_{\mathcal{K}_{\mathcal{A}}} := m_{\mathcal{A}}(R(B_Y); X)$ for $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$. Then, $[\mathcal{K}_{\mathcal{A}}, \|\cdot\|_{\mathcal{K}_{\mathcal{A}}}]$ is a Banach operator ideal (see [**17**, Section 2]). From [**17**, Remarks 1.3 and 1.7], a subset K of X is relatively compact if and only if K is $\mathcal{K}$-compact. In this case,

$$m_{\mathcal{K}}(K; X) = \sup_{x \in K} \|x\|.$$

Thus, $[\mathcal{K}_{\mathcal{K}}, \|\cdot\|_{\mathcal{K}_{\mathcal{K}}}] = [\mathcal{K}, \|\cdot\|]$.

The main notion of this paper is the $\mathcal{K}_{\mathcal{A}}$-AP for Banach spaces, which was introduced by Lassalle and Turco [**17**], namely, a Banach space X is said to have the $\mathcal{K}_{\mathcal{A}}$-AP if

$$\mathcal{K}_{\mathcal{A}}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}$$

for every Banach space Y. The main purpose of this paper is to characterize the $\mathcal{K}_{\mathcal{A}}$-AP with some weakened statements and investigate in which cases the $\mathcal{K}_{\mathcal{A}}$-AP passes from the dual space of a Banach space to the Banach space itself. One may refer to [**2**, **4–7**, **10**, **12–18**, **21**, **24**, **25**] for investigations related with the $\mathcal{K}_{\mathcal{A}}$-AP.

2. Characterizations of the $\mathcal{K}_{\mathcal{A}}$-approximation property. In [**17**], the authors introduced a locally convex topology on the space $\mathcal{L}(X, Y)$ of all bounded operators from X to Y. Let $\mathcal{A}$ be a Banach operator ideal. The *topology $\tau_{s\mathcal{A}}$ on $\mathcal{L}(X, Y)$ of strong uniform convergence on $\mathcal{A}$-compact sets*, which is given by the seminorms

$$q_K(T) = m_{\mathcal{A}}(T(K); Y),$$

where K ranges overall $\mathcal{A}$-compact subsets of X. It was shown in [**17**, Proposition 3.2] that a Banach space X has the $\mathcal{K}_{\mathcal{A}}$-AP if and only if

$$id_X \in \overline{\mathcal{F}(X)}^{\tau_{s\mathcal{A}}},$$

where id_X is the identity map on X and $\mathcal{F}(X)$ is the space of all finite rank operators from X to X.

Delgado and Piñeiro [**4**] introduced an AP via operator ideals, denoted by ($\text{AP}_{\mathcal{A}}$), and studied it using another locally convex topology on the space $\mathcal{L}(X, Y)$ determined by $\mathcal{A}$-compact sets. The *topology $\tau_c(\mathcal{A})$ on $\mathcal{L}(X, Y)$ of uniform convergence on $\mathcal{A}$-compact sets*, which is given by the seminorms

$$p_K(T) = \sup_{x \in K} \|Tx\|,$$

where K ranges overall $\mathcal{A}$-compact subsets of X. In particular, *the topology of uniform convergence on compact sets* is denoted by τ_c. They proved that a Banach space X has the $\text{AP}_{\mathcal{A}}$ if and only if

$$id_X \in \overline{\mathcal{F}(X)}^{\tau_c(\mathcal{A})},$$

if and only if for every Banach space Y,

$$\mathcal{K}_{\mathcal{A}}(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\|\cdot\|}.$$

THEOREM 2.1. *Let $\mathcal{A}$ be a Banach operator ideal and let $\lambda \geq 1$. The following statements are equivalent:*

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(a) *X has the $\mathcal{K}_{\mathcal{A}}$-AP.*
(b) *For every Banach space Y and every injective operator $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$,*

$$R \in \overline{\{SR : S \in \mathcal{F}(X), \|SR\|_{\mathcal{K}_{\mathcal{A}}} \leq \lambda \|R\|_{\mathcal{K}_{\mathcal{A}}}\}}^{\tau_c}.$$

(c) *For every Banach space Y and every $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$, and for every $\delta > 0$,*

$$id_X \in \overline{\{S \in \mathcal{F}(X) : \|SR\|_{\mathcal{K}_{\mathcal{A}}} \leq (\lambda + \delta)\|R\|_{\mathcal{K}_{\mathcal{A}}}\}}^{\tau_c(\mathcal{A})}.$$

