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DOUBLE Γ-CONVERGENCE AND APPLICATION TO ENERGY FUNCTIONALS

MAO-SHENG CHANG

Department of Mathematics, Fu Jen Catholic University, 510 Zhongzheng Road, Xinzhuang District, New Taipei City 24205, Taiwan (mschang@math.fju.edu.tw)

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Abstract We introduce a 'double' version of Γ-convergence, which we have named 'double Γ-convergence', and apply it to obtain the Γ-limit of double-perturbed energy functionals as $p \to 1$ and $p \to +\infty$, respectively. The limit of (p, q)-type capacity as $p \to 1$ and $p \to +\infty$, respectively, is also obtained in this manner.

Keywords: Gamma-convergence; functions of bounded variations; capacity

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1. Introduction

The theory of Γ-convergence [**3**, **6**] was introduced by De Giorgi in the 1970s. One of its essential aspects is that convergence for integral functionals occurs, which assures us that minimizers converge to a minimizer of the Γ-limit functionals (see Proposition 1.1), which are stable under continuous perturbations (see Proposition 1.2). It has become a standard criterion for the study of variational problems.

We say that the functional E_0 is the $\Gamma(L^1(\Omega))$-limit of $\{E_\epsilon\}_{\epsilon>0}$ if, for each $u \in L^1(\Omega)$, the following hold.

(i) If $u_\epsilon \to u$ in $L^1(\Omega)$, then $E_0(u) \leqslant \liminf_{\epsilon\to 0^+} E_\epsilon(u_\epsilon)$.

(ii) There exists a family $\{v_\epsilon\}_{\epsilon>0}$ in $L^1(\Omega)$ such that $v_\epsilon \to u$ in $L^1(\Omega)$ and $\limsup_{\epsilon\to 0^+} E_\epsilon(v_\epsilon) \leqslant E_0(u)$.

The following proposition asserts that Γ-convergence implies convergence of minimizers and minimum values.

Proposition 1.1. *Suppose that F_∞ and each F_n are functionals defined on the Banach space $L^1(\Omega)$ for $n = 1, 2, 3, \dots$. If*

(i) *$\{F_n\}_{n=1}^{\infty}$ Γ-converges in $L^1(\Omega)$ to F_∞,*

(ii) *for each $n \in \mathbb{N}$, u_n is a minimizer of F_n on $L^1(\Omega)$,*

(iii) *there exists a function $u_\infty \in L^1(\Omega)$ such that $u_n \to u_\infty$ in $L^1(\Omega)$ as $n \to \infty$,* *then we have that*

(a) $\lim_{n\to\infty} F_n(u_n) = F_\infty(u_\infty)$,

(b) u_∞ *is a minimizer of* F_∞ *on* $L^1(\Omega)$.

The next proposition states that Γ-convergence is stable under continuous perturbations.

Proposition 1.2. *Suppose that* G, F_∞ *and each* F_n *are functionals defined on the Banach space* $L^1(\Omega)$ *for* $n = 1, 2, 3, \dots$ *. If*

(i) $\{F_n\}_{n=1}^{\infty}$ Γ*-converges in* $L^1(\Omega)$ *to* F_∞,

(ii) G *is continuous on* $L^1(\Omega)$,

then we have that $\{F_n + G\}_{n=1}^{\infty}$ Γ*-converges in* $L^1(\Omega)$ *to* $F_\infty + G$.

For convenience, we refer the reader to the books [**3**, **6**, **10**] for the classical results formulated in Propositions 1.1 and 1.2.

Starting from the pioneering work by Modica and Mortola in [**12**], Modica in [**11**] and Sternberg in [**15**], many papers have been devoted to the study of the Γ-limit of the family of functionals $\{E_\varepsilon\}_{\varepsilon>0}$ with the form

$$E_\varepsilon(u) = \int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon|\nabla u|^2 \,\mathrm{d}x.$$

The nonlinear operator Δ_1 is given by

$$\Delta_1 u \equiv \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right),$$

known as the 1-Laplacian. Since the vector

$$\nu \equiv \frac{\nabla u}{|\nabla u|}$$

is orthogonal to each level set of u, we see that the 1-Laplacian equation [**7**]

$$\Delta_1 u = 0$$

describes 'isotropic diffusion within each level surface, with no diffusion across different level surfaces' by applying the divergence theorem. The 1-Laplacian operator Δ_1 is the variational operator for

$$E(u) \equiv \int_\Omega |\nabla u| \,\mathrm{d}x$$

and the formal limit of the p-Laplacian operator Δ_p as $p \to 1$, where

$$\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

This is related to mathematical topics including the minimal surface, the isoperimetric inequality, the elasticity, the image processing and the relaxation of bounded-variation (BV) functionals.

The infinity Laplacian operator Δ_∞ is given by

$$\Delta_\infty u \equiv \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}.$$

The infinity Laplacian equation

$$\Delta_\infty u = 0$$

was first derived by Aronsson *et al.* [**1**] as the Euler–Lagrange equation for the so-called absolute minimizer u of the L^∞ variational minimizing problem

$$I_\infty(v) \equiv \operatorname*{ess\,sup}_{\Omega} |Dv|$$

among suitable boundary conditions. Furthermore, it was derived as the limit as $p \to \infty$ of the p-Laplacian equation

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} \Delta u + (p-2)|\nabla u|^{p-4} \Delta_\infty u = 0.$$

Dividing $\Delta_p u$ by $(p-2)|\nabla u|^{p-2}$ and letting $p \to \infty$ leads to the partial differential equation

$$\frac{1}{|\nabla u|^2} \Delta_\infty u = 0.$$

For more relevant background and more properties, we refer the reader to [**2**,**5**,**9**,**13**,**14**].

Motivated by in-depth studies on the singular operators Δ_1 and Δ_∞, we introduced a double version of Γ-convergence, which we have named 'double Γ-convergence' and applied it to obtain the Γ-limit of double-perturbed energy

$$E_{\varepsilon,p}(u) = \int_\Omega \frac{1}{\varepsilon} \frac{W(u)}{q} + \varepsilon^{p-1} \frac{|\nabla u|^p}{p} \, \mathrm{d}x$$

as $(\varepsilon, p) \to (0, 1)$ and $(\varepsilon, p) \to (0, +\infty)$, respectively, where $p > 1$, $1/p + 1/q = 1$ and $W(u) = \frac{1}{4} u^2 (1-u)^2$.

For convenience, we define

$$E_{\varepsilon_1,\varepsilon_2}(u) \equiv \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{\varepsilon_2} \frac{W(u)}{P'_{\varepsilon_1}} + \varepsilon_2^{P_{\varepsilon_1}-1} \frac{|\nabla u|^{P_{\varepsilon_1}}}{P_{\varepsilon_1}} \right] \mathrm{d}x & \text{if } u \in W^{1,P_{\varepsilon_1}}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$E'_{\varepsilon_1,\varepsilon_2}(u) \equiv \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{\varepsilon_2} \frac{W(u)}{P_{\varepsilon_1}} + \varepsilon_2^{P'_{\varepsilon_1}-1} \frac{|\nabla u|^{P'_{\varepsilon_1}}}{P'_{\varepsilon_1}} \right] \mathrm{d}x & \text{if } u \in W^{1,P'_{\varepsilon_1}}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where P_{ε_1} and P'_{ε_1} are two real-valued functions of variable ε_1 satisfying the following:

(1°) $P_{\varepsilon_1} > 1$ and $P'_{\varepsilon_1} > 1$ for all $\varepsilon_1 > 0$,

(2°) $1/P_{\varepsilon_1} + 1/P'_{\varepsilon_1} = 1$ for each $\varepsilon_1 > 0$,

(3°) $P_{\varepsilon_1} \to 1$ and $P'_{\varepsilon_1} \to +\infty$ as $\varepsilon_1 \to 0$.

