A FORMULA OF BATEMAN

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1. The formula

$$\sum_{k=0}^{\infty} ((xy)^{\frac{1}{2}}e^{i\phi})^{k-n}\frac{n!}{k!}L_n^{(k-n)}(x)L_n^{(k-n)}(y)=\exp\,((xy)^{\frac{1}{2}}e^{i\phi})L_n\{x+y-2(xy)^{\frac{1}{2}}\cos\phi\} \quad \text{......(1·1)}$$

was stated by Bateman ([**2**], p. 457); a proof is sketched in [**3**], p. 144. Here

$$L_n^{(\alpha)}(x)=\sum_{r=0}^{n}\binom{n+\alpha}{r}\frac{(-x)^{n-r}}{(n-r)!}, \quad \text{......(1·2)}$$

the Laguerre polynomial of degree n, and $L_n(x)=L_n^{(0)}(x)$.

We should like to point out that (1·1) can be proved very rapidly by making use of the following formula due to Bailey ([**1**], p. 219):

$$L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)=\frac{\Gamma(1+\alpha+n)}{n!}\sum_{r=0}^{n}\frac{(xy)^{n-r}L_r^{(\alpha+2n-2r)}(x+y)}{(n-r)!\,\Gamma(1+\alpha+n-r)}, \quad \text{......(1·3)}$$

as well as the simpler formulas ([**3**], p 142)

$$L_n^{(\alpha)}(x-y)=\sum_{r=0}^{n}\frac{y^r}{r!}L_{n-r}^{(\alpha+r)}(x), \quad \text{......(1·4)}$$

$$L_n^{(\alpha)}(x-y)=e^{-y}\sum_{r=0}^{\infty}\frac{y^r}{r!}L_n^{(\alpha+r)}(x), \quad \text{......(1·5)}$$

which are easy consequences of the definition of $L_n^{(\alpha)}(x)$.

Using first (1·3) and then (1·5) and (1·4) we have

$$n!\sum_{k=0}^{\infty}\frac{z^k}{k!}L_n^{(k-n)}(x)L_n^{(k-n)}(y)=\sum_{k=0}^{\infty}z^k\sum_{\substack{r=0\\ r\leqslant k}}^{n}\frac{(xy)^{n-r}L_r^{(k+n-2r)}(x+y)}{(n-r)!(k-r)!}$$

$$=\sum_{r=0}^{n}\frac{(xy)^{n-r}z^r}{(n-r)!}\sum_{k=r}^{\infty}\frac{z^{k-r}}{(k-r)!}L_r^{(k+n-2r)}(x+y)=\sum_{r=0}^{n}\frac{(xy)^{n-r}z^r}{(n-r)!}\sum_{k=0}^{\infty}\frac{z^k}{k!}L_r^{(k+n-r)}(x+y)$$

$$=\sum_{r=0}^{n}\frac{(xy)^{n-r}z^r}{(n-r)!}e^zL_r^{(n-r)}(x+y-z)=z^ne^zL_n\left(x+y-z-\frac{xy}{z}\right). \quad \text{......(1·6)}$$

Thus for $z=(xy)^{\frac{1}{2}}t$ this becomes

$$\sum_{k=0}^{\infty}((xy)^{\frac{1}{2}}t)^{k-n}\frac{n!}{k!}L_n^{(k-n)}(x)L_n^{(k-n)}(y)=\exp\,((xy)^{\frac{1}{2}}t)L_n\left\{x+y-(xy)^{\frac{1}{2}}\left(t+\frac{1}{t}\right)\right\}. \quad \text{......(1·7)}$$

For $t=e^{i\phi}$, (1·7) is identical with (1·1).

2. The identity (1·6) evidently implies that

$$z^nL_n\left(x+y-z-\frac{xy}{z}\right)=n!\,e^{-z}\sum_{k=0}^{\infty}\frac{z^k}{k!}L_n^{(k-n)}(x)L_n^{(k-n)}(y)$$

$$=n!\sum_{k=0}^{\infty}\frac{z^k}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}L_n^{(r-n)}(x)L_n^{(r-n)}(y).$$

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Since by (1·2) $L_n^{(\alpha)}(x)$ is of degree n in α (as well as in x) it follows that

$$\Delta_k = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} L_n^{(r-n)}(x) L_n^{(r-n)}(y) = 0 \qquad (2·1)$$

for $k > 2n$. Consequently we get the following polynomial identity equivalent to (1·6):

$$z^n L_n\left(x + y - z - \frac{xy}{z}\right) = n! \sum_{k=0}^{2n} \frac{z^k}{k!} \Delta_k, \qquad (2·2)$$

with Δ_k defined by the first half of (2·1). We remark that

$$\sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} L_n^{(\alpha+r)}(x) = L_{n-k}^{(\alpha+k)}(x) \quad (n \geqslant k)$$

and in particular

$$\sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} L_n^{(r-n)}(x) = L_{n-k}^{(k-n)}(x) = \frac{(-x)^{n-k}}{(n-k)!} \quad (n \geqslant k), \qquad (2·3)$$

but there seems to be no equally simple formula for Δ_k. Using (1·2) we get

$$\Delta_k = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \sum_{u=0}^{n} \binom{r}{u} \frac{(-x)^{n-u}}{(n-u)!} \sum_{v=0}^{n} \binom{r}{v} \frac{(-y)^{n-v}}{(n-v)!}$$

$$= \sum_{u,\, v=0}^{n} \frac{(-x)^{n-u}(-y)^{n-v}}{(n-u)!\,(n-v)!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r}\binom{r}{u}\binom{r}{v}.$$

Using Vandermonde's theorem it is not difficult to show that

$$\sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r}\binom{r}{u}\binom{r}{v} = \frac{k!}{(k-u)!\,(k-v)!\,(u+v-k)!} \qquad (2·4)$$

for $k \leqslant u+v$; for $k > u+v$ or for u or $v > k$ the sum vanishes. Therefore

$$\Delta_k = \sum_{\substack{u,\, v=0 \\ u+v \geqslant k}}^{\min(k,n)} \frac{k!}{(k-u)!\,(k-v)!\,(u+v-k)!} \, \frac{(-x)^{n-u}(-y)^{n-v}}{(n-u)!\,(n-v)!}, \qquad (2·5)$$

which may be compared with (2·3).

