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PAPER

Wb-sober spaces and the core-coherence of dcpo models

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Abstract

In this paper, we introduce a new class of T_0 spaces called wb-sober spaces, which is strictly larger than the class of open well-filtered spaces. Unlike open well-filtered spaces, wb-sober spaces are defined more intuitively by requiring certain special subsets, termed wb-irreducible closed sets, to have singleton closures. We establish several key results about these spaces, including (1) every open well-filtered space is wb-sober, but not vice versa; (2) every strongly core-coherent wb-sober space is open well-filtered; (3) a space is core-compact iff its irreducible closed sets are wb-irreducible, providing a characterization of core-compactness; (4) every core-compact wb-sober space is sober, thereby generalizing the Jia-Jung problem. In addition, we investigate the core-coherence of the Xi-Zhao model. We prove that a T_1 space contains finite number of isolated points iff its Xi-Zhao model is core-coherent iff its Xi-Zhao model is strongly core-coherent. Based on this result, we then propose a general approach to constructing a non-routine open well-filtered but not well-filtered dcpo.

Keywords: Core-coherence; core-compact space; open well-filtered space; poset model; Scott topology; wb-sober space

1. Introduction

In domain theory, sober spaces and well-filtered spaces are two of the most important classes of topological spaces. Well-filtered spaces were introduced by Heckmann, who posed the question of whether every well-filtered Scott space of a directed complete poset is sober (Heckmann, 1990,1992). This question has sparked extensive research into the relationship between sobriety and well-filteredness, as well as their respective properties (see Ho et al., 2018; Jia et al., 2016; Kou, 2001; Liu et al., 2020; Shen et al., 2020; Wu et al., 2020; Xi and Lawson, 2017; Xi and Zhao, 2017; Xu, 2021; Xu et al., 2020b; Zhao et al., 2019). A recent problem (usually called Jia-Jung problem) in this area is whether every core-compact well-filtered space is sober, as posed by Jia and Jung (2018, Question 2.5.19, p. 44). This question was first positively solved by Lawson et al. (2020, Theorem 3.1). Some other results on this topic are as follows:

(i) Every first-countable well-filtered space is sober (Xu et al., 2020a, Theorem 4.2);
(ii) Every first-countable ω-well-filtered d-space is sober (Xu et al., 2021, Theorem 6.7);
(iii) Every core-compact open well-filtered space is sober (Shen et al., 2020, Theorem 4.7).

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In Section 3, we explore further aspects of the Jia-Jung problem. Specifically, consider the following implication for a topological property M :

$$\text{core-compact} + M \Rightarrow \text{sober}.$$

Currently, it is known that the weakest topological property satisfying this implication is the open well-filteredness. This leads us to the following question:

- Is there any property M that is weaker than open well-filteredness? To what extent can it be weakened?

To address this question, it is necessary to understand the contribution of core-compactness to making a space sober. In Section 3, we introduce the notion of wb-irreducible sets, and use it to obtain a characterization of core-compactness (in the perspective of sobriety) as follows:

- A topological space is core-compact iff its irreducible sets are wb-irreducible.

From this, we naturally define wb-sober spaces by requiring wb-irreducible closed sets to be closures of singletons. We then prove the following results:

- Every open well-filtered space is wb-sober, but not vice versa. Moreover, every strongly core-coherent wb-sober space is open well-filtered.
- Every core-compact wb-sober space is sober, generalizing the previous result (iii).
- The category of wb-sober spaces with continuous mappings is not reflective in the category of T_0 spaces.

The following section is dedicated to exploring the core-coherence of the Xi-Zhao model. A poset P is called a *poset model* of a topological space X if X is homeomorphic to the set Max(P) of maximal elements of P, equipped with the relative topology of the Scott topology of P. The Scott topology is one of the most significant intrinsic topologies on posets in domain theory. Generally, the Scott space of a poset is only T_0. However, by considering the space of its maximal points, a richer variety of spaces can be obtained. Poset model theory has been extensively studied by various researchers and has played a crucial role in linking posets and T_1 spaces. One related result is due to Xi and Zhao (2009) and Zhao and Xi (2018), who proved that every T_1 space X has a dcpo model, denoted by $\mathbf{D(X)}$ and usually called the Xi-Zhao model. Given a topological property M, one can pose the following problem:

- When a T_1 space X satisfies M, is it true that $\mathbf{D(X)}$ equipped with the Scott topology also satisfies M

We say that a property M is *preserved by the Xi-Zhao model* if the above statement is true. It has been shown by He et al. (2019); Xi and Zhao (2017); Zhao and Xi (2018) that sobriety, well-filteredness, and Choquet completeness are all preserved by the Xi-Zhao model. Recently, Chen and Li (2022) proved that the Rudin property and weak sobriety are also preserved by the Xi-Zhao model, but core-compactness is not.

In this paper, we address the following question:

- Is the core-coherent or strongly core-coherent property preserved by the Xi-Zhao model?

We provide a negative answer to this question in Section 4. Furthermore, we establish a necessary and sufficient condition for the Xi-Zhao model to be core-coherent. Specifically, we prove the following:

- A T_1 space X has a finite number of isolated points iff $\mathbf{D(X)}$ is strongly core-coherent iff $\mathbf{D(X)}$ is core-coherent.

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Moreover, we show that for T_1 spaces, the properties of having finite isolated points and being strongly core-coherent are independent.

In the final section of the paper, we focus on developing a general approach to constructing open well-filtered (and therefore wb-sober) but not well-filtered dcpos, utilizing the Xi-Zhao model.

The concept of open well-filtered spaces, introduced by Shen et al. (2020) recently, forms a larger class than that of well-filtered spaces. One notable result of open well-filtered spaces is that every core-compact open well-filtered space is sober, which strengthens the result of Jia-Jung problem Jia (2018) (a positive answer is originally given by Lawson et al. (2020)). A recent related result is given by Chen and Li, who proved that the Xi-Zhao model of every T_1 space is open well-filtered (Chen and Li, 2023, Corollary 4.7).

We have observed the simplicity and validity of utilizing open well-filteredness to solve the Jia-Jung problem Shen et al. (2020). However, it is important to note that the routine condition: for Scott open sets U, V,

$$U \ll V \;\Leftrightarrow\; U = \emptyset,$$

often occurs in dcpos which are not core-compact with respect to the Scott topology. Such kind of dcpos cannot fully capture the characteristic of open well-filteredness, and limit our understanding of the entire class of open well-filtered spaces. Thus, we are motivated to ask the following question:

- Does there exist a non-routine open well-filtered, but not well-filtered dcpo?

In Section 4.3, we provide a positive answer by proving that:

- The Xi-Zhao model **D(X)** is non-routine open well-filtered whenever X has at least one isolated point.

This result offers a method for constructing a non-routine open well-filtered, but not well-filtered dcpo.

2. Preliminary

This section is devoted to a brief review of some basic concepts and notations that will be used in the paper. For more details, see Engelking (1989); Gierz et al. (2003); Goubault-Larrecq (2013).

Let P be a poset. For a subset A of P, we shall adopt the following standard notations:

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\};\ \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

For each $x \in X$, we write $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$. A subset A of P is called a *lower* (resp., an *upper*) *set* if $A = \downarrow A$ (resp., $A = \uparrow A$). An element x is *maximal* (resp., *minimal*) in $S \subseteq P$, if $S \cap \uparrow x = \{x\}$ (resp., $S \cap \downarrow x = \{x\}$). The set of all maximal (resp., minimal) elements of S is denoted by $\mathrm{Max}(S)$ (resp., $\mathrm{Min}(S)$). A nonempty subset D of P is *directed* (resp., *filtered*) if every two elements in D have an upper (resp., lower) bound in D. A subset F of P is called a *filter* if F is an upper and filtered set. P is called a *directed complete poset*, or a *dcpo* for short, if every directed subset of P has a supremum.

For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P for which $\bigvee D$ exists, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$.

A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology on P, called the *Scott topology* and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

Let X be a T_0 space. We use $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) to denote the set of all open (resp., closed) subsets of X, $\mathcal{O}^*(X) = \mathcal{O}(X) \setminus \{\emptyset\}$, and $\mathcal{C}^*(X) = \mathcal{C}(X) \setminus \{\emptyset\}$.

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A subset A of X is called *saturated* if A equals the intersection of all open sets containing it. The specialization order $\leq$ on X, is defined by $x \leq y$ iff $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator.

Remark 2.1. Let X be a T_0 space.

(1) A subset A of X is saturated iff $A = \uparrow A$ with respect to the specialization order. In particular, $\uparrow x = \bigcap\{U : x \in U \in \mathcal{O}(X)\}$ (Gierz et al. 2003, Exercise O-5.14).
(2) If X is a T_1 space, then for any $x, y \in X$, $x \leq y$ iff $x = y$ (Goubault-Larrecq, 2013, Proposition 4.2.3). Therefore, $\{x\} = \uparrow x = \bigcap\{U : x \in U \in \mathcal{O}(X)\}$.

Definition 2.2. *Let X be a T_0 space. A subset A of X is irreducible if $A \neq \emptyset$, and for any $U_1, U_2 \in \mathcal{O}(X)$, whenever $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$, it follows that $A \cap U_1 \cap U_2 \neq \emptyset$. The space X is called sober if for each irreducible closed set A, there exists a (unique) point $x \in X$ such that $A = \mathrm{cl}(\{x\})$.*

Definition 2.3. *A T_0 space X is called well-filtered if for any filtered family $\mathcal{F}$ of nonempty compact saturated subsets of X and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{F}$.*

Definition 2.4 (Shen et al., 2020, Definition 4.1). *Let X be a T_0 space.*

(1) A subfamily $\mathcal{F}$ of $\mathcal{O}(X)$ is called $\ll$-filtered if for each $U_1, U_2 \in \mathcal{F}$, there is $U_3 \in \mathcal{F}$ such that $U_3 \ll U_1$ and $U_3 \ll U_2$ in $(\mathcal{O}(X), \subseteq)$.
(2) The space X is called open well-filtered if for each $\ll$-filtered family $\mathcal{F} \subseteq \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies $V \subseteq U$ for some $V \in \mathcal{F}$.

Remark 2.5. Let $U, V \in \mathcal{O}(X)$. Then, $U \not\ll V$ iff there exists a filtered subfamily $\mathcal{F} \subseteq \mathcal{C}(X)$ such that $V \cap \bigcap \mathcal{F} = \emptyset$ and $U \cap C \neq \emptyset$ for any $C \in \mathcal{F}$.

Let X be a T_0 space, $A \subseteq X$ and $\emptyset \neq \mathcal{F} \subseteq \mathcal{O}(X)$. Let

$$m(\mathcal{F}) = \mathrm{Min}\{C \in \mathcal{C}(X) : C \cap U \neq \emptyset \text{ for all } U \in \mathcal{F}\}.$$

In other words, $A \in m(\mathcal{F})$ iff it is a minimal (under the inclusion order) closed set that has a nonempty intersection with every member of $\mathcal{F}$.