The function P'_{ε_1} is said to be the Lebesgue conjugate function of P_{ε_1}. For example, if $P_{\varepsilon_1} = 1 + \varepsilon_1$, then $P'_{\varepsilon_1} = 1 + 1/\varepsilon_1$; if $P_{\varepsilon_1} = \mathrm{e}^{\varepsilon_1}$, then $P'_{\varepsilon_1} = \mathrm{e}^{\varepsilon_1}/(\mathrm{e}^{\varepsilon_1} - 1)$. Here, ε_1 and ε_2 are positive parameters. Our main results are the following.

(i) $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E in $L^1(\Omega)$ provided that $\varepsilon_2 \cdot P'_{\varepsilon_1} \to 0$ as $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.

(ii) $\{E'_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to $\sigma \cdot E$ in $L^1(\Omega)$ provided that $\varepsilon_2 \cdot P_{\varepsilon_1} \to 0$ as $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.

Here, $\sigma \equiv \int_0^1 W(t)\,\mathrm{d}t$ and

$$E(u) \equiv \begin{cases} \displaystyle\int_\Omega |\nabla u|\,\mathrm{d}x & \text{if } W(u) = 0 \text{ almost everywhere (a.e.) and } u \in \mathrm{BV}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

This paper is aimed at continuing the work of [**4**] and studying the Γ-limit of double-perturbed energy functionals through the method of 'double Γ-convergence'. 'Double Γ-convergence' is a natural generalization of the notion of Γ-convergence. To the best of our knowledge, this is the first generalization of Γ-convergence theory in this field. We anticipate that the 'double' version of Γ-convergence can be applied to solve more important problems.

The paper has the following structure. In § 2, we introduce a 'double' version of Γ-convergence, which we have named 'double Γ-convergence'. In § 3, we prove that $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E in $L^1(\Omega)$. In § 4, we prove that $\{E'_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to σE in $L^1(\Omega)$. Finally, we present the asymptotic behaviour of (p, q)-type capacities in § 5.

2. Double Γ-convergence

Definition 2.1. Let (X, d) be a metric space endowed with a metric d. The sequence $\{x_{m,n}\}_{m=1,\,n=1}^{\infty,\,\infty}$ is said to converge to x in X if

$$\lim_{m\to\infty}\Big(\lim_{n\to\infty} x_{m,n}\Big) = x$$

in this order only. We denote it by $x_{m,n} \to x$ as $m \to \infty$ and $n \to \infty$, or by $x_{m,n} \to x$.

Definition 2.2. Let X be a metric space and let $E_{\varepsilon_1,\varepsilon_2} \colon X \mapsto [0, \infty]$ be a family of functionals. Assume that $E \colon X \mapsto [0, \infty]$. We say that $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E in X if the following statements hold for each $u \in X$.

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(i) (The lim inf inequality.) If $u_{\varepsilon_1,\varepsilon_2} \to u$ in X, then

$$E(u) \leqslant \liminf_{\varepsilon_1 \to 0} \Big(\liminf_{\varepsilon_2 \to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) \Big).$$

(ii) (The lim sup inequality.) There exists a double sequence $\{v_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ in X such that $v_{\varepsilon_1,\varepsilon_2} \to u$ in X and

$$\limsup_{\varepsilon_1 \to 0} \Big(\limsup_{\varepsilon_2 \to 0} E_{\varepsilon_1,\varepsilon_2}(v_{\varepsilon_1,\varepsilon_2}) \Big) \leqslant E(u).$$

In this case we define E as the double $\Gamma(X)$-limit of $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\varepsilon_2>0}$.

Remark 2.3. For each $x \in X$, define

$$S_x \equiv \Big\{ \liminf_{n\to\infty} \liminf_{m\to\infty} f_{n,m}(x_{n,m}) \;\Big|\; x_{n,m} \to x \text{ in } X \Big\},$$
$$T_x \equiv \Big\{ \limsup_{n\to\infty} \limsup_{m\to\infty} f_{n,m}(x_{n,m}) \;\Big|\; x_{n,m} \to x \text{ in } X \Big\}.$$

The following statements are then equivalent:

(i) $\{f_{n,m}\}_{n,m=1}^{\infty}$ double Γ-converges to f in X,

(ii) $\inf S_x = \inf T_x = f(x)$ for each $x \in X$.

Remark 2.4. If the double Γ-limit of $\{f_{n,m}\}_{n,m=1}^{\infty}$ exists, then it is unique.

Definition 2.5. Let $\{x_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ be a family of elements in X and let $x \in X$. We say that $x_{\varepsilon_1,\varepsilon_2} \to x$ in X as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ if, for the entire sequence $\{(\varepsilon_{1_m}, \varepsilon_{2_n})\}_{m=1,n=1}^{\infty}$ converging to $(0,0)$ (i.e. both $\varepsilon_{1_m} \to 0$ as $m \to \infty$ and $\varepsilon_{2_n} \to 0$ as $n \to \infty$), we have that $x_{\varepsilon_{1_m},\varepsilon_{2_n}} \to x$ as $m \to \infty$ and $n \to \infty$.

Theorem 2.6. *Suppose that E and each $E_{\varepsilon_1,\varepsilon_2}$ are functionals defined on a metric space X for each $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. If*

(1) *$\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E in X,*

(2) *$u_{\varepsilon_1,\varepsilon_2}$ is a minimizer of $E_{\varepsilon_1,\varepsilon_2}$ on X for each $\varepsilon_1 > 0$, $\varepsilon_2 > 0$,*

(3) *there exists a function $u \in X$ such that*

$$u_{\varepsilon_1,\varepsilon_2} \to u \text{ in } X \quad \text{as } \varepsilon_1 \to 0^+ \text{ and } \varepsilon_2 \to 0^+,$$

then we have that

(a) $\lim_{\varepsilon_1\to 0}(\liminf_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})) = \lim_{\varepsilon_1\to 0}(\limsup_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})) = E(u)$,

(b) *u is a minimizer of E on X.*

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Proof. This follows from the fact that $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E at u. Following the assumption (3) and the lim inf inequality, Definition 2.2 (i), this yields that

$$E(u) \leqslant \liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})\Big). \tag{2.1}$$

By the lim sup inequality, Definition 2.2 (ii), there exists a sequence $\{v_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ in X such that

$$v_{\varepsilon_1,\varepsilon_2} \to u \text{ in } X \quad \text{as } \varepsilon_1 \to 0,\ \varepsilon_2 \to 0,$$

and

$$\limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(v_{\varepsilon_1,\varepsilon_2})\Big) \leqslant E(u). \tag{2.2}$$

Let $\alpha_{\varepsilon_1} \equiv \liminf_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(v_{\varepsilon_1,\varepsilon_2})$ and $\beta_{\varepsilon_1} \equiv \limsup_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(v_{\varepsilon_1,\varepsilon_2})$ for each $\varepsilon_1 > 0$. Then, $\alpha_{\varepsilon_1} \leqslant \beta_{\varepsilon_1}$ for all $\varepsilon_1 > 0$, and

$$\limsup_{\varepsilon_1\to 0} \alpha_{\varepsilon_1} \leqslant \limsup_{\varepsilon_1\to 0} \beta_{\varepsilon_1} \leqslant E(u). \tag{2.3}$$

By assumption (2), $E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) \leqslant E_{\varepsilon_1,\varepsilon_2}(v_{\varepsilon_1,\varepsilon_2})$ for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$.