(d) *For every Banach space Y and every $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$, and for every $\delta > 0$ and every finite-dimensional subspace F of X, there exists an $S \in \mathcal{F}(X)$ with $\|SR\| \leq (\lambda + \delta)\|R\|_{\mathcal{K}_{\mathcal{A}}}$ such that $\|Sx - x\| \leq \delta \|x\|$ for every $x \in F$.*
(e) *For every Banach space Y and every $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$, and for every $\delta > 0$ and every finite-dimensional subspace F of X, there exists an $S \in \mathcal{F}(X)$ with $\|SR\| \leq (\lambda + \delta)\|R\|_{\mathcal{K}_{\mathcal{A}}}$ such that $Sx = x$ for every $x \in F$.*

In order to prove Theorem 2.1, we show that (a)⇒(b)⇒(c)⇒(d)⇒(e)⇒(a). First, it was shown in [**17**, Proposition 3.1] that a Banach space X has the $\mathcal{K}_{\mathcal{A}}$-AP if and only if for every Banach space Y and every $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$,

$$R \in \overline{\{SR : S \in \mathcal{F}(X)\}}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}},$$

which is equivalent to

$$R \in \overline{\{SR : S \in \mathcal{F}(X), \|SR\|_{\mathcal{K}_{\mathcal{A}}} \leq \|R\|_{\mathcal{K}_{\mathcal{A}}}\}}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}.$$

Hence, (a)⇒(b) follows. To show that (b)⇒(c), we need the following lemma which originates from a representation of Grothendieck [**11**] for the dual space $(\mathcal{L}(X, Y), \tau_c)^*$ (cf. [**19**, Proposition 1.e.3]). See [**1**] for the definition and properties of $\mathcal{A}$-null sequences.

LEMMA 2.2 ([**4**]). *The dual space $(\mathcal{L}(X, Y), \tau_c(\mathcal{A}))^*$ consists of all functionals of the form*

$$f(T) = \sum_{n=1}^{\infty} y_n^*(Tx_n),$$

where (x_n) is an $\mathcal{A}$-null sequence and (y_n^*) is an absolutely summable sequence in Y^*.

Suppose that A is a balanced, convex and compact subset of X. Let X_A be a linear span of A, which is normed by the Minkowski functional of A. Then, it is well known that X_A is a Banach space and the set A is the unit ball of X_A (cf. [**23**, Lemma 4.11]). Let $j_A : X_A \to X$ be the inclusion map.

Proof of theorem 2.1(b)⇒(c). Let Y be a Banach space and let $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$. We may assume that $\|R\|_{\mathcal{K}_{\mathcal{A}}} = 1$. Let $\delta > 0$. We use Lemma 2.2 to apply the separation theorem. Let

$$f := \sum_{n=1}^{\infty} x_n^*(\cdot\, x_n) \in (\mathcal{L}(X), \tau_c(\mathcal{A}))^*,$$

where (x_n) is an $\mathcal{A}$-null sequence and (x_n^*) is an absolutely summable sequence in X^*. Note that the set $\{x_n\}_{n=1}^{\infty}$ is $\mathcal{A}$-compact (cf. [**17**, Proposition 1.4]). We may assume that

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$m_{\mathcal{A}}(\{x_n\}_{n=1}^{\infty}; X) = \delta/\lambda$. Let A be a balanced and closed convex hull of the set

$$\frac{\{x_n\}_{n=1}^{\infty} \bigcup R(B_Y)}{m_{\mathcal{A}}(\{x_n\}_{n=1}^{\infty} \bigcup R(B_Y); X)}.$$

Then, we see that $j_A \in \mathcal{K}_{\mathcal{A}}(X_A, X)$ and $\|j_A\|_{\mathcal{K}_{\mathcal{A}}} = 1$. Consider

$$g := \sum_{n=1}^{\infty} x_n^*(\cdot\, x_n) \in (\mathcal{L}(X_A, X), \tau_c)^*.$$

Then, by (b)

$$\begin{aligned} \mathrm{Re} f(id_X) &= \mathrm{Re} g(j_A) \\ &\leq \sup\{\mathrm{Re} g(Sj_A) : S \in \mathcal{F}(X), \|Sj_A\|_{\mathcal{K}_{\mathcal{A}}} \leq \lambda \|j_A\|_{\mathcal{K}_{\mathcal{A}}}\} \\ &= \sup\{\mathrm{Re} f(S) : S \in \mathcal{F}(X), \|Sj_A\|_{\mathcal{K}_{\mathcal{A}}} \leq \lambda \|j_A\|_{\mathcal{K}_{\mathcal{A}}}\}. \end{aligned}$$