If in (2·2) we replace z by xy/z we get

$$\left(\frac{xy}{z}\right)^n L_n\left(x + y - z - \frac{xy}{z}\right) = n! \sum_{k=0}^{2n} \frac{(xy/z)^k}{k!} \Delta_k.$$

It follows that

$$\frac{\Delta_k}{k!} = (xy)^{n-k} \frac{\Delta_{2n-k}}{(2n-k)!} \quad (0 \leqslant k \leqslant 2n), \qquad (2·6)$$

so that (2·2) becomes

$$z^n L_n\left(x + y - z - \frac{xy}{z}\right) = n! \sum_{k=0}^{n-1} (z^k + (xy)^{k-n} z^{2n-k}) \frac{\Delta_k}{k!} + z^n \Delta_n. \qquad (2·7)$$

For $z = (xy)^{\frac{1}{2}} t$ we get the more symmetrical result

$$(xy)^{n/2} L_n\left\{x + y - (xy)^{\frac{1}{2}}\left(t + \frac{1}{t}\right)\right\} = n! \sum_{k=0}^{n-1} (xy)^{k/2} (t^{k-n} + t^{n-k}) \frac{\Delta_k}{k!} + (xy)^{n/2} \Delta_n, \qquad (2·8)$$

or, if we prefer,

$$(xy)^{n/2} L_n\{x + y - 2(xy)^{\frac{1}{2}} \cos\phi\} = n! \sum_{k=0}^{n-1} (xy)^{k/2} \frac{\Delta_k}{k!} 2\cos(n-k)\phi + (xy)^{n/2} \Delta_n. \qquad (2·9)$$

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3. The formula (2·9) can be generalized in the following way. Differentiation with respect to ϕ yields

$$-(xy)^{(n+1)/2} L_n'\{x+y-2(xy)^{\frac{1}{2}}\cos\phi\} = n!\sum_{k=0}^{n-1}(xy)^{k/2}(n-k)\frac{\Delta_k}{k!}\frac{\sin(n-k)x}{\sin x}.$$

Now let $C_n^{(\lambda)}$ denote the ultraspherical polynomial defined by

$$(1-2xz+z^2)^{-\lambda} = \sum_{n=0}^{\infty} z^n C_n^{(\lambda)}(x),$$

so that

$$C_n^{(1)}(\cos\phi) = \frac{\sin(n+1)\phi}{\sin\phi}, \quad \frac{d}{dx}C_n^{(\lambda)}(x) = 2\lambda C_{n-1}^{(\lambda+1)}(x).$$

We have also

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n+1}^{(\alpha+1)}(x).$$

Thus

$$(xy)^{(n+1)/2} L_{n-1}^{(1)}\{x+y-2(xy)^{\frac{1}{2}}z\} = n!\sum_{k=0}^{n-1}(xy)^{k/2}(n-k)\frac{\Delta_k}{k!}C_{n-k-1}^{(1)}(z).$$

Repeated differentiation with respect to z now leads to

$$(xy)^{(n+\lambda)/2} L_{n-\lambda}^{(\lambda)}\{x+y-2(xy)^{\frac{1}{2}}z\} = n!(\lambda-1)!\sum_{k=0}^{n-\lambda}(xy)^{k/2}(n-k)\frac{\Delta_k}{k!}C_{n-k-\lambda}^{(\lambda)}(z), \qquad (3\cdot1)$$

where λ is an arbitrary positive integer. If we replace n by $n+\lambda$, then, by (2·1), Δ_k becomes

$$\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}L_{n+\lambda}^{(r-n-\lambda)}(x)\,L_{n+\lambda}^{(r-n-\lambda)}(y). \qquad (3\cdot2)$$

Since

$$L_n^{(-k)}(x) = (-x)^k\frac{(n-k)!}{n!}L_{n-k}^{(k)}(x) \quad (0\leqslant k\leqslant n),$$

(3·2) may be written

$$\frac{(xy)^{n+\lambda}}{((n+\lambda)!)^2}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}(r!)^2 L_r^{(n+\lambda-r)}(x)\,L_r^{(n+\lambda-r)}(y).$$

We accordingly rewrite (3·1) as

$$\frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda)}L_n^{(\lambda)}\{x+y-2(xy)^{\frac{1}{2}}z\} = \sum_{k=0}^{n}(xy)^{(n+k)/2}(n+\lambda-k)\frac{\Delta_k^{(\lambda)}}{k!}C_{n-k}^{(\lambda)}(z), \qquad (3\cdot3)$$

where

$$\Delta_k^{(\lambda)} = \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}(r!)^2 L_r^{(n+\lambda-r)}(x)\,L_r^{(n+\lambda-r)}(y). \qquad (3\cdot4)$$

The formula (3·3) has been proved for λ a positive integer. However, since each side is a polynomial in λ, it follows that (3·3) holds for all λ.

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