Let $\alpha^0_{\varepsilon_1} \equiv \liminf_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})$ and $\beta^0_{\varepsilon_1} \equiv \limsup_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})$ for all $\varepsilon_1 > 0$. We then have that $\alpha^0_{\varepsilon_1} \leqslant \alpha_{\varepsilon_1}$, $\beta^0_{\varepsilon_1} \leqslant \beta_{\varepsilon_1}$ and $\alpha^0_{\varepsilon_1} \leqslant \beta^0_{\varepsilon_1}$ for all $\varepsilon_1 > 0$. Following (2.1), this yields that

$$\begin{aligned}
E(u) &\leqslant \liminf_{\varepsilon_1\to 0} \alpha^0_{\varepsilon_1} \leqslant \limsup_{\varepsilon_1\to 0} \alpha^0_{\varepsilon_1} \leqslant \limsup_{\varepsilon_1\to 0} \alpha_{\varepsilon_1} \leqslant \limsup_{\varepsilon_1\to 0} \beta_{\varepsilon_1} \leqslant E(u),\\
E(u) &\leqslant \liminf_{\varepsilon_1\to 0} \alpha^0_{\varepsilon_1} \leqslant \liminf_{\varepsilon_1\to 0} \beta^0_{\varepsilon_1} \leqslant \limsup_{\varepsilon_1\to 0} \beta^0_{\varepsilon_1} \leqslant \limsup_{\varepsilon_1\to 0} \beta_{\varepsilon_1} \leqslant E(u).
\end{aligned}$$

Therefore, we obtain that

$$\left.\begin{aligned}
\lim_{\varepsilon_1\to 0} \alpha^0_{\varepsilon_1} &= E(u),\\
\lim_{\varepsilon_1\to 0} \beta^0_{\varepsilon_1} &= E(u).
\end{aligned}\right\} \tag{2.4}$$

Next, we claim that u is a minimizer of E on X, given any $v \in X$. Since $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E at v, there exists a sequence $\{\omega_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ in X such that

$$\omega_{\varepsilon_1,\varepsilon_2} \to v \text{ in } X \quad \text{as } \varepsilon_1 \to 0,\ \varepsilon_2 \to 0,$$

and

$$\limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(\omega_{\varepsilon_1,\varepsilon_2})\Big) \leqslant E(v). \tag{2.5}$$

By assumption (2),

$$E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) \leqslant E_{\varepsilon_1,\varepsilon_2}(\omega_{\varepsilon_1,\varepsilon_2}) \quad \text{for all } \varepsilon_1 > 0 \text{ and } \varepsilon_2 > 0.$$

Thus, we have that

$$\limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})\Big) \leqslant \limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(\omega_{\varepsilon_1,\varepsilon_2})\Big). \tag{2.6}$$

Combining (2.4), (2.5) and (2.6) yields that $E(u) \leqslant E(v)$. □

Theorem 2.7. *Suppose that* $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ *double* Γ*-converges to* E *in* X*, and that* $F\colon X\to\bar{\mathbb{R}}$ *is continuous on* X*. Then,* $\{E_{\varepsilon_1,\varepsilon_2}+F\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ *double* Γ*-converges to* $(E+F)$ *in* X*.*

Proof. Let $u\in X$. Following the assumption that $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E at u and the lim inf inequality (i), given any $\{u_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ in X with $u_{\varepsilon_1,\varepsilon_2}\to u$ in X, we have that

$$E(u)\leqslant \liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})\Big). \tag{2.7}$$

By the double continuity of F at u,

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0} F(u_{\varepsilon_1,\varepsilon_2})=F(u). \tag{2.8}$$

Let $\gamma_{\varepsilon_1}\equiv \lim_{\varepsilon_2\to 0}F(u_{\varepsilon_1,\varepsilon_2})$ for all $\varepsilon_1>0$. Then (2.8) means that $\lim_{\varepsilon_1\to 0}\gamma_{\varepsilon_1}=F(u)$. Moreover,

$$\liminf_{\varepsilon_1\to 0}\gamma_{\varepsilon_1}=\limsup_{\varepsilon_1\to 0}\gamma_{\varepsilon_1}=F(u). \tag{2.9}$$

Note that

$$\liminf_{\varepsilon_2\to 0}F(u_{\varepsilon_1,\varepsilon_2})=\limsup_{\varepsilon_2\to 0}F(u_{\varepsilon_1,\varepsilon_2})=\gamma_{\varepsilon_1}. \tag{2.10}$$

Combining (2.8) with (2.9), we have that

$$\liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0}F(u_{\varepsilon_1,\varepsilon_2})\Big)=F(u) \tag{2.11}$$

and

$$\limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0}F(u_{\varepsilon_1,\varepsilon_2})\Big)=F(u). \tag{2.12}$$

Thus,

$$\begin{aligned}(E+F)(u)&=E(u)+F(u)\\ &\leqslant \liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0}E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})\Big)\\ &\qquad +\liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0}F(u_{\varepsilon_1,\varepsilon_2})\Big)\quad \text{(by (2.7) and (2.11))}\\ &\leqslant \liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0}E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})+\liminf_{\varepsilon_2\to 0}F(u_{\varepsilon_1,\varepsilon_2})\Big)\\ &\leqslant \liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0}(E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})+F(u_{\varepsilon_1,\varepsilon_2}))\Big)\\ &=\liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0}(E_{\varepsilon_1,\varepsilon_2}+F)(u_{\varepsilon_1,\varepsilon_2})\Big).\end{aligned} \tag{2.13}$$

By the lim sup inequality, Definition 2.2 (ii), there exists $\{\omega_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ in X such that $\omega_{\varepsilon_1,\varepsilon_2}\to u$ in X as $\varepsilon_1\to 0$, $\varepsilon_2\to 0$, and

$$\limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0}E_{\varepsilon_1,\varepsilon_2}(\omega_{\varepsilon_1,\varepsilon_2})\Big)\leqslant E(u). \tag{2.14}$$

By (2.14) and (2.12), we have that

$$\begin{aligned}
\limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0}(E_{\varepsilon_1,\varepsilon_2}+F)(\omega_{\varepsilon_1,\varepsilon_2})\Big)
&= \limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0}(E_{\varepsilon_1,\varepsilon_2}(\omega_{\varepsilon_1,\varepsilon_2})+F(\omega_{\varepsilon_1,\varepsilon_2}))\Big)\\
&\leqslant \limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0}E_{\varepsilon_1,\varepsilon_2}(\omega_{\varepsilon_1,\varepsilon_2})+\limsup_{\varepsilon_2\to 0}F(\omega_{\varepsilon_1,\varepsilon_2})\Big)\\
&\leqslant \limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0}E_{\varepsilon_1,\varepsilon_2}(\omega_{\varepsilon_1,\varepsilon_2})\Big)+\limsup_{\varepsilon_1\to 0}\Big(\limsup_{\varepsilon_2\to 0}F(\omega_{\varepsilon_1,\varepsilon_2})\Big)\\
&\leqslant E(u)+F(u)\\
&= (E+F)(u).
\end{aligned}$$

□

Remark 2.8. Supposing that the family of functionals $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ is independent of the parameter ε_2, Proposition 1.1 is regarded as a special case of Theorem 2.6, and Proposition 1.2 is regarded as a special case of Theorem 2.7.

Remark 2.9. Multiple Γ-convergence structures can be easily established by our method.

3. The limit $P_{\varepsilon_1}\to 1$

We consider energy functionals of the form

$$E_{\varepsilon_1,\varepsilon_2}(u)\equiv\begin{cases}\displaystyle\int_\Omega\left[\frac{1}{\varepsilon_2}\frac{W(u(x))}{P'_{\varepsilon_1}}+\varepsilon_2^{P_{\varepsilon_1}-1}\frac{|\nabla u(x)|^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}\right]\mathrm{d}x & \text{if } u\in W^{1,p_{\varepsilon_1}}(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

where $W(t)=\frac{1}{4}t^2(1-t)^2$, and define the functional E as

$$E(u)\equiv\begin{cases}\displaystyle\int_\Omega|\nabla u|\,\mathrm{d}x & \text{if } W(u)=0 \text{ a.e. and } u\in \mathrm{BV}(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

where $\int_\Omega|\nabla u|\,\mathrm{d}x$ denotes the total variation of u on Ω.