Now, if $S \in \mathcal{F}(X)$ and $\|Sj_A\|_{\mathcal{K}_{\mathcal{A}}} \leq \lambda \|j_A\|_{\mathcal{K}_{\mathcal{A}}} = \lambda$, then

$$\begin{aligned} \|SR\|_{\mathcal{K}_{\mathcal{A}}} &= m_{\mathcal{A}}(SR(B_Y); X) \\ &= m_{\mathcal{A}}(Sj_A R(B_Y); X) \\ &= m_{\mathcal{A}}(\{x_n\}_{n=1}^{\infty} \cup R(B_Y); X) m_{\mathcal{A}}(Sj_A(R(B_Y)/m_{\mathcal{A}}(\{x_n\}_{n=1}^{\infty} \cup R(B_Y); X)); X) \\ &\leq (m_{\mathcal{A}}(\{x_n\}_{n=1}^{\infty}; X) + m_{\mathcal{A}}(R(B_Y); X)) \|Sj_A\|_{\mathcal{K}_{\mathcal{A}}} \\ &\leq (\frac{\delta}{\lambda} + 1)\lambda = \delta + \lambda. \end{aligned}$$

Thus,

$$\mathrm{Re} f(id_X) \leq \sup\{\mathrm{Re} f(S) : S \in \mathcal{F}(X), \|SR\|_{\mathcal{K}_{\mathcal{A}}} \leq \lambda + \delta\}.$$

This completes the proof. □

Note that every bounded subset of a finite-dimensional subspace of a Banach space is $\mathcal{A}$-compact for every Banach operator ideal $\mathcal{A}$. From this, Theorem 2.1(c)⇒(d) follows.

Proof of theorem 2.1(d)⇒(e). Let Y be a Banach space and let $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$. Let $\delta > 0$ and let F be a finite-dimensional subspace of X. Let $P : X \to F$ be a projection. Let $\gamma > 0$ be such that $\gamma(1 + \|P\|) \leq \delta$.

By (d) there exists an $S \in \mathcal{F}(X)$ with $\|SR\|_{\mathcal{K}_{\mathcal{A}}} \leq (\lambda + \gamma)\|R\|_{\mathcal{K}_{\mathcal{A}}}$ so that

$$\|Sx - x\| \leq \gamma \|x\|$$

for every $x \in F$. Let $S_0 := S + (id_X - S)P \in \mathcal{F}(X)$. Then,

$$S_0 x = Sx + x - Sx = x$$

for every $x \in F$ and

$$\|S_0 R\|_{\mathcal{K}_{\mathcal{A}}} \leq \|SR\|_{\mathcal{K}_{\mathcal{A}}} + \|(id_X - S)P\| \|R\|_{\mathcal{K}_{\mathcal{A}}} \leq (\lambda + \gamma + \gamma\|P\|)\|R\|_{\mathcal{K}_{\mathcal{A}}} \leq (\lambda + \delta)\|R\|_{\mathcal{K}_{\mathcal{A}}}.$$

□

Proof of theorem 2.1(e)⇒(a). The prototype of this proof is the proof of [**17**, Proposition 3.3]. Let K be an $\mathcal{A}$-compact subset of X and $\varepsilon > 0$. By [**17**, Proposition 1.8], there exist a $T \in \mathcal{A} \circ \mathcal{K}(\ell_1, X)$ and a relatively compact subset M of ℓ_1 such that $K \subset T(M)$. By [**17**, Proposition 2.1], $\mathcal{A} \circ \mathcal{K}(\ell_1, X)$ is isometrically equal to $\mathcal{K}_{\mathcal{A}}(\ell_1, X)$.

Using [**23**, Lemma 4.11], there exists the Banach space $X_A \subset \ell_1$ such that A is a compact subset of ℓ_1 and M is a compact subset of X_A. Let $n_0 \in \mathbb{N}$ be such that

$$\sup_{a \in A} \|P_{n_0} a - a\|_1 \leq \frac{\varepsilon}{\|T\|_{\mathcal{K}_{\mathcal{A}}}(\lambda + \varepsilon + 1)},$$

where $P_{n_0} : \ell_1 \to \ell_1$ is the basis projection.