Lemma 2.6 (Shen et al., 2020, Lemma 2.4). *Let X be a T_0 space and $\mathcal{F}$ be a $\ll$-filtered subfamily of $\mathcal{O}(X)$. Then $m(\mathcal{F}) \neq \emptyset$ (using Zorn's Lemma), and every set in $m(\mathcal{F})$ is irreducible.*

Lemma 2.7 (Shen et al., 2022a, Theorem 3.3). *Let X be a T_0 space. Then, the following statements are equivalent:*

(1) X is open well-filtered.
(2) For each $\ll$-filtered family $\mathcal{F} \subseteq \mathcal{O}(X)$, $m(\mathcal{F}) \subseteq \{\mathrm{cl}(\{x\}) : x \in X\}$.

Definition 2.8 (Goubault-Larrecq, 2013, Definition 5.2.3). *A T_0 space X is called core-compact if $(\mathcal{O}(X), \subseteq)$ is a continuous poset.*

Remark 2.9. The following implications hold:

(1) Every sober space is well-filtered (Gierz et al., 2003, Theorem II-1.21); every well-filtered space is open well-filtered (Shen et al., 2020, Theorem 4.7).
(2) Every core-compact open well-filtered space is sober (Shen et al., 2020, Remark 4.2).

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Throughout this paper, we use $\mathbb{N}$ to denote the set of all positive integers, and let ω represent the first infinite ordinal. Clearly, $\omega = \{0\} \cup \mathbb{N}$, and a set A is finite iff $|A| < \omega$.

3. On Wb-Sober Spaces

In this section, we introduce a new class of topological spaces, called wb-sober spaces. In contrast to the definition of open well-filtered spaces, the wb-sober spaces are defined in terms of some special sets (called wb-irreducible sets), which is more consistent with the definition of sober spaces. Another advantage is that although the wb-sobriety is weaker than open well-filteredness, but the wb-sober spaces share many important properties with the open well-filtered spaces, especially in their link to sober spaces.

Definition 3.1. *Let X be a topological space.*

(1) A subset A of X is called way-below irreducible (wb-irreducible for short) if $A \neq \emptyset$, and for any $U_1, U_2 \in \mathcal{O}(X)$, whenever $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$, there exists $V \in \mathcal{O}(X)$ such that $V \ll U_1$, $V \ll U_2$, and $A \cap V \neq \emptyset$.

(2) We call X wb-sober if it is T_0, and for each wb-irreducible closed set A, there exists a (unique) $x \in X$ such that $A = \mathrm{cl}(\{x\})$.

Next, we provide some equivalent conditions for wb-irreducibility, which will be used in the sequel.

Lemma 3.2. *Let X be a T_0 space and $A \subseteq X$. Then, the following statements are equivalent:*

(1) A is wb-irreducible.

(2) For any $U_1, U_2 \in \mathcal{O}(X)$, if $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$, then there exists $V \in \mathcal{O}(X)$ such that $V \ll U_1 \cap U_2$ and $A \cap V \neq \emptyset$.

(3) A is irreducible, and for each $U \in \mathcal{O}(X)$, if $A \cap U \neq \emptyset$, then there exists $V \in \mathcal{O}(X)$ such that $V \ll U$ and $A \cap V \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2): Suppose A is wb-irreducible and $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$. Since A is wb-irreducible, there exists an open set W such that $W \ll U_1$, $W \ll U_2$, and $A \cap W \neq \emptyset$. Again, by the wb-irreducibility of A, there exists $V \in \mathcal{O}(X)$ such that $V \ll W$ and $A \cap V \neq \emptyset$. Since $V \ll W \ll U_1$ and $V \ll W \ll U_2$, it follows that $V \ll W \subseteq U_1 \cap U_2$. Therefore, $V \ll U_1 \cap U_2$, as desired. Hence, conclusion (2) holds.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are straightforward. □

The notion of wb-irreducibility helps one elucidate the core-compactness:

Proposition 3.3. *Let X be a T_0 space. Then, the following statements are equivalent:*

(1) X is core-compact.

(2) Every irreducible subset of X is wb-irreducible.

(3) For each $x \in X$, $\mathrm{cl}(\{x\})$ is wb-irreducible.

Proof. (1) $\Rightarrow$ (2): Let A be an irreducible set in X, and suppose $U \in \mathcal{O}(X)$ such that $A \cap U \neq \emptyset$. Choose a point $x \in A \cap U$. Since X is core-compact, we have $U = \bigcup\{V \in \mathcal{O}(X) : V \ll U\}$. Thus, there exists some $V_0 \in \mathcal{O}(X)$ such that $x \in V_0 \ll U$. Consequently, $x \in A \cap V_0 \neq \emptyset$. Therefore, by Lemma 3.2, A is wb-irreducible.

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The implications (2) ⇒ (3) and (3) ⇒ (1) are straightforward. □

The following result, together with Theorem 3.11, which will be presented later, generalizes Theorem 4.7 in Shen et al. (2020). It is immediate from Proposition 3.3.

Theorem 3.4. *A core-compact space is wb-sober iff it is sober.*

To further investigate the relationship between sober spaces and wb-sober spaces, we introduce the concept of strong core-coherence. First, we review the definition of core-coherence.

Definition 3.5 (Goubault-Larrecq, 2013, Definition 5.2.18, Proposition 5.2.19). *A topological space X is said to be core-coherent if for any $U_1, U_2, U \in \mathcal{O}(X)$, whenever $U \ll U_1$ and $U \ll U_2$, it holds that $U \ll U_1 \cap U_2$ (or equivalently, for any $U_1, U_2, V_1, V_2 \in \mathcal{O}(X)$, whenever $V_1 \ll U_1$ and $V_2 \ll U_2$, it holds that $V_1 \cap V_2 \ll U_1 \cap U_2$).*

Definition 3.6. *A topological space X is said to be strongly core-coherent if for any $U, U_1, U_2 \in \mathcal{O}(X)$, whenever $U_1 \ll U_2$, it holds that $U \cap U_1 \ll U \cap U_2$.*

As the name suggests, every strongly core-coherent is core-coherent, which can be easily verified. However, the converse is not true in general, as demonstrated by the following examples.

Example 3.7.

(1) The set $\mathbb{R}$ of real numbers under the usual topology is core-coherent but not strongly core-coherent.
*— It is known that locally compact T_2 spaces are core-coherent (see (*Goubault-Larrecq, 2013, *Fact 5.2.22 and Lemma 5.2.22)). Since $\mathbb{R}$ is a locally compact T_2 space, it follows that $\mathbb{R}$ is core-coherent. Additionally, note that $(0, 1) \ll \mathbb{R}$, as we can insert a compact subset, such as $[0, 1]$, between them. However, $(0, 1) \not\ll (0, 1) = (0, 1) \cap \mathbb{R}$, implying that $\mathbb{R}$ is not strongly core-coherent.*

(2) The unit interval $[0, 1]$, as the subspace of $\mathbb{R}$, is core-coherent but not strongly core-coherent.
— Using a similar argument as for $\mathbb{R}$, we can conclude that $[0, 1]$ is core-coherent. Additionally, note that $[0, 1]$ is a compact space, so $[0, 1] \ll [0, 1]$. However, $(0, 1) \not\ll (0, 1)$, it follows that $[0, 1]$ is not strongly core-coherent.

(3) Let $P = \mathbb{N} \cup \{a, b\}$, where $a, b \notin \mathbb{N}$ and $a \neq b$. The partial order $\leq$ on P is defined as follows:

  (i) $a \leq b \leq n$ for each $n \in \mathbb{N}$;
  (ii) $m \leq n$ iff $n \leq m$ in $\mathbb{N}$ under the usual order for each $m, n \in \mathbb{N}$.

The poset P can be represented by Figure 1. The space ΣP is a core-coherent space but not strongly core-coherent.
*— Indeed, P is a bounded complete continuous dcpo, and posets of this kind are always core-coherent under the Scott topology (see (*Goubault-Larrecq, 2013, *p. 200, line 5)). Additionally, consider the open sets $U = \mathbb{N}$, $U_1 = \mathbb{N} \cup \{b\} = \uparrow b$, and $U_2 = P = \uparrow a$. Clearly, $U_1 \ll U_2$ in $\sigma(X)$, but $U \cap U_1 = \mathbb{N} \not\ll \mathbb{N} = U \cap U_2$. Therefore, ΣP is not strongly core-coherent.*

Proposition 3.8. *Let X be a T_0 space, and define $\Uparrow V = \{U \in \mathcal{O}(X) : V \ll U\}$ for any $V \in \mathcal{O}(X)$. Then, the following conditions are equivalent:*

(1) X is strongly core-coherent.
(2) For each $V \in \mathcal{O}(X)$, if $\Uparrow V \neq \emptyset$, then $W \ll W$ for any $W \subseteq V$.
(3) For each $V \in \mathcal{O}(X)$, if $V \ll X$, then $W \ll W$ for any $W \subseteq V$.

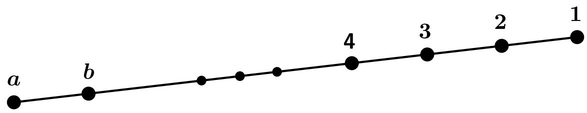


Figure 1. The poset $P = \mathbb{N} \cup \{a, b\}$.

Proof. (1) $\Rightarrow$ (2): Suppose $\twoheaduparrow V \neq \emptyset$ and $W \subseteq V$. Then, there exists $V_1 \in \mathcal{O}(X)$ such that $V \ll V_1$ (thus, $V \subseteq V_1$). Since X is strongly core-coherent, we have that $W = W \cap V \ll W \cap V_1 = W$, as desired.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Suppose $U, U_1, U_2 \in \mathcal{O}(X)$ and $U_1 \ll U_2$. From $U_1 \ll U_2 \subseteq X$, it follows that $U_1 \ll X$. By condition (3), it holds that $U \cap U_1 \ll U \cap U_1 \subseteq U \cap U_2$, which implies that $U \cap U_1 \ll U \cap U_2$. Therefore, X is strongly core-coherent. $\square$

As an immediate consequence of Proposition 3.8, we have:

Corollary 3.9. *The strongly core-coherent compact spaces are exactly the Noetherian spaces (i.e., every open subset of which is compact).*

From the preceding discussion on strong core-coherence, it is evident that this property is generally difficult to satisfy for arbitrary topological spaces. However, an interesting result that we will establish in the next section is that Xi-Zhao models are readily strongly core-coherent.

Next, we examine the relationship between open well-filtered spaces and wb-sober spaces.

Lemma 3.10. *Let X be a T_0 space and $A \subseteq X$, and define*

$$\mathcal{F}_A = \{U \in \mathcal{O}(X) : U \cap A \neq \emptyset\}.$$

Consider the following conditions:

(1) The set A is wb-irreducible.

(2) The family $\mathcal{F}_A$ is a $\ll$-filtered family such that $\mathrm{cl}(A) \in m(\mathcal{F}_A)$.

(3) There exists a $\ll$-filtered subfamily $\mathcal{F}$ of $\mathcal{O}(X)$ such that $\mathrm{cl}(A) \in m(\mathcal{F})$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). Moreover, if X is strongly core-coherent, then all three conditions are equivalent.

