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ON THE GENERALITY OF THE AP-INTEGRAL

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1. Introduction and definitions. In 1955 Taylor [**6**] constructed an AP-integral sufficiently strong to integrate Abel summable series with coefficients $o(n)$. He showed that the AP-integral includes the special Denjoy integral and further that, when applied to trigonometric series, the AP-integral is more powerful than the SCP-integral of Burkill [**1**] and the P^2-integral of James [**3**]. The present paper shows that the AP-integral includes the SCP-integral, and, under natural assumptions, the P^2-integral.

After completing this manuscript I was advised by Skvorcov that he had shown [**5**] under more general conditions that the P^2-integral is included in the AP-integral. The proof in the present paper seems to have some value in its own right and is considerably shorter.

Since the definition of the AP-integral is essentially for a function defined in $(0, 2\pi]$ and elsewhere by 2π-periodicity, we shall consider SCP-integrable and P^2-integrable functions defined similarly. As Skvorcov [**4**] has pointed out, however, P^2-integrability on $[0, 2\pi]$ and on $[-2\pi, 0]$ does not necessarily imply integrability on $[-2\pi, 2\pi]$. To illustrate this, consider the function defined on $[0, 2\pi]$ by

$$f(x) = \begin{cases} 0, & x \in (2/\pi, 2\pi], \\ (-1/x^3)\cos x^{-1}, & x \in (0, 2/\pi], \end{cases}$$

and in $[-2\pi, 0]$ by periodicity. This function is P^2-integrable on $[0, 2\pi]$ and on $[-2\pi, 0]$ but it is not P^2-integrable on $[-2/\pi, 2/\pi]$ (see [**4**]).

We shall make use of the following results proved by Skvorcov [**4**].

I. *Let the function $f(x)$ be P^2-integrable on the closed intervals $[a, c]$ and $[c, b]$ and have $F_1(x)$, $F_2(x)$, respectively, for its P^2-integral on these intervals. Then $f(x)$ is P^2-integrable on $[a, b]$ if and only if there exists a number α such that the function*

$$F(x) = \begin{cases} F_1(x) + (\alpha/(c-a))(x-a), & x \in [a, c], \\ F_2(x) + (\alpha/(c-b))(x-b), & x \in [c, b], \end{cases} \tag{1.1}$$

is smooth at the point c. If such a number α exists, then the function $F(x)$ is the P^2-integral of $f(x)$ on $[a, b]$ and

$$F(c) = \alpha = \lim_{h \to 0+} \frac{[F_1(c-h) + F_2(c+h)](b-c)(c-a)}{h(b-a)}. \tag{1.2}$$

II. *In order that a P^2-integral be additive on intervals it is sufficient that*

Condition (A): *the functions $F(x)$ which is the P^2-integral on $[a, b]$ have a left*

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derivative at the end point $b(F_-'(b))$ *and a right derivative at the end point* $a(F_+'(a))$.

The reader is directed to the references, particularly [**3**; **6**] for notation, but some definitions are repeated here for convenience.

The definition of P^2-major and P^2-minor functions $M(x)$ and $m(x)$ of $f(x)$ on $[-2\pi, 2\pi]$ may be given as follows:

(1.3) $M(x)$ and $m(x)$ are continuous in $[-2\pi, 2\pi]$;

(1.4) $M(-2\pi) = M(2\pi) = m(-2\pi) = m(2\pi) = 0$;

(1.5) $\underline{D}^2M(x) \geqq f(x) \geqq \bar{D}^2m(x)$, a.e. in $[-2\pi, 2\pi]$;

(1.6) $\underline{D}^2M(x) > -\infty$, $\bar{D}^2m(x) < +\infty$, for all $x \in [-2\pi, 2\pi]$, except possibly in a denumerable set E;

(1.7) $M(x)$ and $m(x)$ are smooth for all x in E.

