Next, we turn our attention to the parallel product. In joint work with Dzhafarov, Hirschfeldt, Patey, and Pauly, we investigate the infinite pigeonhole principle for different numbers of colors and how these problems behave under Weihrauch reducibility with respect to parallel products.

Finally, we leave the setting of computable reducibilities for the setting of reverse mathematics. First, we define a Σ^1_1 axiom of finite choice and investigate its relationships with other theorems of hyperarithmetic analysis. For one, we show that it follows from Arithmetic Bolzano-Weierstrass. On the other hand, using an elaboration of Steel's forcing with tagged trees, we show that it does not follow from Δ^1_1 comprehension. Second, in joint work with James Barnes and Richard A. Shore, we analyze a theorem of Halin about disjoint rays in graphs. Our main result shows that Halin's theorem is a theorem of hyperarithmetic analysis, making it only the second "natural" (i.e., not formulated using concepts from logic) theorem with this property.

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Andrea Vaccaro, C^****-algebras and the Uncountable*****:** ***A Systematic Study of the Combinatorics of the Uncountable in the Noncommutative Framework***, Università di Pisa, Italy – York University, USA, 2019. Supervised by Ilijas Farah and Alessandro Berarducci (cosupervisor). MSC: 03E75 (primary), 46L05, 03E35. Keywords: C^*-algebras, Naimark's problem, Jensen's diamond, Calkin algebra, forcing, liftings.

Abstract

This dissertation investigates nonseparable C^*-algebras using methods coming from set theory [2]. C^*-algebras are objects usually studied in the framework of functional analysis, and they are defined as self-adjoint, norm-closed subalgebras of $\mathcal{B}(H)$ (the algebra of bounded linear operators on a complex Hilbert space H). The Gelfand transform establishes an equivalence between the category of abelian C^*-algebras and the category of locally compact, Hausdorff spaces, motivating the idea that C^*-algebras are the noncommutative analogues of topological spaces. The thesis consists of three independent chapters.

The first chapter concerns Naimark's problem, an old open question about irreducible representations of C^*-algebras. A representation of a C^*-algebra $\mathcal{A}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ for some Hilbert space H, and it is irreducible if it cannot be decomposed as direct sum of two nontrivial subrepresentations. The representation theory of separable C^*-algebras has been widely investigated by operator algebraists, and its study culminated with Glimm's seminal work on type I C^*-algebras. On the other hand, the representation theory of nonseparable C^*-algebras is characterized by more erratic and extreme behaviors, which do not allow a complete generalization of the results holding in the separable setting. Naimark's problem is an example of this discrepancy. It is well known that the C^*-algebra of the compact operators on a Hilbert space has a unique irreducible representation up to spatial equivalence, the identity. Naimark's problem asks whether this property characterizes the algebras of compact operators up to isomorphism. A *counterexample* (*to Naimark's problem*) is C^*-algebra that is not isomorphic to the algebra of compact operators on some Hilbert space, yet still has only one irreducible representation up to spatial equivalence. Such algebras have to be nonseparable, and in 2004 Akemann and Weaver [1] used Jensen's diamond $\diamondsuit$ in combination with some deep results on the representation theory of separable C^*-algebras to build the first counterexamples. It is not known whether a positive answer to Naimark's problem is consistent with ZFC. This chapter focuses on the problem of finding more characterizing properties for the counterexamples. Some elementary observations on the structural properties of these C^*-algebras seem to suggest that the trace space (a classical C^*-algebraic invariant) of a counterexample is trivial. We prove that this is not the case and we show that, assuming $\diamondsuit$, almost every trace space of a separable C^*-algebra also occurs as the trace space of a counterexample (see also [4]).

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The second chapter revolves around the Calkin algebra $\mathcal{Q}(H)$, the quotient of $\mathcal{B}(H)$ modulo the ideal of compact operators $\mathcal{K}(H)$, for a separable, infinite-dimensional Hilbert space H. The Calkin algebra has been object of intense studies by the researchers in operator algebras for almost 80 years. Over the last 15 years, it has also become fertile ground for applications of set theory in C*-algebras, due to its structural similarities with the boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, of which it is considered the noncommutative analogue. We investigate some structural properties of $\mathcal{Q}(H)$, focusing on the question 'What C*-algebras embed into $\mathcal{Q}(H)$?'. We prove that every C*-algebra, regardless of its density character, can be embedded into $\mathcal{Q}(H)$ in a ccc forcing extension of the universe (this is part of the joint work [3]). This is used to prove that the statement 'Every C*-algebra of density character less than $2^{\aleph_0}$ embeds into $\mathcal{Q}(H)$' is independent from $\mathsf{ZFC} + 2^{\aleph_0} \geq \aleph_\alpha$, for every $\alpha > 2$. In particular, such statement is implied by Martin's axiom. Chapter 2 ends with a generalization (under Martin's axiom) to nonseparable C*-algebras of Voiculescu's theorem, a classical result in the theory of extensions of separable C*-algebras (these results are also in [5]).

The last chapter concerns liftings of abelian subalgebras of coronas of C*-algebras. For a C*-algebra $\mathcal{A}$, the multiplier algebra $\mathcal{M}(\mathcal{A})$ is the 'maximal' unitization of $\mathcal{A}$, while its corona $\mathcal{Q}(\mathcal{A})$ is the quotient $\mathcal{M}(\mathcal{A})/\mathcal{A}$. They are the noncommutative analogues of the Čech–Stone compactification and the corona, respectively, of a locally compact Hausdorff space. Given a set of commuting elements in a corona of a nonabelian, nonunital C*-algebra, we study what obstructions could prevent the existence of a commutative lifting to the multiplier algebra. We show that, while for countable families the only issues arising are of K-theoretic nature, for larger families the size itself becomes an obstacle. Using a combinatorial argument which goes back to Luzin's families, we prove, for a fairly general class of separable, nonabelian, nonunital C*-algebras $\mathcal{A}$, the existence of a set of $\aleph_1$ commuting elements in $\mathcal{Q}(\mathcal{A})$ containing no uncountable subset that lifts to a set of commuting elements in $\mathcal{M}(\mathcal{A})$ (see also [6]).

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Lorenzo Galeotti, ***The Theory of the Generalised Real Numbers and Other Topics in Logic***, Universität Hamburg, Germany, 2019. Supervised by Benedikt Löwe. MSC: 03E15, 03D60, 03F30, 03C55. Keywords: descriptive set theory, generalised Baire spaces, transfinite computability, Peano arithmetics, model theoretic set theory.

Abstract

The *real line* $\mathbb{R}$ and *Baire space* ω^ω are two of the most fundamental objects in descriptive set theory and have been studied extensively. In recent years, set theorists have become increasingly interested in *generalised Baire spaces* κ^κ, i.e., the sets of functions from κ to κ for an uncountable cardinal κ. In the thesis, two κ-analogues of $\mathbb{R}$ are discussed, one