Proof. (1) $\Rightarrow$ (2): We first prove $\mathcal{F}_A$ is a $\ll$-filtered family. Suppose $U_1, U_2 \in \mathcal{F}_A$. Then, $U_1 \cap A \neq \emptyset$ and $U_2 \cap A \neq \emptyset$. Using the wb-irreducibility of A, there is $U_3 \in \mathcal{O}(X)$ such that $U_3 \ll U_1, U_2$ and $A \cap U_3 \neq \emptyset$, which implies that $U_3 \in \mathcal{F}_A$. Hence, $\mathcal{F}_A$ is $\ll$-filtered.

Now we prove that $\mathrm{cl}(A) \in m(\mathcal{F}_A)$. It is clear that $\mathrm{cl}(A)$ has a nonempty intersection with each element of $\mathcal{F}_A$. Now suppose C is a closed subset of $\mathrm{cl}(A)$ such that $C \cap U \neq \emptyset$ for each $U \in \mathcal{F}_A$. We need to prove $C = \mathrm{cl}(A)$. This is true since if $\mathrm{cl}(A) \cap (X \setminus C) \neq \emptyset$, then $X \setminus C \in \mathcal{F}_A$, but $C \cap (X \setminus C) = \emptyset$, which contradicts that C has a nonempty intersection with all elements of $\mathcal{F}_A$. Thus, we have that $\mathrm{cl}(A) \in m(\mathcal{F}_A)$.

(2) $\Rightarrow$ (1): Suppose $U_1, U_2 \in \mathcal{O}(X)$ such that $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$. Then, $U_1, U_2 \in \mathcal{F}_A$ and since $\mathcal{F}_A$ is $\ll$-filtered, there exists $U_3 \in \mathcal{F}_A$ such that $U_3 \ll U_1$ and $U_3 \ll U_2$. From the definition of $\mathcal{F}_A$, it follows that $A \cap U_3 \neq \emptyset$. This shows that A is wb-irreducible.

That (2) $\Rightarrow$ (3) is trivial.

Now suppose X is a strongly core-coherent space. We prove:

(3) $\Rightarrow$ (1): Let $U \in \mathcal{O}(X)$ satisfying that $A \cap U \neq \emptyset$. Since $\mathcal{F}$ is $\ll$-filtered, there exist $W, V \in \mathcal{F}$ such that $W \ll V$. Given that X is strongly core-coherent, it follows from Proposition 3.8 that

$W \cap U \ll W \cap U \subseteq U$, and thus $W \cap U \ll U$. Note that $\mathrm{cl}(A) \in m(\mathcal{F})$, so $A \cap W \neq \emptyset$, and by Lemma 2.6, A is irreducible, implying that $A \cap W \cap U \neq \emptyset$. Therefore, by Lemma 3.2, A is a wb-irreducible set. □

Theorem 3.11.

(1) Every open well-filtered space is wb-sober, but not vice versa.

(2) Every strongly core-coherent wb-sober is open well-filtered.

Proof. (1) Suppose X is an open well-filtered space, and A is a wb-irreducible closed subset of X. Then by Lemma 3.10, there exists a $\ll$-filtered subfamily $\mathcal{F}$ of $\mathcal{O}(X)$ such that $A \in m(\mathcal{F})$. Since X is open well-filtered, by Lemma 2.7 there exists $x \in X$ such that $A = \mathrm{cl}(\{x\})$. The uniqueness of x is trivial since X is T_0. Thus, X is wb-sober. The converse is not true, as demonstrated by Example 4.32 below.

(2) Now assume X is a strongly core-coherent wb-sober space. Suppose $\mathcal{F} \subseteq \mathcal{O}(X)$ is a $\ll$-filtered family. Then by Lemma 3.10, each set in $m(\mathcal{F})$ is wb-irreducible closed, and thus is of the form $\mathrm{cl}(\{x\})$, $x \in X$, by the wb-sobriety of X. Using Lemma 2.7, we have that X is open well-filtered. □

The following corollaries are trivial by Theorem 3.11 and Remark 2.9.

Corollary 3.12. *Let X be a strongly core-coherent T_0 space. Then, the following statements are equivalent:*

(1) X is open well-filtered.

(2) For each wb-irreducible closed set A in X, there exists a unique point $x \in X$ such that $A = \mathrm{cl}(\{x\})$.

Corollary 3.13. *Let*

$$\mathcal{A} = \{locally\ compactness,\ core - compactness\}$$

$$\mathcal{B} = \{sobriety,\ well - filteredness,\ open\ well - filteredness, \text{wb-sobriety}\}$$

Then, for any properties $\mathtt{A}, \mathtt{A}' \in \mathcal{A}$ and $\mathtt{B}, \mathtt{B}' \in \mathcal{B}$, we have the following equivalence relation:

$$\mathtt{A} + \mathtt{B} \Leftrightarrow \mathtt{A}' + \mathtt{B}'.$$

Corollary 3.14. *Let X be a core-compact space. Then, the following statements are equivalent:*

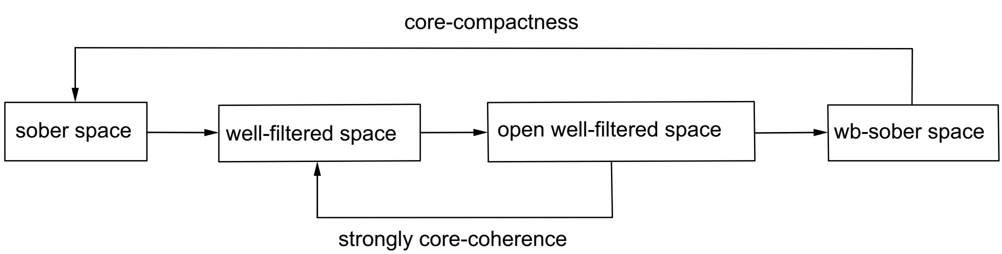
(1) X is wb-sober.

(2) X is open well-filtered.

(3) X is well-filtered.

(4) X is sober.

According to the above arguments, we make a summary as follows:



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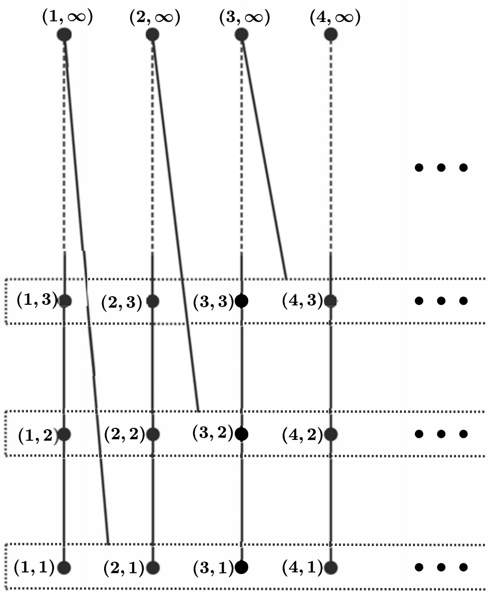

Figure 2. The Johnstone's dcpo $\mathbb{J}$.

$\Sigma\mathbb{J}$	strongly core-coherent	core-compact	well-filtered	open well-filtered	wb-sober
	✓	×	×	✓	✓

Figure 3. Some properties on $\Sigma\mathbb{J}$.

Example 3.15. *Let* $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ *be the Johnstone's dcpo* Johnstone (1981), *which is ordered by* $(m, n) \leq (m', n')$ *iff either* $m = m'$ *and* $n \leq n'$, *or* $n' = \infty$ *and* $n \leq m'$ *(see Figure 2). The following result is useful:*

- $\forall U, V \in \sigma(\mathbb{J})$, $U \ll V$ *iff* $U = \emptyset$ *(see* (Goubault-Larrecq, 2013, *Exercise 5.2.15)).*

We have the following properties (as illustrated in Figure 3):

(i) $\Sigma\mathbb{J}$ *is open well-filtered, and thus wb-sober by Theorem 3.11. However, it is not well-filtered (see* (Goubault-Larrecq, 2013, *Exercise 8.3.9)).*

(ii) Since $\Downarrow U = \{\emptyset\}$ *for each* $U \in \sigma(\mathbb{J})$, *one can easily deduce that* $\Sigma\mathbb{J}$ *is strongly core-coherent but not core-compact.*

Therefore, we deduce that

(1) The properties of strong core-coherence and core-compactness are parallel:
strong core-coherence $\not\Rightarrow$ *core-compactness* $\not\Rightarrow$ *strong core-coherence.*
— The first non-implication can be demonstrated by Johnstone's dcpo, using result (ii) mentioned above. For the second non-implication, refer to Example 3.7 (3), by noting that the Scott space of a continuous dcpo is always core-compact (see (Gierz et al., 2003, *Theorem II-1.14.)).*

(2) Strongly core-coherent open well-filtered spaces need not be well-filtered.
— This follows directed from the above results (i) and (ii) .

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In the next part, we explore the reflectivity of the category of wb-sober spaces. We denote by $\mathbf{Top_0}$ the category of all T_0 spaces and continuous mappings between them, and by **Wb-sober** the full subcategory of $\mathbf{Top_0}$ consisting of all wb-sober spaces.

Let $\mathbb{N}_{\mathrm{cof}}$ denote the space $\mathbb{N}$ endowed with the cofinite topology, where the nonempty open sets are the complements of finite subsets of $\mathbb{N}$. The following lemma is useful.

Lemma 3.16 (Shen et al., 2022b, Theorem 4.3). *If* $\mathbf{K}$ *is a reflective subcategory of* $\mathbf{Top}_0$ *such that* $\Sigma\mathbb{J} \in \mathbf{K}$, *then* $\mathbb{N}_{cof} \in \mathbf{K}$.

It is known that every subset of $\mathbb{N}$ is compact in this topology. Additionally, one can easily show that $\mathbb{N}$ itself is wb-irreducible, and it is not the closure of any singleton in $\mathbb{N}$. Therefore, we have:

Lemma 3.17. $\mathbb{N}_{cof}$ *is not wb-sober.*

From Example 3.15 we know that $\Sigma\mathbb{J} \in$ **Wb-sober**. However, by Lemma 3.17, $\mathbb{N}_{cof} \notin$ **Wb-sober**. Thus by Lemma 3.16, we obtain the following result.

Theorem 3.18. *The category* ***Wb-sober*** *is not a reflective subcategory of* $\mathbf{Top}_0$.

4. The Core-Coherence of the Xi-Zhao Model

We have seen from Theorem 3.11 that strong core-coherence plays a crucial role in linking wb-sobriety and open well-filteredness. The primary focus of this section is to investigate the following question:

- Does the Xi-Zhao model preserve the (strong) core-coherence?

We will present a negative answer to this question by constructing a counterexample in the final part. Moreover, we will provide a sufficient and necessary condition under which the Xi-Zhao model satisfies the (strong) core-coherence. Particularly, an interesting conclusion is that the Xi-Zhao dcpo is strongly core-coherent iff it is core-coherent.

4.1 On the Xi-Zhao model

In this part, we introduce some basic concepts and results of the Xi-Zhao model, which will be used in the sequel.