Theorem 3.1. *Assume that $\varepsilon_2\cdot P'_{\varepsilon_1}\to 0$ as $(\varepsilon_1,\varepsilon_2)\to(0,0)$. Then, $\{E_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E in $L^1(\Omega)$.*

Proof. The Euler–Lagrange equation of $E_{\varepsilon_1,\varepsilon_2}$ is

$$\frac{W'(u)}{P'_{\varepsilon_1}}-\varepsilon_2^{P_{\varepsilon_1}}\operatorname{div}(|\nabla u|^{P_{\varepsilon_1}-2}\nabla u)=0. \tag{3.1}$$

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For $n = 1$, (3.1) becomes

$$\frac{W'(u(t))}{P'_{\varepsilon_1}} - \varepsilon_2^{P_{\varepsilon_1}}(P_{\varepsilon_1} - 1)|u'(t)|^{P_{\varepsilon_1}-2}u''(t) = 0. \tag{3.2}$$

Intending to find a solution u of (3.2) with $u' > 0$ on some interval I, we define f_{ε_1} and F_{ε_1} on $[0, \infty)$ by

$$f_{\varepsilon_1}(t) \equiv t^{P_{\varepsilon_1}-1} \quad \text{and} \quad F_{\varepsilon_1}(t) \equiv \int_0^t f_{\varepsilon_1}(s)\,\mathrm{d}s = \frac{t^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}. \tag{3.3}$$

Multiplying (3.2) by u', we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(W \circ u)(t)}{P'_{\varepsilon_1}} - \varepsilon_2^{P_{\varepsilon_1}}(P_{\varepsilon_1} - 1)(F_{\varepsilon_1} \circ u')(t)\right) = 0.$$

This implies the existence of a positive constant $C_{\varepsilon_1,\varepsilon_2}$ that depends on ε_1 and ε_2 and will be determined later, such that

$$\frac{W \circ u}{P'_{\varepsilon_1}} - \varepsilon_2^{P_{\varepsilon_1}}(P_{\varepsilon_1} - 1)F_{\varepsilon_1} \circ u' = -C_{\varepsilon_1,\varepsilon_2}. \tag{3.4}$$

Since $F_{\varepsilon_1}^{-1}(t) = (P_{\varepsilon_1}t)^{1/P_{\varepsilon_1}}$, (3.4) can be expressed as

$$u'(t) = \frac{1}{\varepsilon_2}(W(u(t)) + P'_{\varepsilon_1}C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}. \tag{3.5}$$

We define $\Psi_{\varepsilon_1,\varepsilon_2} \colon [0, 1] \to \mathbb{R}$ by

$$\Psi_{\varepsilon_1,\varepsilon_2}(t) \equiv \int_0^t \frac{\varepsilon_2}{(W(s) + P'_{\varepsilon_1}C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}}\,\mathrm{d}s,$$

and let $\eta_{\varepsilon_1,\varepsilon_2} \equiv \Psi_{\varepsilon_1,\varepsilon_2}(1)$. Then, $\Psi_{\varepsilon_1,\varepsilon_2}(0) = 0$,

$$\Psi'_{\varepsilon_1,\varepsilon_2}(t) = \varepsilon_2 \cdot \frac{1}{(W(t) + P'_{\varepsilon_1}C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}} > 0 \quad \text{for all } t \in (0, 1)$$

and

$$0 < \eta_{\varepsilon_1,\varepsilon_2} \leqslant \frac{\varepsilon_2}{(P'_{\varepsilon_1}C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}}. \tag{3.6}$$

Clearly, the inverse function $\Psi_{\varepsilon_1,\varepsilon_2}^{-1} \colon [0, \eta_{\varepsilon_1,\varepsilon_2}] \to [0, 1]$ exists and $(\Psi_{\varepsilon_1,\varepsilon_2}^{-1})'$ satisfies

$$(\Psi_{\varepsilon_1,\varepsilon_2}^{-1})'(t) = \frac{1}{\varepsilon_2}(W(\Psi_{\varepsilon_1,\varepsilon_2}^{-1}(t)) + P'_{\varepsilon_1}C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}} \tag{3.7}$$

for all $t \in (0, \eta_{\varepsilon_1,\varepsilon_2})$. Thus, the function $\Psi_{\varepsilon_1,\varepsilon_2}^{-1}$ is one solution of (3.2) with $u' > 0$ on $(0, \eta_{\varepsilon_1,\varepsilon_2})$. Let $\tilde{\Psi}_{\varepsilon_1,\varepsilon_2}^{-1}$ denote the extension function of $\Psi_{\varepsilon_1,\varepsilon_2}^{-1}$, as

$$\tilde{\Psi}_{\varepsilon_1,\varepsilon_2}^{-1}(t) \equiv \begin{cases} 0, & t < 0, \\ \Psi_{\varepsilon_1,\varepsilon_2}^{-1}(t), & t \in [0, \eta_{\varepsilon_1,\varepsilon_2}], \\ 1, & t > \eta_{\varepsilon_1,\varepsilon_2}. \end{cases} \tag{3.8}$$

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Obviously,

$$\tilde{\Psi}^{-1}_{\varepsilon_1,\varepsilon_2}(t) \leqslant \chi_0(t) \leqslant \tilde{\Psi}^{-1}_{\varepsilon_1,\varepsilon_2}(t+\eta_{\varepsilon_1,\varepsilon_2}) \tag{3.9}$$

for all $t \in \mathbb{R}$, where χ_0 is the Heaviside function.

Suppose that $u_{\varepsilon_1,\varepsilon_2} \to u$ in $L^1(\Omega)$, with

$$\liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})\Big) < +\infty. \tag{3.10}$$

By the continuity of W at u and by applying Fatou's lemma twice, we have that

$$\begin{aligned}
0 &\leqslant \int_\Omega W(u)\,\mathrm{d}x \\
&= \int_\Omega \liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0} W(u_{\varepsilon_1,\varepsilon_2})\Big)\,\mathrm{d}x \\
&\leqslant \liminf_{\varepsilon_1\to 0}\int_\Omega \liminf_{\varepsilon_2\to 0} W(u_{\varepsilon_1,\varepsilon_2})\,\mathrm{d}x \\
&\leqslant \liminf_{\varepsilon_1\to 0}\Big(\liminf_{\varepsilon_2\to 0}\int_\Omega W(u_{\varepsilon_1,\varepsilon_2})\,\mathrm{d}x\Big) \\
&\leqslant \liminf_{\varepsilon_1\to 0}\liminf_{\varepsilon_2\to 0}(\varepsilon_2 P'_{\varepsilon_1} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})) \\
&= 0 \quad (\text{by (3.10) and } \varepsilon_2 P'_{\varepsilon_1} \to 0 \text{ as } \varepsilon_1 \to 0 \text{ and } \varepsilon_2 \to 0).
\end{aligned}$$

Thus, we obtain that

$$W(u) = 0 \quad \text{a.e. in } \Omega.$$

So, we have $u \in \{0,1\}$ a.e. in Ω. Let $A \equiv \{x \in \Omega \mid u(x) = 1\}$. Then, $u = \chi_A$ a.e. in Ω. Define

$$h(x) \equiv \begin{cases} -\operatorname{dist}(x,\partial A) & \text{if } x \notin A, \\ \operatorname{dist}(x,\partial A) & \text{if } x \in A. \end{cases}$$

Thus, $u(x) = \chi_A(x) = \chi_0(h(x))$ for all $x \in \Omega$. Following (3.9) we get that

$$\int_\Omega \tilde{\Psi}^{-1}_{\varepsilon_1,\varepsilon_2}(h(x))\,\mathrm{d}x \leqslant \int_\Omega u(x)\,\mathrm{d}x \leqslant \int_\Omega \tilde{\Psi}^{-1}_{\varepsilon_1,\varepsilon_2}(h(x)+\eta_{\varepsilon_1,\varepsilon_2})\,\mathrm{d}x. \tag{3.11}$$