A function $f(x)$, defined on $[-2\pi, 2\pi]$, is called P^2-integrable on $(-2\pi, x, 2\pi)$, $-2\pi < x < 2\pi$, if, for every $\epsilon > 0$, there exists a P^2-major function $M(x)$ and a P^2-minor function $m(x)$ such that $0 \leqq m(x) - M(x) < \epsilon$. The P^2-integral of $f(x)$ at the point x is defined as $-F(x)$, where

$$F(x) = \sup M(x) = \inf m(x)$$

and we write

$$-F(x) = P^2\text{-}\int_{-2\pi,x,2\pi} f(t)\,dt.$$

Now suppose that $f(x)$ is P^2-integrable over $[0, 2\pi]$ and extend $f(x)$ to $[-2\pi, 0]$ by the relation $f(x) = f(x + 2\pi)$. Then if

$$F_1(x) = P^2\text{-}\int_{-2\pi,x,0} f(t)\,dt \qquad \text{and} \qquad F_2(x) = P^2\text{-}\int_{0,x,2\pi} f(t)\,dt,$$

it follows that $F_1(x) = F_2(x + 2\pi)$. Moreover, if $f(t)$ is P^2-integrable on $[-2\pi, 2\pi]$ with

$$F(x) = P^2\text{-}\int_{-2\pi,x,2\pi} f(t)\,dt$$

and $\alpha = F(0)$, we obtain, using **I** above,

$$F(x) = \begin{cases} F_1(x) + (\alpha/2\pi)(x + 2\pi), & x \in [-2\pi, 0], \\ F_2(x) + (\alpha/-2\pi)(x - 2\pi), & x \in [0, 2\pi], \end{cases}$$

where $F(x)$ is smooth (even at $x = 0$) and $F(2\pi) = F(-2\pi) = 0$.

We recall that according to Taylor's definition [**6**], a finite constant M and the real function $\Phi(x)$ form an AP-upper approximating pair if

(1.8) $\Lambda(x) = \Phi(x) - Mx^2/4\pi$ is periodic with period 2π;

(1.9) $\Lambda(x)$ is Lebesgue-integrable and $\mathscr{A}$-continuous for all x;

(1.10) $\Lambda(x)$ is approximately continuous and has the property R^*;

(1.11) $\Phi(-2\pi) = \Phi(2\pi) = 0$;

(1.12) $A\underline{D}^2\Phi(x) \geqq f(x)$ a.e.; $A\underline{D}^2\Phi(x) > -\infty$ except possibly in a denumerable set E;

(1.13) $$\lim_{r\to 1-}\left[(1-r)\frac{\partial^2}{\partial x^2}\Lambda(r,x)\right] = 0, \text{ at all points of } E.$$

A lower approximating pair $(m, \phi(x))$ is defined analogously.

In the case where $\inf M = \sup m = I$, say, $f(x)$ is said to be AP-integrable over $(0, 2\pi)$ and we write

$$\text{AP-}\int_0^{2\pi} f(t)\,dt = I.$$

2. An inclusion relation between the SCP-integral, the P²-integral, and the AP-integral.

THEOREM 2.1. *If $f(x)$ is* P²*-integrable over* $[0, 2\pi]$ *and satisfies condition* (A) *above, then $f(x)$ is* AP*-integrable on* $[0, 2\pi]$ *and*

$$\text{AP-}\int_0^{2\pi} f(t)\,dt = \pi^{-1}\text{P}^2\text{-}\int_{-2\pi,0,2\pi} f(t)\,dt,$$

where $f(t)$ is defined on $[-2\pi, 0]$ by 2π-periodicity.

Proof. Corresponding to $\epsilon > 0$, there exists a P²-major function $M(x)$ on $[0, 2\pi]$ such that if $R(x) = M(x) - F(x)$, it is true that

$$|R(x)| < \epsilon/2, \quad |R_+'(0)| < \epsilon/2, \quad |R_-'(2\pi)| < \epsilon/2 \quad (\text{see [4]}).$$

Since each $R(x)$ is convex on $(0, 2\pi)$ we see also that $R(x)$ is the integral of a non-decreasing function in $(0, 2\pi)$. In other words,

$$M(x) = F(x) + \int_0^x \xi(t)\,dt$$

where $\xi(t_1) \leqq \xi(t_2)$ for $t_1 \leqq t_2$. From this it follows that $D^2M(x) \geqq f(x)$, $x \in [0, 2\pi]$, except for some set A of measure 0.