Let X be a T_1 space. We use $\mathsf{OF}(X)$ to denote the family of all filters on $(\mathcal{O}^*(X), \subseteq)$ having nonempty intersections. For each $A \subseteq X$, let

$$\vec{A} = \{U \in \mathcal{O}^*(X) : A \subseteq U\}.$$

In particular, we simply write $\vec{x}$ for $\overrightarrow{\{x\}}$.

Remark 4.1 (Zhao, 2009, Theorem 1). Let X be a T_1 space.

(1) $\mathrm{Max}(\mathsf{OF}(X)) = \{\vec{x} : x \in X\}$.

(2) The compact elements in $\mathsf{OF}(X)$ are of the form $\vec{U}$, where $U \in \mathcal{O}^*(X)$.

Lemma 4.2. *Let* X *be a* T_0 *space. Then, for any* $x, y \in X$, $\vec{x} = \vec{y}$ *if and only if* $x = y$.

Proof. It is straightforward since X is a T_0 space. □

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Theorem 4.3 (Zhao and Xi, 2018, Lemma 1). *For each T_1 space X, there is a dcpo* $\mathbf{D(X)}$ *such that X is homeomorphic to* $\mathrm{Max}(\mathbf{D(X)})$.

The dcpo model $\mathbf{D(X)}$ in (Zhao and Xi, 2018, Lemma 1), usually called *Xi-Zhao model*, is constructed as follows:

$$\mathbf{D(X)} = \{(\mathcal{F}, \vec{x}) : \mathcal{F} \in \mathsf{OF}(X), x \in X \text{ such that } \mathcal{F} \subseteq \vec{x}\},$$

and $(\mathcal{F}, \vec{x}) \leq (\mathcal{G}, \vec{y})$ in $\mathbf{D(X)}$ iff either $x = y$ (or equivalently, $\vec{x} = \vec{y}$) and $\mathcal{F} \subseteq \mathcal{G}$, or $\mathcal{G} = \vec{y}$ and $\mathcal{F} \subseteq \vec{y}$. Then, $\mathrm{Max}(\mathbf{D(X)}) = \{(\vec{x}, \vec{x}) : x \in X\}$ and the mapping $h : X \longrightarrow \mathrm{Max}(\mathbf{D(X)})$, where $h(x) = (\vec{x}, \vec{x})$ for each $x \in X$, is a homeomorphism.

To avoid confusion, we use the notations $\mathbb{A}, \mathbb{B}, \mathbb{F}, \mathbb{U}, \mathbb{V}, \cdots$ to indicate subsets of $\mathbf{D(X)}$, while $A, B, F, U, V, \cdots$ indicate subsets of X.

Remark 4.4. The dcpo model $\mathbf{D(X)}$ satisfies the following properties:

(1) For any $x \in X$ and $U \in \mathcal{O}(X)$, it is clear that $(\vec{U}, \vec{x}) \in \mathbf{D(X)}$ iff $x \in U$.

(2) For any $x \in U \in \mathcal{O}(X)$ and $y \in V \in \mathcal{O}(X)$ (hence $(\vec{U}, \vec{x}), (\vec{V}, \vec{y}) \in \mathbf{D(X)}$ by (1)), if $x \neq y$, then $\uparrow(\vec{U}, \vec{x}) \cap \uparrow(\vec{V}, \vec{y}) = \{(\vec{z}, \vec{z}) : z \in V \cap U\}$.
— Suppose $(\mathcal{F}, \vec{z}) \in \uparrow(\vec{U}, \vec{x}) \cap \uparrow(\vec{V}, \vec{y})$, that is $(\vec{U}, \vec{x}) \leq (\mathcal{F}, \vec{z})$ and $(\vec{V}, \vec{y}) \leq (\mathcal{F}, \vec{z})$. There are three cases to consider:
Case 1: Suppose $z = x$. From $x \neq y$ it follows that $z \neq y$. Since $(\vec{V}, \vec{y}) \leq (\mathcal{F}, \vec{z})$, we have that $\mathcal{F} = \vec{z}$ and $z \in V$ (or equivalently, $\vec{V} \subseteq \vec{z}$). Note that $z = x \in U$ by our assumption. Therefore, $(\mathcal{F}, \vec{z}) = (\vec{z}, \vec{z})$ and $z \in U \cap V$.
Case 2: Suppose $z = y$. It is similar to Case 1.
Case 3: Suppose $z \neq x$ and $z \neq y$. Since $z \neq y$ and $(\vec{V}, \vec{y}) \leq (\mathcal{F}, \vec{z})$, it follows that $\mathcal{F} = \vec{z}$ and $z \in V$ (or equivalently, $\vec{V} \subseteq \vec{z}$). Similarly, since $z \neq x$ and $(\vec{U}, \vec{x}) \leq (\mathcal{F}, \vec{z})$, it follows that $\mathcal{F} = \vec{z}$ and $z \in U$. Therefore, $(\mathcal{F}, \vec{z}) = (\vec{z}, \vec{z})$ and $z \in U \cap V$.
From the above argument, we can deduce that $\uparrow(\vec{U}, \vec{x}) \cap \uparrow(\vec{V}, \vec{y}) \subseteq \{(\vec{z}, \vec{z}) : z \in V \cap U\}$. The reverse inclusion is clear. Therefore, we get conclusion (2).

(3) For each directed subset $\mathbb{D}$ of $\mathbf{D(X)}$, either there is a largest element in $\mathbb{D}$, or there exists a directed family $\{\mathcal{F}_i : i \in I\} \subseteq \mathsf{OF}(X)$ and $x \in X$ such that $\mathbb{D} = \{(\mathcal{F}_i, \vec{x}) : i \in I\}$ and $\bigvee \mathbb{D} = (\bigvee_{i \in I} \mathcal{F}_i, \vec{x})$ (see (Zhao and Xi, 2018, Remark 2)).

(4) If $\mathbb{C} = \downarrow\{(\mathcal{F}_i, \vec{x}_i) : i \in I\} \subseteq \mathbf{D(X)} \setminus \mathrm{Max}(\mathbf{D(X)})$ such that for any $j, k \in I$, $\vec{x}_j \neq \vec{x}_k$ whenever $j \neq k$, then $\mathbb{C}$ is a Scott closed subset of $\mathbf{D(X)}$ (see (Xi and Zhao, 2017, Remark 1)).

Corollary 4.5. *Let X be a T_1 space. Then, the following are equivalent:*

(1) X is a finite topological space.

(2) $\mathbf{D(X)}$ is a finite dcpo.

(3) $\Sigma\mathbf{D(X)}$ is a compact space.

Proof. (1) $\Rightarrow$ (2): Suppose X is a finite set, that is $|X| < \omega$. We have that $|\mathbf{D(X)}| \leq |\mathsf{OF}(X)| \times |X| \leq 2^{|\mathcal{O}(X)|} \times |X| \leq 2^{2^{|X|}} \times |X| < \omega$. Therefore, $\mathbf{D(X)}$ is a finite set.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Assume, on the contrary, there exists a countable subset $\{x_k : k \in \mathbb{N}\} \subseteq X$, such that for any $k, m \in \mathbb{N}$, $x_k \neq x_m$ whenever $k \neq m$. For each $n \in \mathbb{N}$, define

$$\mathbb{C}_n := \downarrow\{(\vec{X}, \vec{x}_k) : k \geq n\}.$$

By Remark 4.4(4), $\{\mathbb{C}_n : n \in \mathbb{N}\}$ is a filtered family of nonempty Scott closed subset of $\mathbf{D(X)}$, but $\bigcap_{n \in \mathbb{N}} \mathbb{C}_n = \emptyset$. This shows that $\Sigma\mathbf{D(X)}$ is not compact, as desired. □

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Chen and Li (2023, Corollary 4.7) recently verified the open well-filteredness of the Xi-Zhao model. Their result provides a way to construct open well-filtered dcpos, which we will explore further in the sequel. Here is the statement of the theorem:

Theorem 4.6 ((Chen and Li, 2023, Corollary 4.7)). *The Xi-Zhao model of each T_1 space is open well-filtered (hence wb-sober) with respect to the Scott topology. Hence, every T_1 space has an open well-filtered dcpo model.*

4.2 The (strong) core-coherence of $\mathbf{D(X)}$

The cofinite topological space $\mathbb{N}_{\mathrm{cof}}$ often serves as a classic example (or sometimes a counterexample) in the context of the Xi-Zhao model. We observe that $\Sigma\mathbf{D}(\mathbb{N}_{\mathrm{cof}})$ satisfies strong core-coherence (and hence core-coherence), with the proof relying heavily on the fact that $\mathbb{N}_{\mathrm{cof}}$ has a finite number of isolated points (in this case, zero). Inspired by this observation, we aim to extend this result to general T_1 spaces with finitely many isolated points. Specifically, we prove that the Xi-Zhao model $\Sigma\mathbf{D}(X)$ for such spaces is always core-coherent (and even strongly core-coherent). Notably, this conclusion also holds in reverse.

To achieve this, we first introduce the following definition.

Definition 4.7. *Let X be a T_1 space. We define X_{iso} as the set of all isolated points in X, that is,*

$$X_{\mathrm{iso}} = \{x \in X : \{x\} \in \mathcal{O}(X)\}.$$

Lemma 4.8. *Let X be a T_1 space. Then, for any $x \in U \subseteq X_{\mathrm{iso}}$, $\mathbb{U} := \uparrow(\vec{U}, \vec{x})$ is a Scott open subset of* $\mathbf{D(X)}$.

Proof. First, note that $U \in \mathcal{O}(X)$ as $U \subseteq X_{\mathrm{iso}}$. Clearly, $\mathbb{U}$ is an upper set. Assume $\mathbb{D}$ is a directed subset of $\mathbf{D(X)}$ such that $\bigvee \mathbb{D} \in \mathbb{U}$, that is $(\vec{U}, \vec{x}) \leq \bigvee \mathbb{D}$. By Remark 4.4(3), we assume there exists a directed family $\{\mathcal{F}_i : i \in I\} \subseteq \mathsf{OF}(X)$ and $y \in X$ such that $\mathbb{D} = \{(\mathcal{F}_i, \vec{y}) : i \in I\}$ and $(\vec{U}, \vec{x}) \leq \bigvee \mathbb{D} = (\bigvee_{i\in I} \mathcal{F}_i, \vec{y})$. There are two cases:

(i) If $y = x$, then $\vec{U} \subseteq \bigvee_{i\in I} \mathcal{F}_i$. By Remark 4.1(2) and $U \in \mathcal{O}(X)$, there exists $i_0 \in I$ such that $\vec{U} \subseteq \mathcal{F}_{i_0}$, which implies that $(\vec{U}, \vec{x}) \leq (\mathcal{F}_{i_0}, \vec{x}) = (\mathcal{F}_{i_0}, \vec{y}) \in \mathbb{D}$.