Define $H_{\varepsilon_1,\varepsilon_2}(t) \equiv \int_\Omega \tilde{\Psi}^{-1}_{\varepsilon_1,\varepsilon_2}(h(x)+t)\,\mathrm{d}x$ for all $t \in [0,\eta_{\varepsilon_1,\varepsilon_2}]$. By the intermediate value theorem, with $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists $\delta_{\varepsilon_1,\varepsilon_2} \in [0,\eta_{\varepsilon_1,\varepsilon_2}]$ such that

$$H_{\varepsilon_1,\varepsilon_2}(\delta_{\varepsilon_1,\varepsilon_2}) = \int_\Omega u(x)\,\mathrm{d}x. \tag{3.12}$$

Define $u_{\varepsilon_1,\varepsilon_2}\colon \Omega \to \mathbb{R}$ by

$$u_{\varepsilon_1,\varepsilon_2}(x) \equiv \tilde{\Psi}^{-1}_{\varepsilon_1,\varepsilon_2}(h(x)+\delta_{\varepsilon_1,\varepsilon_2}) \tag{3.13}$$

for each $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. We then have (by (3.12)) that

$$\int_\Omega u_{\varepsilon_1,\varepsilon_2}(x)\,\mathrm{d}x = \int_\Omega u(x)\,\mathrm{d}x. \tag{3.14}$$

We define a function $\chi_{\varepsilon_1,\varepsilon_2}(t) \equiv \tilde{\Psi}^{-1}_{\varepsilon_1,\varepsilon_2}(t+\delta_{\varepsilon_1,\varepsilon_2})$ for all $t \in \mathbb{R}$. Then,

$$u_{\varepsilon_1,\varepsilon_2} = \chi_{\varepsilon_1,\varepsilon_2} \circ h \quad \text{on } \Omega. \tag{3.15}$$

Define

$$\Omega_{\delta_{\varepsilon_1,\varepsilon_2}} \equiv \{x \in \Omega \mid -\delta_{\varepsilon_1,\varepsilon_2} \leqslant h(x) \leqslant \eta_{\varepsilon_1,\varepsilon_2} - \delta_{\varepsilon_1,\varepsilon_2}\}.$$

Then, by the Coarea formula, we have that

$$\begin{aligned}
\int_\Omega |u_{\varepsilon_1,\varepsilon_2} - u|\,\mathrm{d}x &= \int_\Omega |\chi_{\varepsilon_1,\varepsilon_2} \circ h - \chi_0 \circ h|\,\mathrm{d}x \\
&= \int_{\Omega_{\delta_{\varepsilon_1,\varepsilon_2}}} |\chi_{\varepsilon_1,\varepsilon_2} \circ h - \chi_0 \circ h|\,\mathrm{d}x \\
&= \int_{\Omega_{\delta_{\varepsilon_1,\varepsilon_2}}} |\chi_{\varepsilon_1,\varepsilon_2} \circ h - \chi_0 \circ h|\,|\nabla h|\,\mathrm{d}x \\
&\qquad (|\nabla h| = 1 \text{ a.e. on } \Omega_{\delta_{\varepsilon_1,\varepsilon_2}} \text{ provided that } \delta_{\varepsilon_1,\varepsilon_2} \text{ is small enough}) \\
&= \int_{-\delta_{\varepsilon_1,\varepsilon_2}}^{\eta_{\varepsilon_1,\varepsilon_2}-\delta_{\varepsilon_1,\varepsilon_2}} |\chi_{\varepsilon_1,\varepsilon_2} - \chi_0|(t) \cdot H^{n-1}(\{x \in \Omega \mid h(x) = t\})\,\mathrm{d}t \\
&\leqslant \eta_{\varepsilon_1,\varepsilon_2} \cdot \sup_{|t| \leqslant \eta_{\varepsilon_1,\varepsilon_2}} H^{n-1}(\{x \in \Omega \mid h(x) = t\})\,\mathrm{d}t \\
&\leqslant \frac{\varepsilon_2}{(P'_{\varepsilon_1} C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}} \cdot \gamma_{\varepsilon_1,\varepsilon_2} \quad \text{(by (3.6))},
\end{aligned} \tag{3.16}$$

where $\gamma_{\varepsilon_1,\varepsilon_2} \equiv \sup_{|t| \leqslant \eta_{\varepsilon_1,\varepsilon_2}} H^{n-1}(\{x \in \Omega \mid h(x) = t\})$.

Next, we evaluate

$$\begin{aligned}
E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) &= \int_{\Omega_{\delta_{\varepsilon_1,\varepsilon_2}}} \left[\frac{1}{\varepsilon_2} \frac{W \circ \chi_{\varepsilon_1,\varepsilon_2} \circ h}{P'_{\varepsilon_1}} + \varepsilon_2^{P_{\varepsilon_1}-1} \frac{(\chi'_{\varepsilon_1,\varepsilon_2})^{P_{\varepsilon_1}} \circ h |\nabla h|^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}\right] \mathrm{d}x \\
&= \int_{\Omega_{\delta_{\varepsilon_1,\varepsilon_2}}} \left[\frac{1}{\varepsilon_2} \frac{W \circ \chi_{\varepsilon_1,\varepsilon_2} \circ h}{P'_{\varepsilon_1}} + \varepsilon_2^{P_{\varepsilon_1}-1} \frac{(\chi'_{\varepsilon_1,\varepsilon_2})^{P_{\varepsilon_1}} \circ h}{P_{\varepsilon_1}}\right] |\nabla h|\,\mathrm{d}x \\
&\qquad (|\nabla h| = 1 \text{ a.e. on } \Omega_{\delta_{\varepsilon_1,\varepsilon_2}} \text{ provided that } \delta_{\varepsilon_1,\varepsilon_2} \text{ is small enough}) \\
&= \int_{-\delta_{\varepsilon_1,\varepsilon_2}}^{\eta_{\varepsilon_1,\varepsilon_2}-\delta_{\varepsilon_1,\varepsilon_2}} \left[\frac{1}{\varepsilon_2} \frac{W \circ \chi_{\varepsilon_1,\varepsilon_2}(t)}{P'_{\varepsilon_1}} + \varepsilon_2^{P_{\varepsilon_1}-1} \frac{(\chi'_{\varepsilon_1,\varepsilon_2})^{P_{\varepsilon_1}}(t)}{P_{\varepsilon_1}}\right] \\
&\qquad\qquad \times H^{n-1}(\{x \in \Omega_{\delta_{\varepsilon_1,\varepsilon_2}} \mid h(x) = t\})\,\mathrm{d}t \\
&\leqslant \gamma_{\varepsilon_1,\varepsilon_2} \int_0^{\eta_{\varepsilon_1,\varepsilon_2}} \left[\frac{1}{\varepsilon_2}\left(\frac{W \circ \Psi^{-1}_{\varepsilon_1,\varepsilon_2} + P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2}}{P'_{\varepsilon_1}}\right) + \varepsilon_2^{P_{\varepsilon_1}-1} \frac{((\Psi^{-1}_{\varepsilon_1,\varepsilon_2})')^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}\right] \mathrm{d}t
\end{aligned}$$

$$= \gamma_{\varepsilon_1,\varepsilon_2} \int_0^{\eta_{\varepsilon_1,\varepsilon_2}} \left[\frac{([(1/\varepsilon_2)(W \circ \Psi_{\varepsilon_1,\varepsilon_2}^{-1} + P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2})]^{1/P'_{\varepsilon_1}})^{P'_{\varepsilon_1}}}{P'_{\varepsilon_1}} + \frac{[\varepsilon_2^{(P_{\varepsilon_1}-1)/P_{\varepsilon_1}} (\Psi_{\varepsilon_1,\varepsilon_2}^{-1})']^{P_{\varepsilon_1}}}{P_{\varepsilon_1}} \right] \mathrm{d}t$$

$$= \gamma_{\varepsilon_1,\varepsilon_2} \int_0^{\eta_{\varepsilon_1,\varepsilon_2}} \left[\frac{1}{\varepsilon_2} (W \circ \Psi_{\varepsilon_1,\varepsilon_2}^{-1} + P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2}) \right]^{1/P'_{\varepsilon_1}} \cdot \varepsilon_2^{1/P'_{\varepsilon_1}} \cdot (\Psi_{\varepsilon_1,\varepsilon_2}^{-1})' \, \mathrm{d}t.$$