Now we make use of the following well-known lemma (see [**1**]).

Given a set A of measure zero, and $\epsilon > 0$, there exists a function $J(x)$, convex and smooth in (a, b), such that $J(a) = 0$, $J(x) \geqq 0$, $D^2J(x) \geqq 0$, in (a, b), $J(b) < \epsilon/2$, and $D^2J(x) = +\infty$ in A.

Using a function $J(x)$ with the above properties we define a new function

$$M_2(x) = M(x) + J(x) - (x/2\pi)J(2\pi).$$

Then $D^2M_2(x) \geqq f(x) > -\infty$ except in A and moreover, in A,

$$\underline{D}^2M_2(x) \geqq \underline{D}^2M(x) + D^2J(x) = +\infty,$$

except on a denumerable set where however $M_2(x)$ is smooth. It is easy to see that the $J(x)$ of the lemma may be constructed so that

$$R_2(x) \equiv M_2(x) - F(x) < \epsilon$$

and $|R_{2+}'(0)| < \epsilon$, $|R_{2-}'(2\pi)| < \epsilon$. Now we extend $f(x)$ and $M_2(x)$ to $[-2\pi, 0]$ by 2π-periodicity, letting $M_1(x) = M_2(x + 2\pi)$.

We introduce a constant formed from the known properties of $M_1(x)$ and $M_2(x)$:

$$M = \pi \lim_{h \to 0+} \frac{M_2(h) + M_1(-h)}{h}.$$

(This limit exists since both $\lim(M_2(h)/h)$ and $\lim(M_1(-h)/h)$ exist because of Condition (A) and the manner in which $M_2(x)$ was constructed.)

The next step is to construct a P^2-major function on $[-2\pi, 2\pi]$ which will then be used to construct an AP-upper approximating pair. To this end, let

$$M_3(x) = \begin{cases} M_1(x) + (M/2\pi)(x + 2\pi), & x \in [-2\pi, 0], \\ M_2(x) - (M/2\pi)(x - 2\pi), & x \in [0, 2\pi]. \end{cases}$$

Then $M_3(0) = M$, $M_3(x)$ is a P^2-major function of $f(x)$ on $[-2\pi, 2\pi]$, $M_3(0) - F(0) < 2\epsilon\pi$, and

$$\Lambda(x) = M_3(x) + M_3(0)(x^2/4\pi^2)$$

is 2π-periodic.

It follows than [**6**, Theorem 4] that $AD^2M_3(x) \geqq f(x)$ a.e. and $AD^2M_3(x) > -\infty$ except on a countable set where however $\Lambda(r, x)$ satisfies condition (1.13) above because of smoothness of $M_3(x)$ (see [**6**, Theorem 5]). The pair $\{-M_3(0)/\pi, M_3(x)\}$ thus form an AP-upper approximating pair for $f(x)$ on $[-2\pi, 2\pi]$. An exactly analogous construction holds for a minor approximating pair, and the statement of the theorem follows.

THEOREM 2.2. *If $f(x)$ is SCP-integrable on $[0, 2\pi]$ with basis B, then $f(x)$ is AP-integrable on $[0, 2\pi]$ and*

$$(\mathrm{SCP}, B) \int_0^{2\pi} f(t)\, dt = \mathrm{AP}\text{-}\int_0^{2\pi} f(t)\, dt.$$

Proof. Extend $f(x)$ and B to $[-2\pi, 0]$ by 2π-periodicity. It is well known that SCP-integrability implies P^2-integrability (see e.g. [**2**]). If

$$F_1(x) = P^2\text{-}\int_{0,x,2\pi} f(t)\, dt \qquad \text{and} \qquad F_2(x) = P^2\text{-}\int_{-2\pi,x,0} f(t)\, dt,$$

it follows from [**2**, Theorem III] that $F_{1+}'(0)$ and $F_{2-}'(0)$ both exist. The proof of the theorem follows from Theorem 2.1 above and [**2**, Theorem II].

Remark. If in the definition of the P^2-integral, smoothness is imposed on major and minor functions at the end points of the interval $[a, b]$ where it is generalized to mean the existence of one-sided derivatives, the P^2-integral is included in the AP-integral by the above argument.

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