(ii) If $y \neq x$, then $\bigvee_{i\in I} \mathcal{F}_i = \vec{y}$ and $y \in U \subseteq X_{\mathrm{iso}}$, which implies that $\{y\} \in \mathcal{O}(X)$. By Remark 4.1(2), there exists $i_0 \in I$ such that $\vec{y} = \mathcal{F}_{i_0}$, so $(\vec{U}, \vec{x}) \leq (\bigvee_{i\in I} \mathcal{F}_i, \vec{y}) = (\vec{y}, \vec{y}) = (\mathcal{F}_{i_0}, \vec{y}) \in \mathbb{D}$.

All these show that $\mathbb{D} \cap \mathbb{U} \neq \emptyset$. Therefore, $\mathbb{U} \in \sigma(\mathbf{D(X)})$, as desired. □

Lemma 4.9. *Let X be a T_1 space. Then, the following statements hold.*

(1) For each $x \in X$, $\{x\} \in \mathcal{O}(X)$ iff $\{(\vec{x}, \vec{x})\} \in \sigma(\mathbf{D(X)})$. In other words,

$$\Sigma\mathbf{D(X)}_{\mathrm{iso}} = \{(\vec{x}, \vec{x}) : x \in X_{\mathrm{iso}}\}.$$

Therefore, $|X_{\mathrm{iso}}| = |\Sigma\mathbf{D(X)}_{\mathrm{iso}}|$. In particular, $|X_{\mathrm{iso}}| < \omega$ iff $|\Sigma\mathbf{D(X)}_{\mathrm{iso}}| < \omega$.

(2) $\Sigma\mathbf{D(X)}_{\mathrm{iso}} \ll \Sigma\mathbf{D(X)}_{\mathrm{iso}}$ in $\sigma(\mathbf{D(X)})$ iff X_{iso} is finite.

Proof. (1) *Necessity:* Suppose $\{x\}$ is an open subset of X. We need to prove $\{(\vec{x}, \vec{x})\}$ is a Scott open subset of $\mathbf{D(X)}$. First, it is clear that $\{(\vec{x}, \vec{x})\}$ is an upper set, as the point $(\vec{x}, \vec{x})$ is maximal. Assume $\mathbb{D}$ is a directed subset of $\mathbf{D(X)}$ such that $\bigvee \mathbb{D} = (\vec{x}, \vec{x})$. By Remark 4.4(3), we may assume that $\mathbb{D} = \{(\mathcal{F}_i, \vec{y}) : i \in I\}$, where $\{\mathcal{F}_i : i \in I\} \subseteq \mathsf{OF}(X)$ is directed and $y \in X$, so $(\bigvee_{i\in I} \mathcal{F}_i, \vec{y}) = \bigvee \mathbb{D} = (\vec{x}, \vec{x})$. It follows that $\vec{x} = \vec{y}$ (so $x = y$ by Lemma 4.2), and $\bigvee_{i\in I} \mathcal{F}_i = \vec{x}$. Note that $\{x\} \in \mathcal{O}(X)$, so by Remark 4.1(2), we have that $\vec{x}$ is compact in $\mathsf{OF}(X)$, so there exists $i_0 \in I$ such that $\mathcal{F}_{i_0} = \vec{x}$. This shows that $(\vec{x}, \vec{x}) = (\mathcal{F}_{i_0}, \vec{x}) = (\mathcal{F}_{i_0}, \vec{y}) \in \mathbb{D}$. Therefore, $\{(\vec{x}, \vec{x})\}$ is a Scott open set.

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Sufficiency: Assume $\{(\vec{x},\vec{x})\} \in \sigma(\mathbf{D(X)})$. By using the T_1 separation of X, one can verify that $\{(\vec{U},\vec{x}) : x \in U \in \mathcal{O}(X)\}$ is a directed subset of $\mathbf{D(X)}$ whose supremum equals $(\vec{x},\vec{x})$. Then by using the Scott openness of $\{(\vec{x},\vec{x})\}$, there exists $U_0 \in \mathcal{O}(X)$ such that $(\vec{x},\vec{x}) = (\vec{U_0},\vec{x})$. It follows that $\vec{x} = \vec{U_0}$, which implies that $\{x\} = \bigcap \vec{x} = \bigcap \vec{U_0} = U_0 \in \mathcal{O}(X)$ by Remark 2.1. Thus, $\{x\} \in \mathcal{O}(X)$, as desired.

(2) Suppose $\Sigma\mathbf{D(X)}_{\text{iso}} \ll \Sigma\mathbf{D(X)}_{\text{iso}}$. By (1), the family $\{\{(\vec{x},\vec{x})\} : (\vec{x},\vec{x}) \in \Sigma\mathbf{D(X)}_{\text{iso}}\}$ forms an open cover of $\Sigma\mathbf{D(X)}_{\text{iso}}$, so there exists a finite subset $\mathbb{F} \subseteq \Sigma\mathbf{D(X)}_{\text{iso}}$ such that $\Sigma\mathbf{D(X)}_{\text{iso}} \subseteq \mathbb{F}$. It follows that $\Sigma\mathbf{D(X)}_{\text{iso}} = \mathbb{F}$, so $\Sigma\mathbf{D(X)}_{\text{iso}}$ is finite. Using result (1), we have that X_{iso} is finite. The converse is trivial. □

Lemma 4.10. *Let X be a T_1 space and $\mathbb{U} \in \sigma(\mathbf{D(X)})$. Then, the following statements hold.*

(1) $\mathrm{Max}(\mathbb{U}) = \mathbb{U} \cap \mathrm{Max}(\mathbf{D(X)})$.

(2) For each $(\mathcal{F},\vec{x}) \in \mathbb{U}$, there exists $V \in \mathcal{O}(X)$ such that $(\mathcal{F},\vec{x}) \geq (\vec{V},\vec{x}) \in \mathbb{U}$. In other words,

$$\mathbb{U} = \uparrow\{(\vec{V},\vec{x}) \in \mathbb{U} : x \in V \in \mathcal{O}(X)\}.$$

(3) For each $V \in \mathcal{O}(X)$ and $x \in X$, if $(\vec{V},\vec{x}) \in \mathbb{U}$, then $x \in V \subseteq \{y \in X : (\vec{y},\vec{y}) \in \mathrm{Max}(\mathbb{U})\}$.

(4) If $|\mathrm{Max}(\mathbb{U})| < \omega$, then there is a finite subset $\mathbb{F}$ of $\mathbf{D(X)}$ such that $\mathbb{U} = \uparrow\mathbb{F}$.

Proof. (1) It is clear since $\mathbb{U}$ is an upper set.

(2) Suppose $(\mathcal{F},\vec{x}) \in \mathbb{U}$. Note that $\{(\vec{V},\vec{x}) : V \in \mathcal{F}\}$ is a directed subset of $\mathbf{D(X)}$ whose supremum is $(\mathcal{F},\vec{x})$, and since $(\mathcal{F},\vec{x}) \in \mathbb{U} \in \sigma(\mathbf{D(X)})$, there is $V_0 \in \mathcal{F}$ such that $(\vec{V_0},\vec{x}) \in \mathbb{U}$. It follows that $(\mathcal{F},\vec{x}) \geq (\vec{V_0},\vec{x})$, completing the proof.

(3) Suppose $(\vec{V},\vec{x}) \in \mathbb{U}$. First, it is clear that $x \in V$, as $(\vec{V},\vec{x}) \in \mathbf{D(X)}$. Let $y \in V$. Then we have that $(\vec{y},\vec{y}) \geq (\vec{V},\vec{x}) \in \mathbb{U} = \uparrow\mathbb{U}$, which follows that $(\vec{y},\vec{y}) \in \mathbb{U} \cap \mathrm{Max}(\mathbf{D(X)}) = \mathrm{Max}(\mathbb{U})$. This proves (3).

(4) Suppose there exists $m \in \mathbb{N}$ such that $\mathrm{Max}(\mathbb{U}) = \{(\vec{x}_k,\vec{x}_k) : 1 \leq k \leq m\}$. Let $\mathbb{F} := \{(\vec{V},\vec{x}) \in \mathbb{U} : x \in V \in \mathcal{O}(X)\}$. For each $(\vec{V},\vec{x}) \in \mathbb{F}$, from (3) it follows that $x \in V \subseteq \{x_1, x_2, \cdots, x_m\}$. Thus, we have that $|\mathbb{F}| \leq 2^m \times m$, and hence $\mathbb{F}$ is a finite set. By (2), $\mathbb{U} = \uparrow\mathbb{F}$, completing the proof. □

Lemma 4.11. *Let X be a T_1 space and $x \in U \in \mathcal{O}(X)$. Then, $U \neq \{x\}$ iff $\vec{U} \neq \vec{x}$.*

Proof. Suppose $U \neq \{x\}$. Note that U is nonempty since $x \in U$. Then, there exists $y \in U$ such that $y \neq x$. Since X is a T_1 space, $U \setminus \{y\}$ is an open set that contains x. Consequently, $U \setminus \{y\} \in \vec{x}$, and clearly $U \setminus \{y\} \notin \vec{U}$. Therefore, $\vec{U} \neq \vec{x}$. The converse is trivial. □

Proposition 4.12. *Let X be a T_1 space and $\mathbb{U} \in \sigma(\mathbf{D(X)})$. If $|X_{\text{iso}}| < \omega$, then the following statements are equivalent:*

(1) $\mathbb{U} \ll \mathbf{D(X)}$.

(2) $\mathbb{U} \ll \mathbb{V}$ *for some* $\mathbb{V} \in \sigma(\mathbf{D(X)})$.

(3) $\mathrm{Max}(\mathbb{U})$ *is a finite set.*

(4) There exists a finite subset $\mathbb{F}$ of $\mathbf{D(X)}$ such that $\mathbb{U} = \uparrow\mathbb{F}$.

Proof. (1) ⇔ (2): It is trivial.

(1) ⇒ (3): Assume, on the contrary, $\mathrm{Max}(\mathbb{U})$ is infinite. Then there exists a subset $\mathbb{M} := \{(\vec{x}_k,\vec{x}_k) : k \in \mathbb{N}\}$ of $\mathrm{Max}(\mathbb{U})$ such that for any $m, n \in \mathbb{N}$, $x_m \neq x_n$ whenever $m \neq n$. Given that $|X_{\text{iso}}| < \omega$, we may, without loss of generality, assume that no singleton $\{x_k\}$ is open (i.e., $x_k \notin X_{\text{iso}}$). If this were not the case, we could redefine $\mathbb{M}' := \mathbb{M} \setminus \{(\vec{x},\vec{x}) : \vec{x} \in X_{\text{iso}}\}$. Since X_{iso} is finite, $\mathbb{M}'$ would still be infinite.