Now, let us consider the finite-dimensional subspace $TP_{n_0} j_A(X_A)$ of X. Then, by (e) there exists an $S \in \mathcal{F}(X)$ with $\|ST\|_{\mathcal{K}_{\mathcal{A}}} \leq (\lambda + \varepsilon)\|T\|_{\mathcal{K}_{\mathcal{A}}}$ such that

$$STP_{n_0} j_A = TP_{n_0} j_A.$$

We now have

$$\begin{aligned} m_{\mathcal{A}}((S - id_X)(K); X) &\leq \|(S - id_X)Tj_A\|_{\mathcal{A}} \\ &\leq \|STj_A - STP_{n_0} j_A\|_{\mathcal{A}} + \|STP_{n_0} j_A - Tj_A\|_{\mathcal{A}} \\ &\leq \|ST\|_{\mathcal{A}} \|j_A - P_{n_0} j_A\| + \|T\|_{\mathcal{A}} \|P_{n_0} j_A - j_A\| \\ &\leq \|P_{n_0} j_A - j_A\|(\|ST\|_{\mathcal{A}\circ\mathcal{K}} + \|T\|_{\mathcal{A}\circ\mathcal{K}}) \\ &= \sup_{a \in A} \|P_{n_0} a - a\|_1 (\|ST\|_{\mathcal{K}_{\mathcal{A}}} + \|T\|_{\mathcal{K}_{\mathcal{A}}}) \leq \varepsilon. \end{aligned}$$

This completes the proof. □

We introduce a topology on $\mathcal{K}_{\mathcal{A}}(Y, X)$, which is weaker than the topology induced by the norm $\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}$. For a net (T_α) in $\mathcal{K}_{\mathcal{A}}(Y, X)$ and $T \in \mathcal{K}_{\mathcal{A}}(Y, X)$, we say that T_α converges to T in *the topology* $\tau_{cc}(m_{\mathcal{A}})$ if

$$\lim_\alpha m_{\mathcal{A}}((T_\alpha - T)(K); X) = 0$$

for every compact subset K of Y.

THEOREM 2.3. *For a Banach operator ideal $\mathcal{A}$, a Banach space X has the $\mathcal{K}_{\mathcal{A}}$-AP if (and only if) for every quotient space Z of ℓ_1 and every injective operator $R \in \mathcal{K}_{\mathcal{A}}(Z, X)$,*

$$R \in \overline{\mathcal{F}(Z, X)}^{\tau_{cc}(m_{\mathcal{A}})}.$$

Proof. Let K be an $\mathcal{A}$-compact subset of X and let $\varepsilon > 0$. By [**1**, Theorem 1.1], there exists an $\mathcal{A}$-null sequence (x_n) in X such that

$$K \subset \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_1} \right\}.$$

According to [**1**, Lemma 1.2], there exists a sequence (β_n) of positive numbers with $\beta_n \leq 1$ and $\beta_n \longrightarrow 0$ such that $(z_n) := (x_n/\beta_n)$ is an $\mathcal{A}$-null sequence.

Now, we define the maps $D_\beta : \ell_1 \to \ell_1$ and $M_{\hat{z}} : \ell_1 \to X$ by $D_\beta(\alpha_n) = (\beta_n \alpha_n)$ and $M_{\hat{z}}(\alpha_n) = \sum_n \alpha_n z_n$, respectively. The injective operator $\overline{M_{\hat{z}}} : \ell_1/\ker(M_{\hat{z}}) \to X$ is defined

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by $\overline{M_{\hat{z}}}((\alpha_n) + \ker(M_{\hat{z}})) = M_{\hat{z}}(\alpha_n)$ and then $M_{\hat{z}} = \overline{M_{\hat{z}}}q$, where $q : \ell_1 \to \ell_1/\ker(M_{\hat{z}})$ is the quotient operator.

$$\ell_1 \xrightarrow{D_\beta} \ell_1 \xrightarrow{q} \ell_1/\ker(M_{\hat{z}}) \xrightarrow{\overline{M_{\hat{z}}}} X.$$

We see that D_β is compact and $M_{\hat{z}}$ is $\mathcal{A}$-compact. Since the ideal of $\mathcal{A}$-compact operators is surjective (cf. [**17**, Proposition 2.1]), $\overline{M_{\hat{z}}}$ is $\mathcal{A}$-compact.

