AXIOMS FOR SEMI-LATTICES

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A semi-lattice (Birkhoff, Lattice Theory, p. 18, Ex. 1) is an algebra $\langle A, . \rangle$ with a single binary operation satisfying: (1) $x = xx$, (2) $xy = yx$, and (3) $(xy)z = x(yz)$. In this note we show that the three identities may be reduced to two but cannot be reduced to one.

It is easy to see that (2), (3) imply (4) $(uv)((wx)(yz)) = ((vu)(xw))(zy)$. Setting $w = y = u$ and $x = z = v$ in (4) and using (1) we get $uv = vu$. Setting $v = u$, $x = w$, and $z = y$ in (4) and using (1) we get $u(wy) = (uw)y$. And so (1) and (4) imply (2) and (3).

If a single identity is sufficient to define the notion of semi-lattice it must be of form $x = \ldots$. Any identity not of that form is satisfied by, e.g. the algebra $\langle \{0, 1\}, . \rangle$ where $00 = 01 = 10 = 11 = 0$, which is not a semi-lattice.

Now suppose we have a semi-lattice with two distinct elements a, b. Let $c = ab$. Either $c \neq a$ or $c \neq b$. We suppose the latter. Then $bb = b$ and $bc = cb = cc = c$. Thus any identity holding in a semi-lattice with at least two elements must have the same variables occurring on each side of the equality sign. For suppose "x" occurs on the left but not on the right. Setting $x = c$ and all other variables equal to b yields the contradiction $c = b$.

Thus a single sufficing identity would have to be of form $x = f(x)$. Clearly such an identity will not imply (2), for the algebra $\langle \{0, 1\}, . \rangle$ where $00 = 01 = 0$ and $10 = 11 = 1$ satisfies $x = f(x)$ for any f but is not commutative.

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