For each $k \in \mathbb{N}$, note that $\{(\vec{U}, \vec{x}_k) : x_k \in U \in \mathcal{O}(X)\}$ is a directed subset of $\mathbf{D(X)}$ whose supremum is $(\vec{x}_k, \vec{x}_k)$. Since $(\vec{x}_k, \vec{x}_k)$ belongs to the Scott open set $\mathbb{U}$, there exists $U_k \in \mathcal{O}(X)$ such that $(\vec{U}_k, \vec{x}_k) \in \mathbb{U}$. Since $\{x_k\}$ is not open (i.e., $x_k \notin X_{\text{iso}}$), it holds that $U_k \neq \{x_k\}$, and from Lemma 4.11 it follows that $\vec{U}_k \neq \vec{x}_k$, that is, $(\vec{U}_k, \vec{x}_k) \in \mathbf{D(X)} \setminus \text{Max}(\mathbf{D(X)})$. Therefore, $\{(\vec{U}_k, \vec{x}_k) : k \in \mathbb{N}\} \subseteq \mathbf{D(X)} \setminus \text{Max}(\mathbf{D(X)})$. For each $n \in \mathbb{N}$, define

$$\mathbb{C}_n := \downarrow\{(\vec{U}_k, \vec{x}_k) : k \geq n\}.$$

By Remark 4.4(4), $\{\mathbb{C}_n : n \in \mathbb{N}\}$ is a filtered family of Scott closed subset of $\mathbf{D(X)}$. It holds that $\bigcap_{n\in\mathbb{N}} \mathbb{C}_n = \emptyset$, but $(\vec{U}_n, \vec{x}_n) \in \mathbb{C}_n \cap \mathbb{U} \neq \emptyset$, which implies that $\mathbb{U} \not\ll \mathbf{D(X)}$. Therefore, the conclusion holds.

(3) $\Rightarrow$ (4): It follows immediately from Lemma 4.10(4).

(4) $\Rightarrow$ (1): It is straightforward. □

Theorem 4.13. *Let X be a T_1 space. Then, the following conditions are equivalent:*

(1) X_{iso} is a finite set.

(2) $\Sigma\mathbf{D(X)}$ is a strongly core-coherent space.

(3) $\Sigma\mathbf{D(X)}$ is a core-coherent space.

Proof. (1) $\Rightarrow$ (2): Suppose $\mathbb{U}_1, \mathbb{U}_2, \mathbb{V} \in \sigma(\mathbf{D(X)})$ such that $\mathbb{U}_1 \ll \mathbb{U}_2$. Then, $\mathbb{V} \cap \mathbb{U}_1 \subseteq \mathbb{U}_1 \ll \mathbb{U}_2 \subseteq \mathbf{D(X)}$, which implies that $\mathbb{V} \cap \mathbb{U}_1 \ll \mathbf{D(X)}$. By Proposition 4.12, there exists a finite subset $\mathbb{F}$ of $\mathbf{D(X)}$ such that $\mathbb{V} \cap \mathbb{U}_1 = \uparrow\mathbb{F}$. Since $\mathbb{V} \cap \mathbb{U}_1 = \uparrow\mathbb{F} \subseteq \mathbb{V} \cap \mathbb{U}_2$, it holds that $\mathbb{V} \cap \mathbb{U}_1 \ll \mathbb{V} \cap \mathbb{U}_2$. This shows that $\Sigma\mathbf{D(X)}$ is strongly core-coherent.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): If $|X_{\text{iso}}| \leq 2$, then the conclusion holds. Assume $|X_{\text{iso}}| \geq 2$, and then we can choose $x, y \in X_{\text{iso}}$ with $x \neq y$. Note that $X_{\text{iso}} \in \mathcal{O}(X)$, and then by Lemma 4.8, $\mathbb{U} := \uparrow(\vec{X}_{\text{iso}}, \vec{x}) \in \sigma(\mathbf{D(X)})$ and $\mathbb{V} := \uparrow(\vec{X}_{\text{iso}}, \vec{y}) \in \sigma(\mathbf{D(X)})$. It is clear that both $\mathbb{U}$ and $\mathbb{V}$ are compact saturated sets in $\sigma(\mathbf{D(X)})$, so $\mathbb{U} \ll \mathbb{U}$ and $\mathbb{V} \ll \mathbb{V}$ in $\sigma(\mathbf{D(X)})$. Since $\Sigma\mathbf{D(X)}$ is core-coherent, by Remark 4.4(2) we have that

$$\Sigma\mathbf{D(X)}_{\text{iso}} = \{(\vec{x}, \vec{x}) : x \in X_{\text{iso}}\} = \mathbb{U} \cap \mathbb{V} \ll \mathbb{U} \cap \mathbb{V} = \{(\vec{x}, \vec{x}) : x \in X_{\text{iso}}\} = \Sigma\mathbf{D(X)}_{\text{iso}}.$$

By Lemma 4.9(2), X_{iso} is a finite set, completing the proof. □

The following corollary is clear by Lemma 4.9, Theorems 4.6, 3.11 and 4.13.

Corollary 4.14. *Let X be a T_1 space. Then, the following statements are equivalent:*

(1) X_{iso} is a finite set.

(2) $\Sigma\mathbf{D(X)}_{\text{iso}}$ is a finite set.

(3) $\Sigma\mathbf{D(X)}$ is a core-coherent space.

(4) $\Sigma\mathbf{D(X)}$ is a strongly core-coherent space.

(5) $\Sigma\mathbf{D(X)}$ is a core-coherent open well-filtered space.

(6) $\Sigma\mathbf{D(X)}$ is a strongly core-coherent open well-filtered space.

(7) $\Sigma\mathbf{D(X)}$ is a core-coherent wb-sober space.

(8) $\Sigma\mathbf{D(X)}$ is a strongly core-coherent wb-sober space.

By Theorem 4.6 and Corollary 4.14, the following result is clear.

Theorem 4.15. *Every T_1 space having finite number of isolated points has a (strongly) core-coherent open well-filtered dcpo model.*

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Considering that $|(\mathbb{N}_{cof})_{\mathrm{iso}}| = 0 < \omega$, the following result holds as a direct application of Theorem 4.13.

Corollary 4.16. *The Scott space* $\Sigma\mathbf{D}(\mathbb{N}_{cof})$ *is (strongly) core-coherent and open well-filtered.*

4.3 A non-routine open well-filtered dcpo

Definition 4.17. *A* T_0 *space* X *is said to be routine (with respect to the way-below relation* $\ll$ *on* $\mathcal{O}(X)$*) if for any* $U, V \in \mathcal{O}(X)$*, whenever* $U \ll V$*,* $U = \emptyset$*.*

Remark 4.18.

(1) It is trivial to verify that every routine space is strongly core-coherent (thus core-coherent) and open well-filtered:

$$\text{open well-filtered} \quad \Leftarrow \quad \text{routine} \quad \Rightarrow \quad \text{strongly core-compact.}$$

(2) The Scott space of the Johnstone's dcpo is routine, as shown in Example 3.15.

Theorem 4.19. *Let* X *be a* T_1 *space. Then, the following statements are equivalent:*

(1) $X_{\mathrm{iso}} = \emptyset$.

(2) $\Sigma\mathbf{D(X)}$ *is a routine space.*

Proof. (1) $\Rightarrow$ (2): Assume, on the contrary, that $\Sigma\mathbf{D(X)}$ is non-routine. Then, there exist $\mathbb{U}, \mathbb{V} \in \sigma(\mathbf{D(X)})$ such that $\emptyset \neq \mathbb{U} \ll \mathbb{V}$. Then, it is clear that $\mathbb{U} \ll \mathbf{D(X)}$. By Proposition 4.12, $\mathrm{Max}(\mathbb{U}) = \mathbb{U} \cap \mathrm{Max}\mathbf{D(X)}$ is a finite set, and we may assume $\mathrm{Max}(\mathbb{U}) := \{(\vec{x}, \vec{x}) : x \in F\}$, where F is a finite subset of X. Then, by Remark 4.4(1), $F = h^{-1}(\mathrm{Max}(\mathbb{U})) \in \mathcal{O}(X)$. Note that $F \neq \emptyset$ because $\mathbb{U} \neq \emptyset$. Choose a point $x \in F \neq \emptyset$. Then, as X is T_1 space, the finite set $F \setminus \{x\}$ is closed, so the singleton set $\{x\} = F \setminus (F \setminus \{x\})$ is open, contradicting $X_{\mathrm{iso}} = \emptyset$. Therefore, $\Sigma\mathbf{D(X)}$ is a routine space.

(2) $\Rightarrow$ (1): Assume, on the contrary, there exists a point $x \in X_{\mathrm{iso}}$. Then, by Lemma 4.8, $\emptyset \neq \uparrow(\vec{X}_{\mathrm{iso}}, \vec{x}) \in \sigma(\mathbf{D(X)})$ and $\uparrow(\vec{X}_{\mathrm{iso}}, \vec{x}) \ll \uparrow(\vec{X}_{\mathrm{iso}}, \vec{x}) \subseteq \mathbf{D(X)}$. Thus, $\emptyset \neq \uparrow(\vec{X}_{\mathrm{iso}}, \vec{x}) \ll \mathbf{D(X)}$, contradicting assumption (2). Therefore, $X_{\mathrm{iso}} = \emptyset$. □

The following result is clear from the above theorem, which strengthens Corollary 4.16.

Corollary 4.20. *The Scott space* $\Sigma\mathbf{D}(\mathbb{N}_{cof})$ *is a routine space.*

By Corollary 4.14, Theorems 4.6 and 4.19, we have the following results.

Theorem 4.21. *Every* T_1 *space containing at least one isolated point has a non-routine open well-filtered dcpo model.*

Theorem 4.22. *Every* T_1 *space containing a finite nonempty set of isolated points has a non-routine strongly core-coherent open well-filtered dcpo model.*

By Theorem 3 in Xi and Zhao (2017) (saying that a T_1 space is well-filtered iff its Xi-Zhao model is well-filtered), we obtain the following corollary, which provides a general approach to construct a non-routine open well-filtered but not well-filtered dcpo.

Corollary 4.23. *Let* X *be a* T_1 *space. If* X *is not well-filtered and* $X_{\mathrm{iso}} \neq \emptyset$*, then* $\mathbf{D(X)}$ *is a non-routine open well-filtered, but not well-filtered dcpo.*

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4.4 Some examples

Before presenting the example, we will introduce some concepts and properties related to the sum (or co-product) of topological spaces, which will be used later.

Definition 4.24. *Suppose X and Y are two disjoint topological spaces, that is, $X \cap Y = \emptyset$. The set $Z = X \cup Y$ with the topology*

$$\mathcal{O}(Z) = \{U \subseteq Z : U \cap X \in \mathcal{O}(X), U \cap Y \in \mathcal{O}(Y)\}$$

is called the sum (co-product) of the spaces X and Y, denoted by $X \oplus Y$.

Lemma 4.25 ((Engelking, 1989, Theorem 2.2.7)). *The sum $X \oplus Y$ is T_0 (resp., T_1) whenever X and Y are T_0 (resp., T_1) spaces.*

Lemma 4.26. *Let X and Y be two disjoint topological spaces, and $U, V \in \mathcal{O}(X \oplus Y)$. Then, the following conditions are equivalent:*

(1) $U \ll V$ in $\mathcal{O}(X \oplus Y)$.

(2) $U \cap X \ll V \cap X$ in $\mathcal{O}(X)$ and $U \cap Y \ll V \cap Y$ in $\mathcal{O}(Y)$.