Now, by the assumption, there exists an $U \in \mathcal{F}(\ell_1/\ker(M_{\hat{z}}), X)$ such that

$$m_{\mathcal{A}}((U - \overline{M_{\hat{z}}})(qD_\beta(B_{\ell_1})); X) \leq \frac{\varepsilon}{2}.$$

We may assume that $U = \sum_{k=1}^{m} y_k^* \otimes x_k$, where $y_k^* \in (\ell_1/\ker(M_{\hat{z}}))^*$ and $x_k \in X$ for each $k = 1, \ldots, m$ and $\sum_{k=1}^{m} \|x_k\| = 1$. Since $\overline{M_{\hat{z}}}$ is injective, $(\ell_1/\ker(M_{\hat{z}}))^* = \overline{\overline{M_{\hat{z}}}^*(X^*)}^{\text{weak}^*} = \overline{\overline{M_{\hat{z}}}^*(X^*)}^{\tau_c}$. The second equality follows from $(Z^*, weak^*)^* = (Z^*, \tau_c)^*$ for every Banach space Z (cf. [**20**, Theorem 2.7.8]). Then, for each $k = 1, \ldots, m$, we can choose an $x_k^* \in X^*$ such that

$$\sup_{y \in qD_\beta(B_{\ell_1})} |y_k^*(y) - \overline{M_{\hat{z}}}^*(x_k^*)(y)| \leq \frac{\varepsilon}{2}.$$

We show that $S = \sum_{k=1}^{m} x_k^* \otimes x_k \in \mathcal{F}(X)$ is the desired operator approximating to id_X. Since, for every $(\alpha_n) \in \ell_1$,

$$(S\overline{M_{\hat{z}}}qD_\beta - UqD_\beta)(\alpha_n) = \sum_{k=1}^{m} (((\overline{M_{\hat{z}}}^* x_k^*)qD_\beta - y_k^* qD_\beta)(\alpha_n))x_k,$$

we have

$$\begin{aligned}
& m_{\mathcal{A}}((S - id_X)(K); X) \\
& \leq m_{\mathcal{A}}((S - id_X)(\overline{M_{\hat{z}}}qD_\beta(B_{\ell_1})); X) \\
& = m_{\mathcal{A}}((S\overline{M_{\hat{z}}}qD_\beta - UqD_\beta + UqD_\beta - \overline{M_{\hat{z}}}qD_\beta)(B_{\ell_1}); X) \\
& = \|S\overline{M_{\hat{z}}}qD_\beta - UqD_\beta + UqD_\beta - \overline{M_{\hat{z}}}qD_\beta\|_{\mathcal{K}_{\mathcal{A}}} \\
& \leq \left\| \sum_{k=1}^{m} ((\overline{M_{\hat{z}}}^* x_k^*)qD_\beta - y_k^* qD_\beta) \otimes x_k \right\|_{\mathcal{K}_{\mathcal{A}}} + m_{\mathcal{A}}((UqD_\beta - \overline{M_{\hat{z}}}qD_\beta)(B_{\ell_1}); X) \\
& \leq \sum_{k=1}^{m} m_{\mathcal{A}}((((\overline{M_{\hat{z}}}^* x_k^*)qD_\beta - y_k^* qD_\beta) \otimes x_k)(B_{\ell_1}); X) + \frac{\varepsilon}{2} \\
& = \sum_{k=1}^{m} \|x_k\| \sup_{y \in qD_\beta(B_{\ell_1})} |y_k^*(y) - \overline{M_{\hat{z}}}^*(x_k^*)(y)| + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

□

3. A duality of the $\mathcal{K}_{\mathcal{A}}$-approximation property. One may refer to [**3**, **22**] for definitions and information of operator ideals. Given a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, we denote by $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\min}$, $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\max}$, $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\text{sur}}$, $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\text{inj}}$, $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\text{adj}}$ and $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{\text{dual}}$, the *minimal kernel*, *maximal hull*, *surjective hull*, *injective hull*, *adjoint*

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ideal and *dual ideal*, respectively. For operator ideals $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ and $[\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$, in this paper, $\mathcal{A} = \mathcal{B}$ means that $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}] = [\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$, and $\mathcal{A} \subset \mathcal{B}$ means the norm one inclusion.

A Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is called *right-accessible* if for all finite-dimensional normed space M, Banach space Y, $T \in \mathcal{L}(M, Y)$ and $\varepsilon > 0$, there exist a finite-dimensional subspace N of Y and an $S \in \mathcal{L}(M, N)$ such that $T = I_N S$ and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$, where $I_N : N \to Y$ is the inclusion map. Note that $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is right-accessible if and only if

$$\mathcal{A}^{\min} = \mathcal{A} \circ \overline{\mathcal{F}}$$

(see [**3**, Proposition 25.2(2)]).

