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CORRECTION TO 'CONJUGACY CLASS SIZE CONDITIONS WHICH IMPLY SOLVABILITY'

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The authors have realised that in the proof of [1, Theorem C] there is a mistake, although the conclusion of the theorem is correct. The proof of the theorem is divided into two cases: when there exist no p-elements of index p^a in the group G and when they do exist. In the second case of the proof, the authors apply [1, Lemma 2.3] to the abelian normal subgroup $O_{p'}(G)$. However, the conclusion is false, so the final contradiction is not achieved as it is argued in the paper. In order to solve this, we provide here a correction of this case.

Proof of Theorem C. The first two paragraphs of the original proof of Theorem C in [1] are correct. We assume then that there are p-elements of index p^a in G and that p divides n.

Now suppose that there is a p-element z such that $|z^G| = p^a$. Then, if y is a p'-element of primary order of $C_G(z)$, we can get

$$C_G(zy) = C_G(z) \cap C_G(y) \subseteq C_G(z).$$

It follows that y has index 1 or n in $C_G(z)$, and if there are p'-elements of primary orders of both indexes 1 and n in $C_G(z)$, it follows that $n = p^a q^b$ by [1, Lemma 2.1]. By [1, Lemma 2.2], we conclude that G is a $\{p, q\}$-group (or a p-group if $b = 0$) up to central factors. So the proof is finished in this case. Now suppose that every p'-element of primary order of $C_G(z)$ has index 1. Accordingly, we can write $C_G(z) = P_z \times H_z$, where H_z is an abelian p-complement. As $C_G(z)$ has index p^a in G, we deduce that G has abelian p-complements, so in particular G is solvable as it is the product of two nilpotent subgroups. We show that $C_G(z)$ is abelian and we will see that this leads to a contradiction. If $y \in P_z \setminus Z(G)_p$, then $H_z \subseteq C_G(y)$, so necessarily y has index p^a. Notice that every $h \in H_z \setminus Z(G)$ of prime power order has index p^a, and then $C_G(h) = C_G(z)$.

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As a result, $H_z \subseteq C_G(h) \cap C_G(y)$, and this implies that $|C_G(h) : C_G(h) \cap C_G(y)| = 1$, that is, $C_G(h) \subseteq C_G(y)$. Consequently, P_z is abelian, and so $C_G(z)$ is abelian too, as we wanted to show.

For every $x \in H_z \setminus Z(G)$ of prime power order, we deduce that $C_G(z) = C_G(x)$. We can then apply [1, Lemma 2.4] to $O_q(G)$ for every prime q dividing the order of G, and we get $F(G) \subseteq C_G(z)$, where $F(G)$ denotes the Fitting subgroup. Since $C_G(z)$ is abelian, then $C_G(z) \subseteq C_G(F(G)) \subseteq F(G)$. Therefore, $C_G(z) = F(G)$. In particular, we have proved that G has an abelian normal p-complement, and also that $O_p(G)$ is abelian.

We now show that there are p-elements of index $p^a n$ in G. Let x be an element of index $p^a n$ and write $x = x_p x_{p'}$, where x_p and $x_{p'}$ are the p-part and the p'-part of x, respectively. Moreover, $x_{p'}$ is primary order. If $x_{p'} \in Z(G)$, then by the above paragraph $C_G(x_{p'})$ is abelian and contains the p-complement of G. As a consequence $C_G(x) = C_G(x_{p'})$, which means that x has index p^a, which is a contradiction. Hence, $x_{p'} \in Z(G)$ and $|x_p^G| = |x^G| = p^a n$, as we wanted to prove.

Take P to be a Sylow p-subgroup of G. Since we are assuming that p divides n, this forces that $|P/Z(G)_p| > p^a n_p$. Let us take $u \in G^*$ of index n. Arguing as in the above paragraph, one can easily prove that u can be assumed to be a p-element. Moreover, up to conjugacy, we can assume that $u \in P \setminus O_p(G)$. It is easy to see that $C_{O_p(G)}(u) = Z(G)_p$ and then

$$|u^{O_p(G)}| = |O_p(G) : Z(G)_p| \leq n_p.$$

On the other hand,

$$|P : Z(G)_p| = |P : O_p(G)||O_p(G) : Z(G)_p| \leq p^a n_p.$$

However, this contradicts the fact that $|P/Z(G)p| > p^a n_p$, so this case cannot occur. Now the theorem is proved. □

Reference

[1] Q. Kong and Q. Liu, 'Conjugacy class size conditions which imply solvability', *Bull. Aust. Math. Soc.* **88** (2013), 297–300.

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