Proof. (1) $\Rightarrow$ (2): We only verify $U \cap X \ll V \cap X$, as the proof for $U \cap Y \ll V \cap Y$ is similar. Suppose $\{V_i : i \in I\} \subseteq \mathcal{O}(X)$ is an open cover of $V \cap X$. Then, $V = (V \cap X) \cup (V \cap Y) \subseteq (V \cap X) \cup Y \subseteq \bigcup_{i \in I} (V_i \cup Y)$, which implies that $\{V_i \cup Y : i \in I\} \subseteq \mathcal{O}(X \oplus Y)$. As $U \ll V$, there exists a finite subset $J \subseteq I$ such that $U \subseteq \bigcup_{j \in J} (V_j \cup Y)$. Note that $X \cap Y = \emptyset$ and $V_i \subseteq X$ for all $i \in I$, so $U \cap X \subseteq \bigcup_{j \in J} ((V_j \cap X) \cup (Y \cap X)) = \bigcup_{j \in J} (V_j \cap X) = \bigcup_{j \in J} V_j$. This shows that $U \cap X \ll V \cap X$.

(2) $\Rightarrow$ (1): Suppose $\{V_i : i \in I\} \subseteq \mathcal{O}(X \oplus Y)$ is an open cover of V. Then, it is clear that $\{V_i \cap X : i \in I\} \subseteq \mathcal{O}(X)$ is an open cover of $V \cap X$ and $\{V_i \cap Y : i \in I\} \subseteq \mathcal{O}(Y)$ is an open cover of $V \cap Y$. Since $U \cap X \ll V \cap X$ and $U \cap Y \ll V \cap Y$, there exist finite subsets $J_1, J_2 \subseteq I$ such that $U \cap X \subseteq \bigcup_{j \in J_1} V_j \cap X$ and $U \cap Y \subseteq \bigcup_{j \in J_2} V_j \cap Y$. Let $J := J_1 \cup J_2$, which is a finite set. Then, $U = (U \cap X) \cup (U \cap Y) \subseteq (\bigcup_{j \in J_1} V_j \cap X) \cup (\bigcup_{j \in J_2} V_j \cap Y) \subseteq \bigcup_{j \in J} V_j$. This shows that $U \ll V$. □

Lemma 4.27. *Let X and Y be two disjoint topological spaces, and $K \subseteq X \oplus Y$. Then, the following conditions are equivalent:*

(1) K is a compact set in $X \oplus Y$.

(2) $K \cap X$ and $K \cap Y$ are compact sets in X and Y, respectively.

Proof. The proof is analogous to Lemma 4.26. □

In the following, we provide an example of non-routine open well-filtered but not well-filtered dcpo.

Example 4.28. *Recall that $\mathbb{N}$ is the set of all positive integers. Let $Y := \{\frac{1}{k} : k \in \mathbb{N} \setminus \{1\}\} \cup \{0\}$ be the subspace of the real line with the usual topology. Note that $Y \cap \mathbb{N} = \emptyset$. Let $X := Y \oplus (\mathbb{N})_{cof}$ be the sum of spaces Y and $(\mathbb{N})_{cof}$.*

(1) X is a strongly core-coherent (and hence core-coherent) T_1 space.
— As Y and $(\mathbb{N})_{cof}$ are both T_1 spaces, by Lemma 4.25, we know that X is a T_1 space. Before proving X is strongly core-coherent, we give a property on Y.

Claim: *If $U \ll V$ in $\mathcal{O}(Y)$ and $0 \notin V$, then U is a finite set.*

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This is trivial since each single set $\{\frac{1}{k}\}$, $k \geq 2$, is open in Y. Therefore, $\{\{x\} : x \in V\}$ is an open cover of V, so there exists a finite set $F \subseteq V$ such that $U \subseteq F$. Hence, U is finite.
To show X is strongly core-coherent, assume $U_1, U_2, V \in \mathcal{O}(X)$ such that $U_1 \ll U_2$. We need to prove that $U_1 \cap V \ll U_2 \cap V$, and we will prove this by using Lemma 4.26. Since every set in $(\mathbb{N})_{cof}$ is compact, we have that $U_1 \cap V \cap \mathbb{N} \ll U_2 \cap V \cap \mathbb{N}$. It remains to prove that $U_1 \cap V \cap Y \ll U_2 \cap V \cap Y$. For this purpose, suppose $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ is an open cover of $U_2 \cap V \cap Y$. There are two cases:
(c1) $0 \in U_2 \cap V \cap Y$. Then, there exists i_0 such that $0 \in U_{i_0}$. Observe that the complement set $Y \setminus U$ of every open neighborhood U of 0 in Y is finite, so $F := (U_1 \cap V \cap Y) \setminus U_{i_0} \subseteq Y \setminus U_{i_0}$ is finite. Note that $F \subseteq \bigcup_{i \in I\setminus\{i_0\}} U_i$, there exists a finite subset $J \subseteq I \setminus \{i_0\}$ such that $F \subseteq \bigcup_{j\in J} U_j$. Then, we have that $U_1 \cap V \cap Y \subseteq F \cup U_{i_0} \subseteq \bigcup_{j \in J\cup\{i_0\}} U_j$.
(c2) $0 \notin U_2 \cap V \cap Y$. Since $U_1 \ll U_2$, by Lemma 4.26, $U_1 \cap Y \ll U_2 \cap Y$ in $\mathcal{O}(Y)$. From the claim it follows that $U_1 \cap Y$ is a finite set, so is $U_1 \cap V \cap Y$. Thus, there exists a finite subset $J \subseteq I \setminus \{i_0\}$ such that $U_1 \cap V \cap Y \subseteq \bigcup_{j\in J} U_j$.
From (c1) and (c2), we deduce that $U_1 \cap V \cap Y \ll U_2 \cap V \cap Y$ in $\mathcal{O}(Y)$. Thus, by Lemma 4.26, $U_1 \cap V \ll U_2 \cap V$ in $\mathcal{O}(X)$. Therefore, X is strongly core-coherent.

(2) *Σ**D(X)** is not core-coherent, and hence is not strongly core-coherent.*
It is trivial by Theorem 4.13 and that $X_{\text{iso}} := Y \setminus \{0\}$ is infinite.
(3) *Σ**D(X)** is a non-routine open well-filtered space.*
*— Note that $X_{\text{iso}} := Y \setminus \{0\} \neq \emptyset$. Thus, by Theorem 4.19, Σ**D(X)** is a non-routine open well-filtered space.*
(4) *Σ**D(X)** is not well-filtered.*
— Let $A_k = \{n \in \mathbb{N} : n \geq k\}$ for each $k \in \mathbb{N}$. Then, the family $\{A_k : k \in \mathbb{N}\}$ is a filtered family of compact (automatically saturated) sets in $(\mathbb{N})_{cof}$ and, by Lemma 4.27, in X as well. Since $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$ and $A_k \neq \emptyset$ for any $k \in \mathbb{N}$, it follows that X is not well-filtered. By Theorem 3 in Xi and Zhao (2017) *(saying that a T_1 space is well-filtered iff its Xi-Zhao model is well-filtered), we conclude that Σ**D(X)** is not well-filtered.*

We can deduce the following two main corollaries from the example above.

Corollary 4.29. *There exists a non-routine open well-filtered, but not well-filtered dcpo.*

Corollary 4.30. *Neither core-coherence nor strong core-coherence is preserved by the Xi-Zhao model.*

Next, we show that for a T_1 space X, the properties of $|X_{\text{iso}}| < \omega$ and being (strongly) core-coherent are independent, as illustrated by the following example.

Example 4.31.

(1) *$|X_{\text{iso}}| < \omega \not\Rightarrow X$ is strongly core-coherent.*

— Consider the set $[0, 1]$ with the usual topology, that is, the base open sets are of the forms $[0, r)$ and $(s, 1]$ for $s, t \in [0, 1]$. It is clear that $[0, 1]_{\text{iso}} = \emptyset$, so $|[0, 1]_{\text{iso}}| < \omega$. However, it is not strongly core-coherent, as shown in Example 3.7.

(2) *X is strongly core-coherent $\not\Rightarrow |X_{\text{iso}}| < \omega$.*

— Consider the space X in Example 4.28. It has been proved that X is strongly core-coherent. Additionally, we have $|X_{\text{iso}}| = |Y_{\text{iso}}| + |(\mathbb{N}_{cof})_{\text{iso}}| = \omega + 0 = \omega \not< \omega$.

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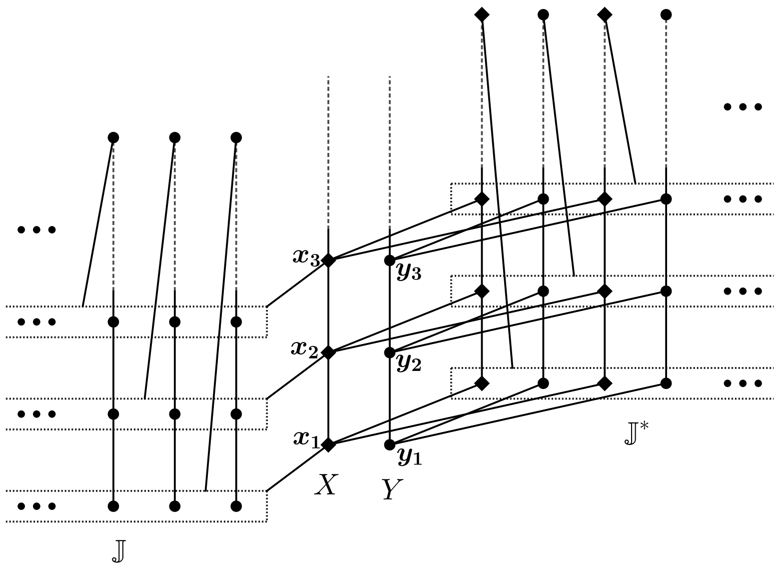

Figure 4. The poset P in Example 4.32.

At the end of the paper, we present a counterexample demonstrating that wb-sober spaces need not be open well-filtered, which also serves as part of the proof of Theorem 3.11.

Example 4.32. *Let* $X = \{x_n : n \in \mathbb{N}\}$ *and* $Y = \{y_n : n \in \mathbb{N}\}$ *be two chains with the following orders:*

$$x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} < \cdots$$

and

$$y_1 < y_2 < y_3 < \cdots < y_n < y_{n+1} < \cdots .$$

Let $\mathbb{J}^* = \mathbb{N} \times (\mathbb{N} \cup \{\infty\}) \times \{*\}$ *be the dcpo that is order-isomorphic to the Johnstone's dcpo* $\mathbb{J}$ *(see Example 3.15) via the mapping* $(m, n, *) \mapsto (m, n)$, $\forall (m, n, *) \in \mathbb{J}^*$.

Let $P = \mathbb{J} \cup X \cup Y \cup \mathbb{J}^*$. *We define a partial order* $\leq$ *on* P *as follows:*

(i) *all* $\mathbb{J}$, X, Y *and* $\mathbb{J}^*$ *are subposets of* P;
(ii) $\forall n \in \mathbb{N}$, $(k, n) \leq x_n \leq (2m - 1, n, *)$ *for all* $k, m \in \mathbb{N}$;
(iii) $\forall n \in \mathbb{N}$, $y_n \leq (2m, n, *)$ *for all* $m \in \mathbb{N}$.