It was shown in [**17**, Proposition 2.1] that

$$\mathcal{K}_{\mathcal{A}} = (\mathcal{A} \circ \overline{\mathcal{F}})^{\mathrm{sur}}.$$

Then, we have the following Lemma.

LEMMA 3.1. *Let $\mathcal{A}$ be a Banach operator ideal. Then,*

$$(\mathcal{A}^{\min})^{\mathrm{sur}} = \mathcal{K}_{\mathcal{A}^{\min}}$$

and, if $\mathcal{A}$ is right-accessible, then

$$(\mathcal{A}^{\min})^{\mathrm{sur}} = \mathcal{K}_{\mathcal{A}}.$$

Now, let us consider the dual space of $\mathcal{L}(X, Y)$ equipped with the topology $\tau_{s\mathcal{A}}$, which was investigated in [**18**]. We note that $\varphi \in (\mathcal{L}(X, Y), \tau_{s\mathcal{A}})^*$ if and only if there exist a $C > 0$ and an $\mathcal{A}$-compact subset of X such that

$$|\varphi(T)| \leq C m_{\mathcal{A}}(T(K); Y)$$

for every $T \in \mathcal{L}(X, Y)$.

LEMMA 3.2 ([**18**, Corollary 4.3]). *Suppose that $\mathcal{A}$ is a maximal, right-accessible Banach operator ideal. Let X and Y be Banach spaces. If $\varphi \in (\mathcal{L}(X, Y), \tau_{s\mathcal{A}})^*$, then there exist $S \in \mathcal{A}^{\min}(\ell_1, X)$ and $R \in (\mathcal{A}^{\mathrm{adj}})^{\min}(Y, \ell_1)$ such that*

$$\varphi(T) = tr(RTS)$$

for every $T \in \mathcal{L}(X, Y)$.

In view of the proof of [**18**, Theorem 4.2], for every $T \in \mathcal{L}(X, Y)$, $RTS \in \mathcal{N}(\ell_1, \ell_1)$, where $\mathcal{N}$ is the ideal of nuclear operators, and since ℓ_1 has the metric AP, the trace functional tr on $\mathcal{N}(\ell_1, \ell_1)$ is well defined.

THEOREM 3.3. *Suppose that $\mathcal{A}$ is a maximal, right-accessible Banach operator ideal such that $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is surjective. Let X and Y be Banach spaces. If $\varphi \in (\mathcal{L}(X, Y), \tau_{s\mathcal{A}})^*$, then there exists a $\psi \in (\mathcal{L}(Y^*, X^*), \tau_{s(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}})^*$ such that*

$$\varphi(T) = \psi(T^*)$$

for every $T \in \mathcal{L}(X, Y)$.

Proof. This proof is motivated from the proof of [**18**, Theorem 4.7].

Let $\varphi \in (\mathcal{L}(X, Y), \tau_{s\mathcal{A}})^*$. Let $S \in \mathcal{A}^{\min}(\ell_1, X)$ and $R \in (\mathcal{A}^{\mathrm{adj}})^{\min}(Y, \ell_1)$ be the operators in Lemma 3.2 such that

$$\varphi(T) = tr(RTS)$$

for every $T \in \mathcal{L}(X, Y)$.

Now, by [**3**, Corollary 22.8.1],

$$R \in (((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{dual}})^{\min}(Y, \ell_1) = (((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\min})^{\mathrm{dual}}(Y, \ell_1).$$

Thus, $R^* \in ((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\min}(\ell_\infty, Y^*)$. Since

$$S \in \mathcal{A}(\ell_1, X) = (((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{adj}})^{\mathrm{dual}}(\ell_1, X),$$

$S^* \in ((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{adj}}(X^*, \ell_\infty)$.

By [**3**, Corollary 21.3], $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is right-accessible. Then, by an application of [**3**, Propositions 25.4.1 and 25.4.2],

$$((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{adj}} \circ ((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\min} \subset \mathcal{N}.$$

Thus, $S^* U R^* \in \mathcal{N}(\ell_\infty, \ell_\infty)$ for every $U \in \mathcal{L}(Y^*, X^*)$.