The last equality follows from (3.7), and the sign of the equality holds in Young's inequality. Therefore, we have that

$$E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) \leqslant \gamma_{\varepsilon_1,\varepsilon_2} \int_0^1 (W(t) + P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2})^{1/P'_{\varepsilon_1}} \, \mathrm{d}t \tag{3.17}$$

by the change of variables formula. Moreover,

$$\limsup_{\varepsilon_1 \to 0} \Big(\limsup_{\varepsilon_2 \to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) \Big) \leqslant \limsup_{\varepsilon_1 \to 0} \left(\limsup_{\varepsilon_2 \to 0} \gamma_{\varepsilon_1,\varepsilon_2} \int_0^1 (W(t) + P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2})^{1/P'_{\varepsilon_1}} \, \mathrm{d}t \right). \tag{3.18}$$

It is crucial for our proof to find the positive constant $C_{\varepsilon_1,\varepsilon_2}$ related to ε_1 and ε_2, such that

$$\lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \frac{\varepsilon_2}{(P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}} = 0 \tag{3.19}$$

and

$$\limsup_{\varepsilon_1 \to 0} \limsup_{\varepsilon_2 \to 0} \int_0^1 (W(t) + P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2})^{1/P'_{\varepsilon_1}} \, \mathrm{d}t = 1. \tag{3.20}$$

Combining (3.19) and (3.6), we obtain that

$$\lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \gamma_{\varepsilon_1,\varepsilon_2} = H^{n-1}(\partial A \cap \Omega) = \mathrm{Per}_{\Omega}(A). \tag{3.21}$$

Following from (3.16), (3.19) and (3.21),

$$u_{\varepsilon_1,\varepsilon_2} \to u \text{ in } L^1(\Omega) \quad \text{as } \varepsilon_1 \to 0 \text{ and } \varepsilon_2 \to 0. \tag{3.22}$$

By (3.18), (3.20) and (3.21), we obtain that

$$\limsup_{\varepsilon_1 \to 0} \Big(\limsup_{\varepsilon_2 \to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) \Big) \leqslant \mathrm{Per}_{\Omega}(A) = \int_{\Omega} |\nabla u| \, \mathrm{d}x = E(u). \tag{3.23}$$

We choose $C_{\varepsilon_1,\varepsilon_2} = 1/P'_{\varepsilon_1}$. Then, (3.19) holds and

$$\int_0^1 (W(t) + P'_{\varepsilon_1} \cdot C_{\varepsilon_1,\varepsilon_2})^{1/P'_{\varepsilon_1}} \, \mathrm{d}t = \int_0^1 (W(t) + 1)^{1/P'_{\varepsilon_1}} \, \mathrm{d}t.$$

This is since

$$\lim_{\varepsilon_1\to 0^+}(W(t)+1)^{1/P'_{\varepsilon_1}} = 1 \quad \text{for each } t\in[0,1],$$

and

$$0 \leqslant (W(t)+1)^{1/P'_{\varepsilon_1}} \leqslant 2$$

for all $t\in[0,1]$ provided that ε_1 is small enough. Using Lebesgue's dominated convergence theorem, it follows that (3.20) holds. Hence, the lim sup inequality, Definition 2.2 (ii), is achieved.

Suppose that $u_{\varepsilon_1,\varepsilon_2}\to u$ in $L^1(\Omega)$. By Young's inequality, we have that

$$\begin{aligned} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) &= \int_\Omega \left[\frac{(((1/\varepsilon_2)W\circ u_{\varepsilon_1,\varepsilon_2})^{1/P'_{\varepsilon_1}})^{P'_{\varepsilon_1}}}{P'_{\varepsilon_1}} + \frac{((\varepsilon_2^{P_{\varepsilon_1}-1})^{1/P_{\varepsilon_1}}|\nabla u_{\varepsilon_1,\varepsilon_2}|)^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}\right]\mathrm{d}x \\ &\geqslant \int_\Omega \left(\frac{1}{\varepsilon_2}W\circ u_{\varepsilon_1,\varepsilon_2}\right)^{1/P'_{\varepsilon_1}}\cdot(\varepsilon_2^{P_{\varepsilon_1}-1})^{1/P_{\varepsilon_1}}\cdot|\nabla u_{\varepsilon_1,\varepsilon_2}|\,\mathrm{d}x \\ &= \int_\Omega W^{1/P'_{\varepsilon_1}}\circ u_{\varepsilon_1,\varepsilon_2}\cdot|\nabla u_{\varepsilon_1,\varepsilon_2}|\,\mathrm{d}x. \end{aligned} \tag{3.24}$$

Define $\varPhi_{\varepsilon_1}:\mathbb{R}\to\mathbb{R}$ by

$$\varPhi_{\varepsilon_1}(t) \equiv \begin{cases} 0 & \text{if } t<0, \\ \displaystyle\int_0^t W^{1/P'_{\varepsilon_1}}(s)\,\mathrm{d}s & \text{if } 0\leqslant t\leqslant 1, \\ \displaystyle\int_0^1 W^{1/P'_{\varepsilon_1}}(s)\,\mathrm{d}s & \text{if } t>1. \end{cases}$$

Then,

$$\begin{gathered} \varPhi_{\varepsilon_1}\circ u_{\varepsilon_1,\varepsilon_2}\in W^{1,P_{\varepsilon_1}}(\Omega) \quad \text{for each } u_{\varepsilon_1,\varepsilon_2}, \\ |\nabla(\varPhi_{\varepsilon_1}\circ u_{\varepsilon_1,\varepsilon_2})| = |\varPhi'_{\varepsilon_1}\circ u_{\varepsilon_1,\varepsilon_2}|\,|\nabla u_{\varepsilon_1,\varepsilon_2}| = \chi_{\{0\leqslant u_{\varepsilon_1,\varepsilon_2}\leqslant 1\}}\cdot W^{1/P'_{\varepsilon_1}}\circ u_{\varepsilon_1,\varepsilon_2}|\nabla u_{\varepsilon_1,\varepsilon_2}|, \\ \varPhi_{\varepsilon_1}\circ u_{\varepsilon_1,\varepsilon_2}\in \mathrm{BV}(\Omega) \quad (\text{since } W^{1,1}(\Omega)\subset \mathrm{BV}(\Omega)), \end{gathered} \tag{3.25}$$

and, for each $\varepsilon_1>0$,

$$\varPhi_{\varepsilon_1}\circ u_{\varepsilon_1,\varepsilon_2}\to\varPhi_{\varepsilon_1}\circ u \text{ in } L^1(\Omega) \quad \text{as } \varepsilon_2\to 0^+.$$

By lower semicontinuity of the variation measure, we have that

$$\int_\Omega |\nabla(\varPhi_{\varepsilon_1}\circ u)|\,\mathrm{d}x \leqslant \liminf_{\varepsilon_2\to 0^+}\int_\Omega |\nabla(\varPhi_{\varepsilon_1}\circ u_{\varepsilon_1,\varepsilon_2})|\,\mathrm{d}x. \tag{3.26}$$