The following conditions are clear by (ii):

(iv) $\forall n \in \mathbb{N}$, $(k, n) \leq (2m - 1, n, *)$ *for all* $k, m \in \mathbb{N}$;
(v) $(k, \infty) = \bigvee_{n \in \mathbb{N}} (k, n) \leq \bigvee_{n \in \mathbb{N}} (2m - 1, n, *) = (2m - 1, \infty, *)$ *for all* $k, m \in \mathbb{N}$.

The poset P *is illustrated in Figure 4. In particular, the order on the n-level of* P *is illustrated in Figure 5.*

(1) ΣP *is not open well-filtered.*
— *For each* $n \in \mathbb{N}$, *it is easy to verify that* $\uparrow \{x_n, y_n\}$ *is a Scott open subset of* P. *Moreover,* $\uparrow \{x_{n+1}, y_{n+1}\} \ll \uparrow \{x_n, y_n\}$ *in* $(\sigma(P), \subseteq)$, *as these are compact open sets. Thus,* $\mathcal{F} := \{\uparrow \{x_n, y_n\} : n \in \mathbb{N}\}$ *is a* $\ll$ *-filtered subfamily of* $\sigma(P)$. *Define* $U := \bigcup_{n \in \mathbb{N}} \uparrow (n, n, *)$, *which is a Scott open set in* $\Sigma \mathbb{J}^*$. *Note that* $\mathbb{J}^*$ *is a Scott open subset of* P, *and then the Scott topology on* $\mathbb{J}^*$ *agrees*

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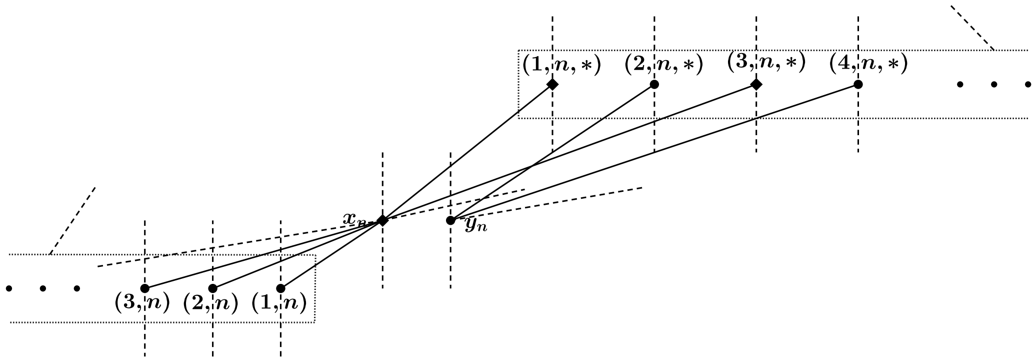

Figure 5. The order on the n-level of P in Example 4.32.

with the relative Scott topology from P (Gierz et al. *2003, Exercise II-1.26.). Thus, U is a Scott open subset of P. We have that $\bigcap \mathcal{F} = \{(n, \infty, *) : n \in \mathbb{N}\} \subseteq U$, while $\uparrow\{x_n, y_n\} \nsubseteq U$ as $(n+1, n, *) \in \uparrow\{x_n, y_n\} \setminus U$ for each $n \in \mathbb{N}$. Therefore, ΣP is not open well-filtered.*

(2) *ΣP is wb-sober.*
We prove this in some steps.

***Step 1:** We show some basic results on P.*

(i) $\forall U \in \sigma(P)$, $U \neq \emptyset$ iff $U \cap \mathbb{J}^* \neq \emptyset$.
— Note that $P = \downarrow\mathbb{J}^*$. Thus, $U \neq \emptyset$ iff $U \cap \downarrow\mathbb{J}^* \neq \emptyset$, which is equivalent to $U \cap \mathbb{J}^* \neq \emptyset$ since $U = \uparrow U$.

(ii) $Y \cup \mathbb{J}^*$ is a Scott open subset of P.
— Note that the complement $P \setminus (Y \cup \mathbb{J}^*) = X \cup \mathbb{J}$ is Scot closed, since it is clear a lower set and closed for the suprema of directed sets.

(iii) Suppose $C_1 = \mathbb{J} \cup X \cup \downarrow\{(2m_i - 1, n_i, *) : i \in I\}$ and $C_2 = \downarrow\{(2m_i, n_i, *) : i \in I\}$, where all m_i, n_i are elements of $\mathbb{N}$ and $m_j \neq m_k$ whenever $j \neq k$ for $j, k \in I$. Then, both C_1 and C_2 are Scott closed subsets of P.
— It is clear.

***Step 2:** $U \ll P \setminus Y$ in $(\sigma(P), \subseteq)$ iff $U = \emptyset$.*
— First, it is clear that Y is Scott closed, and thus $P \setminus Y \in \sigma(P)$. In addition, $\emptyset \ll P \setminus Y$ is trivial. Now assume $U \in \sigma(P)$ and $U \neq \emptyset$. We will prove that $U \not\ll P \setminus Y$ by using Remark 2.5. Since $U \neq \emptyset$, by Step 1 (i), $U \cap \mathbb{J}^* \neq \emptyset$. Note that $\mathbb{J}^*$ is a Scott open subset of P, and thus the Scott topology on $\mathbb{J}^*$ agrees with the relative Scott topology from P (Gierz et al. *2003, Exercise II-1.26.). Then, $U \cap \mathbb{J}^* \in \sigma(\mathbb{J}^*)$, which is of form $\bigcup_{k \in \mathbb{N}\setminus F} \uparrow(k, \phi(k), *)$, where F is a finite subset of $\mathbb{N}$ and $\phi(k) \in \mathbb{N}$ for each $k \in \mathbb{N} \setminus F$. Let n_0 be the greatest number in F. For each $n > n_0$, define $C_n := \downarrow\{(2k, \phi(2k), *) : k \geq n\}$. Then, by Step 1 (iii), $\{C_n : n \in \mathbb{N}\}$ is a filtered family of Scott closed subset of P. It holds that $(P \setminus Y) \cap \bigcap_{n > n_0} C_n \subseteq (P \setminus Y) \cap Y = \emptyset$ and $(2n, \phi(2n), *) \in U \cap C_n \neq \emptyset$ for any $n > n_0$. This shows that $U \not\ll P \setminus Y$, completing the proof.*

***Step 3:** $U \ll Y \cup \mathbb{J}^*$ in $(\sigma(P), \subseteq)$ iff $U = \emptyset$.*
— By Step 1 (ii), $Y \cup \mathbb{J}^* \in \sigma(P)$, and clearly $\emptyset \ll Y \cup \mathbb{J}^*$. Now assume $\emptyset \neq U \in \sigma(P)$. We will prove that $U \not\ll Y \cup \mathbb{J}^*$ by using Remark 2.5. First, using the same argument of Step 2, $U \cap \mathbb{J}^* \in \sigma(\mathbb{J}^*)$, which is of form $\bigcup_{k \in \mathbb{N}\setminus F} \uparrow(k, \phi(k), *)$, where F is a finite subset of $\mathbb{N}$ and $\phi(k) \in \mathbb{N}$ for any $k \in \mathbb{N} \setminus F$. Let n_0 be the greatest number in F. For each $n > n_0$, define $C_n := \mathbb{J} \cup X \cup \downarrow\{(2k-1, \phi(2k-1), *) : k \geq n\}$. Then, by Step 1 (iii), $\{C_n : n \in \mathbb{N}\}$ is a filtered family of Scott closed subset of P. It holds that $(Y \cup \mathbb{J}^*) \cap \bigcap_{n > n_0} C_n = (Y \cup \mathbb{J}^*) \cap \mathbb{J} \cup X = \emptyset$, but

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$(2n-1, \phi(2n-1), *) \in U \cap C_n \neq \emptyset$ for each $n > n_0$. This shows that $U \not\ll Y \cup \mathbb{J}^*$, completing the proof.

Step 4: *ΣP is wb-sober.*
— In fact, there are no wb-irreducible sets in ΣP; consequently, ΣP is wb-sober. Now assume on the contrary that A is a wb-irreducible subset of P.

Claim: *$A \cap (P \setminus Y) = \emptyset$.*

— Assume on the contrary that $A \cap (P \setminus Y) \neq \emptyset$. Since $P \setminus Y \in \sigma(P)$ and A is wb-irreducible, there exists $U \in \sigma(P)$ such that $U \ll (P \setminus Y)$ and $A \cap U \neq \emptyset$. However, by Step 2, $U = \emptyset$, contradicting that $A \cap U \neq \emptyset$. Therefore, $A \cap (P \setminus Y) = \emptyset$.
Using the same argument and Step 3, we obtain that $A \cap (Y \cup \mathbb{J}^*) = \emptyset$, which implies that $A \cap Y = \emptyset$. From the claim, we can deduce that $A = \emptyset$, which leads to a contradiction since wb-irreducible sets are nonempty.

5. Conclusion

(1) This paper delves further into the Jia-Jung problem and the Xi-Zhao model. The latest findings related to the Jia-Jung problem show that every core-compact open well-filtered space is sober, suggesting a connection between core-compactness (and open well-filteredness) and sobriety. Proposition 3.3 in this paper uncovers the link between core-compactness and sobriety, demonstrating that core-compact spaces are precisely those that transform irreducible sets into wb-irreducible sets. Inspired by this result, we introduce the notion of wb-sober spaces and establish their relationship with sober spaces. Specifically, we prove that every open well-filtered space is wb-sober, but not vice versa. Moreover, we show that every core-compact wb-sober space is sober, generalizing Theorem 4.7 of Shen et al. (2020). On the other hand, we provide sufficient and necessary conditions for the Xi-Zhao model to be core-coherent and routine. Furthermore, using the Xi-Zhao model, we propose a general approach to constructing a non-routine open well-filtered but not well-filtered dcpo.

For a T_1 space X, we show a summary of properties for the Xi-Zhao model **D(X)** as follows, where a "✓" indicates that the properties are preserved, and "×" indicates negative cases.

X	**D(X)**	by
sobriety	✓	Zhao and Xi (2018)
well-filteredness	✓	Xi and Zhao (2017)
Choquet completeness	✓	He et al. (2019)
Rudin, weak sobriety	✓	Chen and Li (2022)
core-compactness	×	Chen and Li (2022)
$\|X_{os}\| < \omega_0$	(strong) core-coherent	Theorem 4.14
	open well-filteredness	Chen and Li (2023)
	weak domain	Shen et al. (2019)
	locally quasi-algebraic	Zhao and Xi (2018)

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(2) Given that Example 4.32 presented is a poset rather than a dcpo, and considering that every first-countable well-filtered space (which is automatically a d-space) is sober, we pose the following interesting questions for future research:

Question 5.1.

(1) Is every wb-sober dcpo equipped with the Scott topology open well-filtered?
(2) Under what conditions (other than strong core-coherence and core-compactness) is a wb-sober space open well-filtered?
(3) Is every first-countable wb-sober (or open well-filtered) d-space sober?

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