Now, we can define a linear functional ψ on $\mathcal{L}(Y^*, X^*)$ by

$$\psi(U) = tr(S^* U R^*).$$

Let U be an arbitrary element of $\mathcal{L}(Y^*, X^*)$. Then,

$$|\psi(U)| = |tr(S^* U R^*)| \le \|S^* U R^*\|_{\mathcal{N}}.$$

Since, $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is surjective,

$$S^* U R^* \in (((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{sur}})^{\mathrm{adj}} \circ (((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{sur}})^{\min}(\ell_\infty, \ell_\infty) \subset \mathcal{N}(\ell_\infty, \ell_\infty).$$

Let $\mathcal{B} := (((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{sur}})^{\mathrm{adj}}$. By [**3**, Proposition 25.11] and Lemma 3.1, we have

$$\begin{aligned}
|\psi(U)| &\le \|S^* U R^*\|_{\mathcal{N}} \\
&\le \|S^*\|_{\mathcal{B}} \|U R^*\|_{(((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{sur}})^{\min}} \\
&= \|S^*\|_{\mathcal{B}} \|U R^*\|_{(((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\min})^{\mathrm{sur}}} \\
&= \|S^*\|_{\mathcal{B}} \|U R^*\|_{\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}} \\
&= \|S^*\|_{\mathcal{B}} m_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}(U R^*(B_{\ell_\infty}); X^*).
\end{aligned}$$

Since $R^*(B_{\ell_\infty})$ is $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$-compact, $\psi \in (\mathcal{L}(Y^*, X^*), \tau_{s(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}})^*$, and

$$\varphi(T) = tr(S^* T^* R^*) = \psi(T^*)$$

for every $T \in \mathcal{L}(X, Y)$.

□

From [**18**, Lemma 4.5], we have the following corollary.

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COROLLARY 3.4. *Suppose that $\mathcal{A}$ is a maximal and right-accessible Banach operator ideal such that $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is surjective. If the dual space of a Banach space X has the $\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}$-AP, then X has the $\mathcal{K}_{\mathcal{A}}$-AP.*

COROLLARY 3.5. *Suppose that $\mathcal{A}$ is a surjective, maximal and right-accessible Banach operator ideal. If the dual space of a Banach space X has the $\mathcal{K}_{\mathcal{A}}$-AP, then X has the $\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}$-AP.*

Proof. Let us consider the ideal $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ instead of $\mathcal{A}$ in Corollary 3.4. Then, $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is maximal and right-accessible by [**3**, Corollary 21.3], and

$$(((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{adj}})^{\mathrm{dual}} = \mathcal{A}.$$

Since $\mathcal{A}$ is surjective, by Corollary 3.4, if the dual space of a Banach space X has the $\mathcal{K}_{\mathcal{A}}$-AP, then X has the $\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}$-AP. □

In view of [**18**, Proposition 1.8], we see that $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}^{\mathrm{sur}}}$ (cf. [**1**]). Then, Corollary 3.5 can be reformulated as follows.

COROLLARY 3.6. *Suppose that $\mathcal{A}$ is a maximal and right-accessible Banach operator ideal. If the dual space of a Banach space X has the $\mathcal{K}_{\mathcal{A}}$-AP, then X has the $\mathcal{K}_{((\mathcal{A}^{\mathrm{sur}})^{\mathrm{adj}})^{\mathrm{dual}}}$-AP.*

The notion of p-compactness was introduced by Sinha and Karn [**24**], which stems from Grothendieck's criterion [**11**] of compactness. For $1 \le p < \infty$, a subset K of X is said to be *p-compact* if there exists $(x_n) \in \ell_p(X)$ such that

$$K \subset p\text{-}co(x_n) := \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}} \right\},$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\ell_p(X)$ is the Banach space with the norm $\|\cdot\|_p$ of all X-valued absolutely p-summable sequences. A linear map $T : Y \to X$ is said to be *p-compact* if $T(B_Y)$ is a p-compact subset of X. Delgado, Piñeiro, and Serrano [**5**] defined a norm on the space $\mathcal{K}_p(Y, X)$ of all p-compact operators from Y to X. For $T \in \mathcal{K}_p(Y, X)$, let

$$\|T\|_{\mathcal{K}_p} := \inf \left\{ \|(x_n)\|_p : (x_n) \in \ell_p(X) \text{ and } T(B_Y) \subset p\text{-}co(x_n) \right\}.$$

Then, $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ is a Banach operator ideal [**6**] and $\mathcal{K}_{\mathcal{K}_p} = \mathcal{K}_p$ [**17**].