By (3.10), we have $u = \chi_A$ a.e. in Ω and

$$\begin{aligned}
+\infty &> \liminf_{\varepsilon_1\to 0}\liminf_{\varepsilon_2\to 0} E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2}) \\
&\geqslant \liminf_{\varepsilon_1\to 0}\liminf_{\varepsilon_2\to 0}\int_\Omega W^{1/P'_{\varepsilon_1}}\circ u_{\varepsilon_1,\varepsilon_2}|\nabla u_{\varepsilon_1,\varepsilon_2}|\,\mathrm{d}x \quad \text{(by (3.24))} \\
&\geqslant \liminf_{\varepsilon_1\to 0}\liminf_{\varepsilon_2\to 0}\int_\Omega |\nabla(\Phi_{\varepsilon_1}\circ u_{\varepsilon_1,\varepsilon_2})|\,\mathrm{d}x \quad \text{(by (3.25))} \\
&\geqslant \liminf_{\varepsilon_1\to 0}\int_\Omega |\nabla(\Phi_{\varepsilon_1}\circ u)|\,\mathrm{d}x \quad \text{(by (3.26)).}
\end{aligned} \tag{3.27}$$

We may suppose without loss of generality that $u \in \mathrm{BV}(\Omega)$ (otherwise, $E(u) = +\infty$). We have that

$$\int_{-\infty}^{+\infty} \mathrm{Per}_\Omega(\{x\in\Omega \mid \Phi_{\varepsilon_1}(\chi_A(x)) \leqslant t\})\,\mathrm{d}t = \left[\int_0^1 W^{1/P'_{\varepsilon_1}}(t)\,\mathrm{d}t\right]\cdot \mathrm{Per}_\Omega(A) \tag{3.28}$$

is finite for each $\varepsilon_1 > 0$. Thus, by the coarea formula for BV-functions, $\Phi_{\varepsilon_1}\circ u \in \mathrm{BV}(\Omega)$ and

$$\int_\Omega |\nabla(\Phi_{\varepsilon_1}\circ u)|\,\mathrm{d}x = \left(\int_0^1 W^{1/P'_{\varepsilon_1}}(t)\,\mathrm{d}t\right)\cdot \mathrm{Per}_\Omega(A). \tag{3.29}$$

Applying Lebesgue's dominated convergence theorem again, we have that

$$\lim_{\varepsilon_1\to 0}\int_0^1 W^{1/P'_{\varepsilon_1}}(t)\,\mathrm{d}t = 1. \tag{3.30}$$

Therefore,

$$\liminf_{\varepsilon_1\to 0}\int_\Omega |\nabla(\Phi_{\varepsilon_1}\circ u)|\,\mathrm{d}x = \mathrm{Per}_\Omega(A) = \int_\Omega |\nabla u|\,\mathrm{d}x = E(u). \tag{3.31}$$

Hence, the lim inf inequality, Definition 2.2 (i), is obtained by (3.27) and (3.31). □

4. The limit $P'_{\varepsilon_1} \to \infty$

In this section we consider the asymptotic behaviour of the functionals

$$E'_{\varepsilon_1,\varepsilon_2}(u) \equiv \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{\varepsilon_2}\frac{W(u)}{P_{\varepsilon_1}} + \varepsilon_2^{P'_{\varepsilon_1}-1}\cdot\frac{|\nabla u|^{P'_{\varepsilon_1}}}{P'_{\varepsilon_1}}\right]\mathrm{d}x & \text{if } u \in W^{1,P'_{\varepsilon_1}}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where P'_{ε_1} is the Lebesgue conjugate function of P_{ε_1}. Define the functional E' by

$$E'(u) \equiv \begin{cases} \displaystyle\left(\int_0^1 W(t)\,\mathrm{d}t\right)\cdot\int_\Omega |\nabla u|\,\mathrm{d}x & \text{if } W(u) = 0 \text{ a.e. and } u \in \mathrm{BV}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Assume that $\varepsilon_2 \cdot P_{\varepsilon_1} \to 0$ as $(\varepsilon_1, \varepsilon_2) \to (0,0)$. Then, $\{E'_{\varepsilon_1,\varepsilon_2}\}_{\varepsilon_1>0,\,\varepsilon_2>0}$ double Γ-converges to E' in $L^1(\Omega)$.*

Proof. To prove Theorem 4.1, it suffices to exchange P_{ε_1} and P'_{ε_1} in the explanation of Theorem 3.1 and to choose $C_{\varepsilon_1,\varepsilon_2}$ such that

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\frac{\varepsilon_2}{(P_{\varepsilon_1}\cdot C_{\varepsilon_1,\varepsilon_2})^{1/P'_{\varepsilon_1}}} = 0 \tag{4.1}$$

and

$$\limsup_{\varepsilon_1\to 0}\limsup_{\varepsilon_2\to 0}\int_0^1 (W(t) + P_{\varepsilon_1}\cdot C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}\,\mathrm{d}t = \int_0^1 W(t)\,\mathrm{d}t. \tag{4.2}$$

We choose $C_{\varepsilon_1,\varepsilon_2} \equiv \varepsilon_2$ and let $d_{\varepsilon_1,\varepsilon_2} \equiv P_{\varepsilon_1}\cdot\varepsilon_2$. By assumption, we have that

$$d_{\varepsilon_1,\varepsilon_2} \to 0 \quad \text{as } \varepsilon_1 \to 0 \text{ and } \varepsilon_2 \to 0, \tag{4.3}$$

$$\frac{\varepsilon_2}{d_{\varepsilon_1,\varepsilon_2}} = \frac{1}{P_{\varepsilon_1}} \to 1 \quad \text{as } \varepsilon_1 \to 0 \text{ and } \varepsilon_2 \to 0 \tag{4.4}$$

and

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\frac{\varepsilon_2}{(P_{\varepsilon_1}\cdot C_{\varepsilon_1,\varepsilon_2})^{1/P'_{\varepsilon_1}}} = \lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}(d_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}\cdot\frac{1}{P_{\varepsilon_1}}. \tag{4.5}$$

It follows from (4.3) and (4.4) that we have

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}(d_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}}\cdot\frac{1}{P_{\varepsilon_1}} = 0. \tag{4.6}$$

Since

$$(W(t) + P_{\varepsilon_1}\cdot C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}} = \mathrm{e}^{(1/P_{\varepsilon_1})\ln(W(t)+d_{\varepsilon_1,\varepsilon_2})} \to W(t) \quad \text{for a.e. } t \text{ in } [0,1]$$

and

$$0 \leqslant (W(t) + P_{\varepsilon_1}\cdot C_{\varepsilon_1,\varepsilon_2})^{1/P_{\varepsilon_1}} \leqslant (W(t)+1)^2$$

provided that ε_1 and ε_2 are small enough, by Lebesgue's dominated convergence theorem, (4.2) holds. □

5. The (p,q)-type capacity

Capacity is an effective way to study certain 'small' subsets of $\mathbb{R}^n$. Moreover, capacity is particularly suited to characterizing the fine properties of Sobolev functions. Let D be a convex bounded open set in $\mathbb{R}^n$ with smooth boundary and $1 < p < n$. The p-capacity of D in $\mathbb{R}^n$ can be defined as follows (see [**8**]):

$$\mathrm{Cap}_p(D) \equiv \inf\left\{\int_{\mathbb{R}^n}|\nabla u|^p\,\mathrm{d}x \;\middle|\; u \in C_c^\infty(\mathbb{R}^n),\ u \geqslant 1 \text{ in } D\right\}.$$

A function u is said to be the p-capacitary function of D if u satisfies the following problem:

$$\begin{gathered}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \mathbb{R}^n - D,\\ u = 1 \quad \text{on } \partial D,\\ \lim_{|x|\to+\infty} u(x) = 0.\end{gathered}$$

In this section, we consider the (p, q)-type generalized capacity of D in Ω with the finite perimeter of D in Ω. Let D be compactly contained in Ω, i.e. $\bar{D}$ is compact and $\bar{D} \subset \Omega$. Define the (p, q)-type capacity of D in Ω by

$$\begin{aligned}&\operatorname{Cap}_{\varepsilon_1,\varepsilon_2}(D, \Omega)\\ &\quad\equiv \inf\left\{ \int_\Omega \left[\frac{1}{\varepsilon_2}\frac{W(u)}{P'_{\varepsilon_1}} + \varepsilon_2^{P_{\varepsilon_1}-1}\frac{|\nabla u|^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}\right] \mathrm{d}x \;\middle|\; u \in W_0^{1,P_{\varepsilon_1}}(\Omega) \text{ and } u \geqslant 1 \text{ in } D\right\},\end{aligned}$$

where $W(t) = \frac{1}{4}t^2(1-t)^2$. Evidently, $\operatorname{Cap}_{\varepsilon_1,\varepsilon_2}(D, \Omega)$ can be expressed as

$$\begin{aligned}&\operatorname{Cap}_{\varepsilon_1,\varepsilon_2}(D, \Omega)\\ &\quad\equiv \inf\left\{ \int_\Omega \left[\frac{1}{\varepsilon_2}\frac{W(u)}{P'_{\varepsilon_1}} + \varepsilon_2^{P_{\varepsilon_1}-1}\frac{|\nabla u|^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}\right] \mathrm{d}x \;\middle|\; u \in W_0^{1,P_{\varepsilon_1}}(\Omega) \text{ and } u = 1 \text{ in } D\right\}.\end{aligned}$$

The existence of minimizers of $E_{\varepsilon_1,\varepsilon_2}$ and $E'_{\varepsilon_1,\varepsilon_2}$ can be proved by the direct method of calculus of variations. Suppose that $u_{\varepsilon_1,\varepsilon_2}$ is a minimizer of $E_{\varepsilon_1,\varepsilon_2}$ on $W_0^{1,P_{\varepsilon_1}}(\Omega - \bar{D})$ for each $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Let $\tilde{u}_{\varepsilon_1,\varepsilon_2}$ be the extension function of $u_{\varepsilon_1,\varepsilon_2}$ on Ω,

$$\tilde{u}_{\varepsilon_1,\varepsilon_2}(x) := \begin{cases} u_{\varepsilon_1,\varepsilon_2} & \text{if } x \in \Omega - \bar{D},\\ 1 & \text{if } x \in \bar{D}.\end{cases}$$

Then $\tilde{u}_{\varepsilon_1,\varepsilon_2} \in W_0^{1,P_{\varepsilon_1}}(\Omega)$, $\tilde{u}_{\varepsilon_1,\varepsilon_2} = 1$ on D and

$$\begin{aligned}\operatorname{Cap}_{\varepsilon_1,\varepsilon_2}(D, \Omega) &= \int_\Omega \left[\frac{1}{\varepsilon_2}\frac{W(\tilde{u}_{\varepsilon_1,\varepsilon_2})}{P'_{\varepsilon_1}} + \varepsilon_2^{P_{\varepsilon_1}-1}\cdot\frac{|\nabla \tilde{u}_{\varepsilon_1,\varepsilon_2}|^{P_{\varepsilon_1}}}{P_{\varepsilon_1}}\right] \mathrm{d}x\\ &= \min\{E_{\varepsilon_1,\varepsilon_2}(u) \mid u \in W_0^{1,P_{\varepsilon_1}}(\Omega - \bar{D})\}\\ &= E_{\varepsilon_1,\varepsilon_2}(u_{\varepsilon_1,\varepsilon_2})\\ &= \min\{E_{\varepsilon_1,\varepsilon_2}(u) \mid u \in W_0^{1,P_{\varepsilon_1}}(\Omega - \bar{D}) \text{ and } u = 1 \text{ on } D\}.\end{aligned} \tag{5.1}$$

Assume that $u_{\varepsilon_1,\varepsilon_2} \to u_0$ in $L^1(\Omega)$. Then, $\tilde{u}_{\varepsilon_1,\varepsilon_2} \to \tilde{u}_0$ in $L^1(\Omega)$. Following Theorems 2.6 and 3.1, this yields that

$$\operatorname{Cap}_{\varepsilon_1,\varepsilon_2}(D, \Omega) \to \int_\Omega |\nabla u_0| \,\mathrm{d}x \quad \text{as } \varepsilon_1 \to 0 \text{ and } \varepsilon_2 \to 0, \tag{5.2}$$

where $u_0 \in \mathrm{BV}(\Omega)$, $u_0 = 1$ in D and $W(u_0) = 0$ a.e. in $\Omega - \bar{D}$.

Moreover,

$$\int_{\Omega} |\nabla u_0| \,\mathrm{d}x = \min\left\{ \int_{\Omega} |\nabla u| \,\mathrm{d}x \;\middle|\; u \in \mathrm{BV}(\Omega),\ u = 1 \text{ in } D \text{ and } W(u) = 0 \text{ a.e. in } \Omega - \bar{D} \right\}.$$

Hence, $u_0 = \chi_D$ in Ω and

$$\int_{\Omega} |\nabla u_0| \,\mathrm{d}x = \mathrm{Per}_{\Omega}(D). \tag{5.3}$$

Next, we define

$$\mathrm{Cap}'_{\varepsilon_1,\varepsilon_2}(D,\Omega) \equiv \inf\left\{ \int_{\Omega} \left[\frac{1}{\varepsilon_2} \frac{W(u)}{P_{\varepsilon_1}} + \varepsilon_2^{P'_{\varepsilon_1}-1} \frac{|\nabla u|^{P'_{\varepsilon_1}}}{P'_{\varepsilon_1}} \right] \mathrm{d}x \;\middle|\; u \in W_0^{1,P'_{\varepsilon_1}}(\Omega) \text{ and } u \geqslant 1 \text{ in } D \right\};$$

we can prove, in the same way, that

$$\mathrm{Cap}'_{\varepsilon_1,\varepsilon_2}(D,\Omega) \to \left(\int_0^1 W(t) \,\mathrm{d}t \right) \cdot \mathrm{Per}_{\Omega}(D) \quad \text{as } \varepsilon_1 \to 0 \text{ and } \varepsilon_2 \to 0. \tag{5.4}$$

Hence, the following theorem is obtained.

Theorem 5.1. *Supposing that the assumptions considered in § 3 and § 4 hold, let D be compactly contained in Ω.*

(i) *Suppose that*

(a) $\varepsilon_2 \cdot P'_{\varepsilon_1} \to 0$ *as* $(\varepsilon_1, \varepsilon_2) \to (0,0)$,

(b) $u_{\varepsilon_1,\varepsilon_2}$ *is a minimizer of* $E_{\varepsilon_1,\varepsilon_2}$ *on* $W_0^{1,P_{\varepsilon_1}}(\Omega - \bar{D})$ *for each* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$,

(c) $u_{\varepsilon_1,\varepsilon_2} \to u_0$ *in* $L^1(\Omega)$ *as* $\varepsilon_1 \to 0$ *and* $\varepsilon_2 \to 0$, *for some* $u_0 \in L^1(\Omega)$.

Then, $\mathrm{Cap}_{\varepsilon_1,\varepsilon_2}(D,\Omega) \to \mathrm{Per}_{\Omega}(D)$ *as* $\varepsilon_1 \to 0$ *and* $\varepsilon_2 \to 0$.

(ii) *Suppose that*

(a) $\varepsilon_2 \cdot P_{\varepsilon_1} \to 0$ *as* $(\varepsilon_1, \varepsilon_2) \to (0,0)$,

(b) $u'_{\varepsilon_1,\varepsilon_2}$ *is a minimizer of* $E'_{\varepsilon_1,\varepsilon_2}$ *on* $W_0^{1,P'_{\varepsilon_1}}(\Omega - \bar{D})$ *for each* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$,

(c) $u'_{\varepsilon_1,\varepsilon_2} \to u'_0$ *in* $L^1(\Omega)$ *as* $\varepsilon_1 \to 0$ *and* $\varepsilon_2 \to 0$, *for some* $u'_0 \in L^1(\Omega)$.

Then, $\mathrm{Cap}'_{\varepsilon_1,\varepsilon_2}(D,\Omega) \to (\int_0^1 W(t)\,\mathrm{d}t) \cdot \mathrm{Per}_{\Omega}(D)$ *as* $\varepsilon_1 \to 0$ *and* $\varepsilon_2 \to 0$.

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