For $1 \le p < \infty$, the space $\ell_p^u(X)$, which is a closed subspace of the Banach space $\ell_p^w(X)$ with the norm $\|\cdot\|_p^w$ of all X-valued weakly p-summable sequences, consists of all sequences (x_n) satisfying that

$$\|(0, \ldots, 0, x_m, x_{m+1}, \ldots)\|_p^w \longrightarrow 0$$

as $m \to \infty$ (cf. [**3**, Section 8.2] and [**8**, **9**]). In [**13**], the sequence was called the *unconditionally p-summable sequence*, and the *unconditionally p-compact* (u-p-compact) set and the u-p-compact operator were defined by replacing the space $\ell_p(X)$, in the definition of p-compactness, by the space $\ell_p^u(X)$. The space of all u-p-compact operators from Y to X is denoted by $\mathcal{K}_{up}(Y, X)$ and a norm $\|\cdot\|_{\mathcal{K}_{up}}$ on $\mathcal{K}_{up}(Y, X)$ was defined in [**13**] by

$$\|T\|_{\mathcal{K}_{up}} := \inf \left\{ \|(x_n)\|_p^w : (x_n) \in \ell_p^u(X) \text{ and } T(B_Y) \subset p\text{-}co(x_n) \right\}.$$

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Then, $[\mathcal{K}_{up}, \|\cdot\|_{\mathcal{K}_{up}}]$ is a Banach operator ideal [**13**, Theorem 2.1] and $\mathcal{K}_{up} = (\mathcal{L}_{p^*}^{\min})^{sur}$ [**14**, Proposition 3.1], where $\mathcal{L}_{p^*}$ is the ideal of *p^*-factorable operators*.

A Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is said to *be associated to* a tensor norm α if the canonical map $(\mathcal{A}(M, N), \|\cdot\|_{\mathcal{A}}) \to M^* \otimes_\alpha N$ is an isometry for all finite-dimensional normed spaces M and N. We denote by $/\alpha$ and $\backslash\alpha$, respectively, the *left-injective associate* and *left-projective associate* of α (see [**3**, Sections 20.6 and 20.7]).

For $1 \le p < \infty$, let g_p and d_p be the *Chevet-Saphar tensor norms* (see [**23**, Section 6.2]). It was shown in [**10**, Theorem 3.3] that $\mathcal{K}_p$ is associated with $/d_p$. Consequently, $\mathcal{K}_p^{\max}$ is associated with $/d_p$ and so is surjective and totally accessible (see [**3**, Theorem 20.11(2), the symmetric version of Proposition 21.1(2), Proposition 21.3 and Theorem 21.5]).

For $1 < p < \infty$, it was shown in [**12**] that if the dual space X^* of a Banach space X has the $\mathcal{K}_{up}$-AP, then X has the $\mathcal{K}_p$-AP, and if X^* has the $\mathcal{K}_p$-AP, then X has the $\mathcal{K}_{up}$-AP. In [**15**], it was shown that if X^* has the $\mathcal{K}_{u1}$-AP, then X has the $\mathcal{K}_1$-AP.

Corollary 3.7 ([**18**, Theorem 4.7]). *If the dual space of a Banach space X has the $\mathcal{K}_1$-AP, then X has the $\mathcal{K}_{u1}$-AP.*

Proof. Consider the ideal $\mathcal{K}_1^{\max}$ in Corollary 3.5. Recall that $d_1 = g_1$. Then, the ideal

$$((\mathcal{K}_1^{\max})^{adj})^{dual}$$

is associated with

$$(/g_1)' = \backslash g_1' = \backslash g_1^* = g_\infty$$

(see [**3**, Proposition 20.14]). Since $\mathcal{L}_\infty$ is associated with g_∞, $((\mathcal{K}_1^{\max})^{adj})^{dual} = \mathcal{L}_\infty$. Hence, by Corollary 3.5, if the dual space of a Banach space X has the $\mathcal{K}_{\mathcal{K}_1^{\max}}$-AP, then X has the $\mathcal{K}_{\mathcal{L}_\infty}$-AP. The proof follows since $\mathcal{K}_{\mathcal{K}_1^{\max}} = ((\mathcal{K}_1^{\max})^{\min})^{sur} = (\mathcal{K}_1^{\min})^{sur} = \mathcal{K}_{\mathcal{K}_1} = \mathcal{K}_1$ and $\mathcal{K}_{\mathcal{L}_\infty} = \mathcal{K}_{u1}$. □

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